Chapter 3. The L_p Spaces

3.1 Definition: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let W be a vector space over $\mathbb{F}$. An **inner product** over $\mathbb{F}$ is a function $\langle\ ,\ \rangle : W \times W \to \mathbb{F}$ (meaning that if $u, v \in W$ then $\langle u, v\rangle \in \mathbb{F}$) such that for all $u, v, w \in W$ and all $t \in \mathbb{F}$ we have

(1) (Sesquilinearity) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$, $\langle tu, v\rangle = t\,\langle u, v\rangle$,
$\langle u, v+w\rangle = \langle u, v\rangle + \langle vuw\rangle$, $\langle u, tv\rangle = \overline{t}\,\langle u, v\rangle$,
(2) (Conjugate Symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$, and
(3) (Positive Definiteness) $\langle u, u\rangle \geq 0$ with $\langle u, u\rangle = 0 \iff u = 0$.

For $u, v \in W$, $\langle u, v\rangle$ is called the inner product of u with v. An **inner product space** over $\mathbb{F}$ is a vector space over $\mathbb{F}$ equipped with an inner product. Given two inner product spaces U and V over $\mathbb{F}$, a linear map $L : U \to V$ is called a **homomorphism** of inner product spaces (or we say that L **preserves inner product**) when $\big\langle L(x), L(y)\big\rangle = \langle x, y\rangle$ for all $x, y \in U$.

3.2 Theorem: *Let W be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $u, v \in W$. Then if $\langle x, u\rangle = \langle x, v\rangle$ for all $x \in U$, or if $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in U$ then $u = v$.*

Proof: Suppose that $\langle x, u\rangle = \langle x, v\rangle$ for all $x \in U$. Then $\langle x, u-v\rangle = \langle x, u\rangle - \langle x, v\rangle = 0$ for all $x \in U$. In particular, taking $x = u - v$ we have $\langle u-v, u-v\rangle = 0$, so $u - v = 0$ hence $u = v$. Similarly, if $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in U$ then $u = v$.

3.3 Definition: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let W be a vector space over $\mathbb{F}$. A **norm** on W is a map $\|\ \| : W \to \mathbb{R}$ such that for all $u, v \in W$ and all $t \in \mathbb{F}$ we have

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ with $\big(\|u\| = 0 \iff u = 0\big)$, and
(3) (Triangle Inequality) $\|u+v\| \leq \|u\| + \|v\|$.

For $u \in W$ the real number $\|u\|$ is called the **norm** (or **length**) of u, and we say that u is a **unit vector** when $\|u\| = 1$. A **normed linear space** over $\mathbb{F}$ is a vector space over $\mathbb{F}$ equipped with a norm. Given two normed linear spaces U and V over $\mathbb{F}$, a linear map $L : U \to V$ is called a **homomorphism** of normed linear spaces (or we say that L **preserves norm**) when $\big\|L(x)\big\| = \|x\|$ for all $x \in U$.

3.4 Theorem: *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let W be an inner product space over $\mathbb{F}$. For $u \in W$ define $\|u\| = \sqrt{\langle u, u\rangle}$. Then for all $u, v \in W$ and all $t \in \mathbb{F}$ we have*

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ *with* $\big(\|u\| = 0 \iff u = 0\big)$,
(3) $\|u \pm v\|^2 = \|u\|^2 \pm 2\,\mathrm{Re}\langle u, v\rangle + \|v\|^2$,
(4) (Pythagoras' Theorem) if $\langle u, v\rangle = 0$ *then* $\|u+v\|^2 = \|u\|^2 + \|v\|^2$,
(5) (Parallelogram Law) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$,
(6) (Polarization Identity) if $\mathbb{F} = \mathbb{R}$ *then* $\langle u, v\rangle = \frac{1}{4}\big(\|u+v\|^2 - \|u-v\|^2\big)$ *and*
if $\mathbb{F} = \mathbb{C}$ *then* $\langle u, v\rangle = \frac{1}{4}\big(\|u+v\|^2 + i\|u+iv\|^2 - \|u-v\|^2 - i\|u-iv\|^2\big)$,
(7) (The Cauchy-Schwarz Inequality) $|\langle u, v\rangle| \leq \|u\|\,\|v\|$ *with* $|\langle u, v\rangle| = \|u\|\,\|v\|$ *if and only if* $\{u, v\}$ *is linearly dependent, and*
(8) (The Triangle Inequality) $\big|\|u\| - \|v\|\big| \leq \|u+v\| \leq \|u\| + \|v\|$.

In particular, $\|\ \|$ *is a norm on* W.

Proof: We only prove Part (7) and part of Part (8). To prove Cauchy's Inequality, suppose first that $\{u,v\}$ is linearly dependent. Then one of x and y is a multiple of the other, say $v=tu$ with $t\in\mathbb{F}$. Then $|\langle u,v\rangle| = |\langle u,tu\rangle| = \left|\overline{t}\,\langle u,u\rangle\right| = |t|\,\|u\|^2 = \|u\|\,\|tu\| = \|u\|\,\|v\|$.

Next we suppose that $\{u,v\}$ is linearly independent. Then $1\cdot v+t\cdot u\neq 0$ for all $t\in\mathbb{F}$, so in particular $v-\frac{\langle v,u\rangle}{|u|^2}\,u\neq 0$. Thus we have

$$\begin{aligned}0 &< \left\|v-\tfrac{\langle v,u\rangle}{\|u\|^2}\,u\right\|^2 = \left\langle v-\tfrac{\langle v,u\rangle}{\|u\|^2}\,u\,,\,v-\tfrac{\langle v,u\rangle}{\|u\|^2}\,u\right\rangle\\ &= \langle v,v\rangle-\overline{\tfrac{\langle v,u\rangle}{\|u\|^2}}\,\langle v,u\rangle-\tfrac{\langle v,u\rangle}{\|u\|^2}\,\langle u,v\rangle+\tfrac{\langle v,u\rangle}{\|u\|^2}\,\overline{\tfrac{\langle v,u\rangle}{\|u\|^2}}\,\langle u,u\rangle\\ &= \|v\|^2-\tfrac{|\langle u,v\rangle|^2}{\|u\|^2}\end{aligned}$$

so that $\frac{|\langle u,v\rangle|^2}{\|u\|^2}<\|v\|^2$ and hence $|\langle u,v\rangle|\leq\|u\|\,\|v\|$. This proves Part (7).

Using Parts (3) and (7), and the inequality $|\operatorname{Re}(z)|\leq|z|$ for $z\in\mathbb{C}$ (which follows from Pythagoras' Theorem in $\mathbb{R}^2$), we have

$$\begin{aligned}\|u+v\|^2 &= \|u\|^2+2\operatorname{Re}\langle u,v\rangle+\|v\|^2\leq\|u\|^2+2|\langle u,v\rangle|+\|v\|^2\\ &\leq\|u\|^2+2\|u\|\,\|v\|+\|v\|^2=\big(\|u\|+\|v\|\big)^2.\end{aligned}$$

Taking the square root on both sides gives $\|u+v\|\leq\|u\|+\|v\|$.

3.5 Definition: A **metric** on a set X is a function $d:X\times X\to\mathbb{R}$ such that, for all $x,y,z\in X$ we have

(1) (Positive Definiteness) $d(x,y)\geq 0$ with $d(x,y)=0\iff x=y$,
(2) (Symmetry) $d(x,y)=d(y,x)$ and
(3) (Triangle Inequality) $d(x,z)\leq d(x,y)+d(y,z)$.

A set with a metric is called a **metric space**.

3.6 Definition: A **topology** on a set X is a set $\mathcal{T}$ of subsets of X such that

(1) $\emptyset\in\mathcal{T}$ and $X\in\mathcal{T}$,
(2) if $U\in\mathcal{T}$ and $V\in\mathcal{T}$ then $U\cap V\in\mathcal{T}$, and
(3) if K is a set and $U_k\in\mathcal{T}$ for each $k\in K$ then $\bigcup\limits_{k\in K}U_k\in\mathcal{T}$.

For a subset $A\subseteq X$, we say that A is **open** (in X) when $A\in\mathcal{T}$ and we say that A is **closed** (in X) when $X\setminus A\in\mathcal{T}$. A set with a topology is called a **topological space**.

3.7 Note: Given an inner product on a vector space V over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, Theorem 3.4 shows that we can define an associated norm on V by letting $\|x\|=\sqrt{\langle x,x\rangle}$ for $x\in V$.

Given a norm on a vector space V, verify that we can define an associated metric on any subset $X\subseteq V$ by letting $d(x,y)=\|x-y\|$ for $x,y\in X$.

Given a metric on a set X, verify that we can define an associated topology on X by stipulating that a subset $A\subseteq X$ is open when it has the property that for all $a\in A$ there exists $r>0$ such that $B(a,r)\subseteq A$, where $B(a,r)=\left\{x\in X\,\middle|\,d(x,a)<r\right\}$.

3.8 Definition: Let $\{x_n\}_{n\geq 1}$ be a sequence in a metric space X. We say that the sequence $\{x_n\}$ **converges** in X when there exists $a\in X$ such that $\lim\limits_{n\to\infty} x_n = a$, that is when

$$\exists\, a\!\in\! X\ \forall\,\epsilon\!>\!0\ \exists\, n\!\in\!\mathbb{Z}^+\ \forall k\!\in\!\mathbb{Z}^+\ \big(k\geq n \Longrightarrow d(x_n,a)<\epsilon\big).$$

We say that $\{x_n\}$ is **Cauchy** when

$$\forall\,\epsilon\!>\!0\ \exists\, n\!\in\!\mathbb{Z}^+\ \forall\, k,l\!\in\!\mathbb{Z}^+\ \big(k,l\geq n \Longrightarrow d(x_k,x_l)<\epsilon\big).$$

3.9 Note: Verify that, in a metric space, if a sequence converges then it is Cauchy.

3.10 Definition: A metric space X is called **complete** when, in X, every Cauchy sequence converges. A complete normed linear space is called a **Banach space** and a complete inner-product space is called a **Hilbert space**.

3.11 Theorem: *(The Completeness of $\mathbb{R}^n$) The metric space $\mathbb{R}^n$ is complete.*

Proof: We omit the proof.

3.12 Definition: Let $\mathbb{R}^\omega$ denote the set of all sequences $x=\{x_1,x_2,x_3,\cdots\}$ with each $x_k\in\mathbb{R}$. For $x\in\mathbb{R}^\omega$ and for $1\leq p<\infty$ let

$$\begin{aligned} \|x\|_p &= \Big(\sum_{k=1}^{\infty}|x_k|^p\Big)^{1/p}\ , \text{ and}\\ \|x\|_\infty &= \sup\big\{|x_k|\,\big|\,k\in\mathbb{Z}^+\big\}. \end{aligned}$$

Let

$$\begin{aligned} \ell_p &= \big\{x\in\mathbb{R}^\omega\,\big|\,\|x\|_p<\infty\big\}\ , \text{ and}\\ \ell_\infty &= \big\{x\in\mathbb{R}^\omega\,\big|\,\|x\|_\infty<\infty\big\}. \end{aligned}$$

3.13 Definition: Let $A\subseteq\mathbb{R}$ be measurable. Let $\mathcal{M}(A)$ denote the set of all measurable functions $f:A\to[-\infty,\infty]$. For $f\in\mathcal{M}(A)$ and for $1\leq p<\infty$, let

$$\begin{aligned} \|f\|_p &= \left(\int_A |f|^p\right)^{1/p}\ , \text{ and}\\ \|f\|_\infty &= \inf\left\{a\geq 0\,\middle|\,\lambda\big(|f|^{-1}(a,\infty]\big)=0\right\}. \end{aligned}$$

where $|f|^{-1}(a,\infty]=\big\{x\in A\,\big|\,|f(x)|>a\big\}$. Let

$$\begin{aligned} L_p(A) &= \left\{f\in\mathcal{M}(A)\,\middle|\,\|f\|_p<\infty\right\}\Big/\sim\ \ , \text{ and}\\ L_\infty(A) &= \left\{f\in\mathcal{M}(A)\,\middle|\,\|f\|_\infty<\infty\right\}\Big/\sim \end{aligned}$$

where $\sim$ is the equivalence relation given by $f\sim g \iff f=g$ a.e. in A.

3.14 Remark: The reason that we quotient by the equivalence relation in the above definition is that we want $\|f\|_p$ to define a norm on $L_p(A)$ and the quotient is necessary to ensure that $\|f\|_p$ is positive definite (see Part 6 of Theorem 2.30).

3.15 Lemma: *Let $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$ be measurable. Then $\big\{x \in A \big| |f(x)| > \|f\|_\infty\big\}$ has measure zero.*

Proof: We claim that for all $y > \|f\|_\infty$ we have $\lambda\big(|f|^{-1}(y,\infty]\big) = 0$. Let $y > \|f\|_\infty$. By the definition of $\|f\|_\infty$ we can choose a with $\|f\|_\infty \le a < y$ such that $\lambda\big(|f|^{-1}(a,\infty]\big) = 0$. Since $a < y$ we have $(y,\infty] \subseteq (a,\infty]$, so $|f|^{-1}(y,\infty] \subseteq |f|^{-1}(a,\infty]$, hence $\lambda\big(|f|^{-1}(y,\infty]\big) = 0$, as claimed.

Let $B = \big\{x \in A \,\big|\, |f(x)| > \|f\|_\infty\big\}$ and let $B_n = \big\{x \in A \,\big|\, |f(x)| > \|f\|_\infty + \frac{1}{n}\big\}$ for $n \in \mathbb{Z}^+$. Then each B_n is measurable with $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$, and we have $\bigcup\limits_{n=1}^{\infty} B_n = B$. By the above claim, we have $\lambda(B_n) = 0$ for all $n \in \mathbb{Z}^+$ and so $\lambda(B) = \lim\limits_{n\to\infty} \lambda(B_n) = 0$.

3.16 Definition: For $p, q \in [1,\infty]$ we say that p and q are **conjugate** when $\frac{1}{p} + \frac{1}{q} = 1$ where we use the convention that $\frac{1}{\infty} = 0$ so that 1 and ∞ are conjugate.

3.17 Lemma: *Let $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \ge 0$ we have $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.*

Proof: Note that for $p, q \in (1,\infty)$ we have

$$\tfrac{1}{p} + \tfrac{1}{q} = 1 \iff \tfrac{1}{q} = 1 - \tfrac{1}{p} = \tfrac{p-1}{p} \iff q(p-1) = p \iff p(q-1) = q.$$

For $x, y \ge 0$ we have

$$y = x^{p-1} \iff y^q = x^{q(p-1)} \iff y^q = x^p \iff y^{p(q-1)} = x^p \iff y^{q-1} = x$$

so the functions $f(x) = x^{p-1}$ and $g(y) = y^{q-1}$ are inverses of each other. By considering the area under $y = f(x)$ with $0 \le x \le a$ and the area to the left of $y = f(x)$ with $0 \le y \le b$ we see that

$$ab \le \int_{x=0}^{a} x^{p-1}\,dx + \int_{y=0}^{b} y^{q-1}\,dy = \Big[\tfrac{1}{p}\,x^p\Big]_{x=0}^{a} + \Big[\tfrac{1}{q}\,y^q\Big]_{y=0}^{b} = \tfrac{a^p}{p} + \tfrac{b^q}{q}\,.$$

3.18 Theorem: *(Hölder's Inequality) Let $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $A \subseteq \mathbb{R}$ be measurable.*
(1) For all $x, y \in \mathbb{R}^\omega$ we have $\|xy\|_1 \le \|x\|_p \|y\|_q$.
(2) For all $f, g \in \mathcal{M}(A)$ for which fg is defined, we have $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Proof: To prove Part (1) in the case that $p, q \in (1,\infty)$, let $x, y \in \mathbb{R}^\omega$. If $x = 0$ or $y = 0$ the equality holds, so suppose that $x, y \ne 0$. For each index k, apply the above lemma using $a = \frac{|x_k|}{\|x\|_p}$ and $b = \frac{|y_k|}{\|y\|_q}$ to get

$$\frac{|x_k y_k|}{\|x\|_p \|y\|_q} \le \frac{|x_k|^p}{p\|x\|_p^p} + \frac{|y|^q}{q\|y\|_q^q}.$$

Sum over k to get

$$\frac{\|xy\|_1}{\|x\|_p \|y\|_q} \le \frac{\|x\|_p^p}{p\|x\|_p^p} + \frac{\|y\|_q^q}{q\|y\|_q^q} = \tfrac{1}{p} + \tfrac{1}{q} = 1\,.$$

To prove Part (2) in the case that $p,q\in(1,\infty)$, let $f,g\in\mathcal{M}(\mathbb{R})$. If $\|f\|_p=0$ or $\|g\|_q=0$ then the equality holds (with both sides equal to 0), so suppose that $\|f\|_p,\|g\|_q\neq 0$. For each $x\in A$, apply the above lemma using $a=\frac{|f(x)|}{\|f\|_p}$ and $b=\frac{|g(x)|}{\|g\|_q}$ to get

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}\leq\frac{|f(x)|^p}{p\|f\|_p}+\frac{|g(x)|^q}{q\|g\|_q}.$$

Integrate over A to get

$$\frac{\|fg\|_1}{\|f\|_p\|g\|_q}\leq\frac{\|f\|_p^p}{p\|f\|_p^p}+\frac{\|g\|_q^q}{q\|g\|_q^q}=\tfrac{1}{p}+\tfrac{1}{q}=1\,.$$

To prove Part (1) in the case that $p=1$ and $q=\infty$, let $x,y\in\mathbb{R}^\omega$. Note that $|y_k|\leq\|y\|_\infty$ for all indices k and so

$$\|xy\|_1=\sum_{k=1}^\infty|x_k\|y_k|\leq\sum_{k=1}^\infty|x_k|\,\|y\|_\infty=\|x\|_1\|y\|_\infty.$$

Finally, to prove Part (2) in the case that $p=1$ and $q=\infty$, let $f,g\in\mathcal{M}(A)$. Let $B=\big\{x\in A\big||g(x)|\leq\|g\|_\infty\big\}$ and let $C=\big\{x\in A\big|g(x)|>\|g\|_\infty\big\}$. Note that B and C are disjoint and measurable with $A=B\cup C$ and that $\lambda(C)=0$ by Lemma 3.15. Thus

$$\|fg\|_1=\int_A|f\|g|=\int_B|f\|g|\leq\int_B|f|\,\|g\|_\infty=\int_A|f|\,\|g\|_\infty=\|f\|_1\|g\|_\infty.$$

3.19 Theorem: *(Minkowski's Inequality) Let $p\in[1,\infty]$ and let $A\subseteq\mathbb{R}$ be measurable.*
(1) For all $x,y\in\mathbb{R}^\omega$ we have $\|x+y\|_p\leq\|x\|_p+\|y\|_p$.
(2) For all $f,g\in\mathcal{M}(A)$ for which $f+g$ is defined, we have $\|f+g\|_p\leq\|f\|_p+\|g\|_p$.

Proof: To Prove Part (1) in the case that $p=1$, note that when $x,y\in\mathbb{R}^\omega$ we have

$$\|x+y\|_1=\sum_{k=1}^\infty|x_k+y_k|\leq\sum_{k=1}^\infty|x_k|+|y_k|=\sum_{k=1}^\infty|x_k|+\sum_{k=1}^\infty|y_k|=\|x\|_1+\|y\|_1.$$

To prove Part (2) in the case that $p=1$, note that when $f,g,f+g\in\mathcal{M}(A)$ we have

$$\|f+g\|_1=\int_A|f+g|\leq\int_A|f|+|g|=\int_A|f|+\int_A|g|=\|f\|_1+\|g\|_1.$$

To prove Part (1) in the case that $p\in(1,\infty)$, let $x,y\in\mathbb{R}^\omega$ and let q be the conjugate of p so that $\frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}$. For each index k we have

$$\begin{aligned}|x_k+y_k|^p&=|x_k+y_k|\,|x_k+y_k|^{p-1}\leq\big(|x_k|+|y_k|\big)|x_k+y_k|^{p-1}\\&=|x_k|\,|x_k+y_k|^{p-1}+|y_k|\,|x_k+y_k|^{p-1}.\end{aligned}$$

Sum over k then apply Hölder's Inequality to get

$$\begin{aligned}\|x+y\|_p^p&\leq\Big\||x|\,|x+y|^{p-1}\Big\|_1+\Big\||y|\,|x+y|^{p-1}\Big\|_1\leq\|x\|_p\Big\||x+y|^{p-1}\Big\|_q+\|y\|_p\Big\||x+y|^{p-1}\Big\|_q\\&=\Big(\|x\|_p+\|y\|_q\Big)\Big\||x+y|^{p-1}\Big\|_q=\Big(\|x\|_p+\|y\|_p\Big)\Big(\sum_{k=1}^\infty|x_k+y_k|^{q(p-1)}\Big)^{1/q}\\&=\Big(\|x\|_p+\|y\|_p\Big)\Big(\sum_{k=1}^\infty|x_k+y_k|^p\Big)^{(p-1)/p}=\Big(\|x\|_p+\|y\|_p\Big)\|x+y\|_p^{p-1}.\end{aligned}$$

To prove Part (2) in the case that $p \in (1,\infty)$, let $f,g,f+g \in \mathcal{M}(A)$ and let q be the conjugate of p so that $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$. For each $x \in A$ we have

$$\begin{aligned} |f(x)+g(x)|^p &= |f(x)+g(x)|\,|f(x)+g(x)|^{p-1} \le \big(|f(x)|+|g(x)|\big)|f(x)+g(x)|^{p-1} \\ &= |f(x)|\,|f(x)+g(x)|^{p-1} + |g(x)|\,|f(x)+g(x)|^{p-1}. \end{aligned}$$

Integrate over A then apply Hölder's Inequality to get

$$\begin{aligned} \|f+g\|_p^p &\le \Big\||f|\,|f+g|^{p-1}\Big\|_1 + \Big\||g|\,|f+g|^{p-1}\Big\|_1 \le \|f\|_p \Big\||f+g|^{p-1}\Big\|_q + \|g\|_p \Big\||f+g|^{p-1}\Big\|_q \\ &= \Big(\|f\|_p + \|g\|_p\Big)\Big\||f+g|^{p-1}\Big\|_q = \Big(\|f\|_p + \|g\|_p\Big)\Big(\int_A |f+g|^{q(p-1)}\Big)^{1/q} \\ &= \Big(\|f\|_p + \|g\|_p\Big)\Big(\int_A |f+g|^p\Big)^{(p-1)/p} = \Big(\|f\|_p + \|g\|_p\Big)\|f+g\|_p^{p-1}. \end{aligned}$$

To prove Part (1) in the case that $p = \infty$, note that if $x, y \in \ell_\infty$ then we have

$$\|x+y\|_\infty = \sup_{k\ge 1}|x_k+y_k| \le \sup_{k\ge 1}\big(|x_k|+|y_k|\big) \le \sup_{k\ge 1}|x_k| + \sup_{k\ge 1}|y_k| = \|x\|_\infty + \|y\|_\infty.$$

To prove Part (2) in the case that $p = \infty$, let $f, g \in \mathcal{M}(A)$. For all $x \in A$, note that if $\big|f(x)+g(x)\big| > \|f\|_\infty + \|g\|_\infty$ then $|f(x)|+|g(x)| \ge \big|f(x)+g(x)\big| > \|f\|_\infty + \|g\|_\infty$ and hence either $|f(x)| > \|f\|_\infty$ or $|g(x)| > \|g\|_\infty$. This shows that

$$\big\{x\in A\big||f(x)+g(x)| > \|f\|_\infty + \|g\|_\infty\big\} \subseteq \big\{x\in A\big||f(x)| > \|f\|_\infty\big\} \cup \big\{x\in A\big||g(x)| > \|g\|_\infty\big\}.$$

By Lemma 3.15, the two sets on the right both have measure zero, and so the set on the left has measure zero. By the definition of $\|f+g\|_\infty$ it follows that $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

3.20 Corollary: *Let $p \in [1,\infty]$ and let $A \subseteq \mathbb{R}$ be measurable. Then ℓ_p and $L_p(A)$ are normed linear spaces using their p-norms. Also, ℓ_2 is an inner product space using $\langle x,y\rangle = \sum_{k=1}^\infty x_k y_k$, and $L_2(A)$is an inner product space using $\langle f,g\rangle = \int_A fg$.*

Proof: We leave the proof for the ℓ_p spaces as an exercise, and provide the proof for $L_p(A)$. Let $\mathcal{M}(A,[-\infty,\infty])$ be the set of measurable functions $f: A\to[-\infty,\infty]$ (which is not a vector space because addition is not defined) and let $\mathcal{M}(A,\mathbb{R})$ be the vector space of measurable functions $f: A \to \mathbb{R}$. Let $L_p(A) = \big\{f \in \mathcal{M}(A,[-\infty,\infty]\,\big|\,\|f\|_p < \infty\big\}\big/\sim$ and let $L_p(A,\mathbb{R}) = \big\{f \in \mathcal{M}(A,\mathbb{R})\,\big|\,\|f\|_p < \infty\big\}\big/\sim$, where $f \sim g$ when $f = g$ a.e. in A. Note that when $f \in \mathcal{M}(A,[-\infty,\infty])$ with $\|f\|_p < \infty$, we have $|f(x)| < \infty$ for a.e. $x \in A$, so we can identify $L_p(A,[-\infty,\infty])$ with $L_p(A,\mathbb{R})$. Let $W = \big\{f \in \mathcal{M}(A,\mathbb{R})\,\big|\,\|f\|_p < \infty\big\}$ and $V = \big\{f \in W\,\big|\,f = 0 \text{ a.e. in } A\big\}$. Note that W is a subspace of $\mathcal{M}(A,\mathbb{R})$ because of Minkowski's Inequality (if $f,g \in W$ then $f+g\in W$ because $\|f+g\|_p \le \|f\|_p + \|g\|_p$), and note that V is a subspace of W. It follows that $L_p(A,\mathbb{R})$ is a vector space, indeed it is the quotient space $L_p(A,\mathbb{R}) = W/V$. It is easy to see that the p-norm is well-defined on $L_p(A,\mathbb{R})$ and it satisfies all the axioms (with the Triangle Inequality following directly from Minkowski's Inequality). Finally note that when $f,g \in L_2(A,\mathbb{R})$, Hölder's Inequality gives $\int_A |fg| = \big\||f|\,|g|\big\|_1 \le \|f\|_2\|g\|_2 < \infty$ so that fg is integrable, and so the inner product $\langle f,g\rangle = \int_A fg$ is well-defined. It is easy to see that it satisfies the inner product axioms.

3.21 Theorem: *Let $p \in [1, \infty]$ and let $A \subseteq \mathbb{R}$ be measurable. Then the normed linear spaces ℓ_p and $L_p(A)$ are complete.*

Proof: We leave the proof that ℓ_p is complete as an exercise. To prove that $L_p(A)$ is complete in the case that $p < \infty$, let $(f_n)_{n \geq 1}$ be a Cauchy sequence in $L_p(A)$. This means that for all $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $k, l \geq m \implies \|f_k - f_l\|_p < \epsilon$. Choose a subsequence $(f_{n_k})_{k \geq 1}$ with the property that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_p \leq \frac{1}{2^k}$ for all $k \geq 1$. For each $\ell \in \mathbb{Z}^+$, let

$$g_\ell = \sum_{k=1}^{\ell} \left|f_{n_{k+1}} - f_{n_k}\right|$$

and let $g = \lim_{\ell \to \infty} g_\ell$ (note that the limit exists because $(g_\ell(x))_{\ell \geq 1}$ is increasing for all $x \in A$). By Minkowski's Inequality, for all $\ell \in \mathbb{Z}^+$ we have

$$\|g_\ell\|_p \leq \sum_{k=1}^{\ell} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \sum_{k=1}^{\ell} \frac{1}{2^k} < 1.$$

By Fatou's Lemma,

$$\|g\|_p^p = \int_A |g|^p = \int_A \lim_{\ell \to \infty} |g_\ell|^p \leq \liminf_{\ell \to \infty} \int_A |g_\ell|^p = \liminf_{\ell \to \infty} \|g_\ell\|_p^p \leq 1$$

so that $g \in L_p(A)$. Because $\|g\|_p$ is finite, it follows that g is finite a.e. in A, so the sum $\sum \left|f_{n_{k+1}} - f_{n_k}\right|$ converges a.e. in A, hence the sum $\sum \left(f_{n_{k+1}} - f_{n_k}\right)$ converges a.e. in A, and hence the sequence $(f_{n_\ell})_{\ell \geq 1}$ converges a.e. in A because $f_{n_\ell} = f_{n_1} + \sum_{k=1}^{\ell-1} \left(f_{n_{k+1}} - f_{n_k}\right)$. We define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} \lim_{\ell \to \infty} f_{n_\ell}(x) & \text{, if the limit exists in } \mathbb{R}, \text{ and} \\ 0 & \text{, otherwise.} \end{cases}$$

We claim that $f \in L_p(A)$ and that $\lim_{n \to \infty} f_n = f$ in $L_p(A)$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that for all $k, l \geq m$ we have $\|f_k - f_l\|_p \leq \epsilon$. Then for all k such that $n_k \geq m$ we have $\|f_{n_k} - f_m\|_p \leq \epsilon$. By Fatou's Lemma,

$$\begin{aligned} \|f - f_m\|_p^p &= \int_A |f - f_m|^p = \int_A \lim_{k \to \infty} |f_{n_k} - f_m|^p \\ &\leq \liminf_{k \to \infty} \int_A |f_{n_k} - f_m|^p = \liminf_{k \to \infty} \|f_{n_k} - f_m\|_p^p \leq \epsilon^p \end{aligned}$$

so that $\|f - f_m\|_p \leq \epsilon$. This shows that for all $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that for all $n \geq m$ we have $\|f - f_n\|_p \leq \epsilon$. It will follow that $\lim_{n \to \infty} f_n = f$ in $L_p(A)$ once we show that $f \in L_p(A)$. Taking $\epsilon = 1$ and choosing m as above so that $\|f - f_m\| \leq 1$, Minkowski's Inequality gives $\|f\|_p \leq \|f - f_m\|_p + \|f_m\|_p \leq 1 + \|f_m\|_p < \infty$ so that $f \in L_p(A)$, as required.

Now let us prove that $L_\infty(A)$ is complete. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L_\infty(A)$. Let $B_n = \big\{x\in A \big| \, f_n(x) > \|f_n\|_\infty\big\}$ and $C_{k,l} = \big\{x\in A \big| \, |f_k(x)-f_l(x)| > \|f_k-f_l\|_\infty\big\}$. By Lemma 3.15, the sets B_n and $C_{k,l}$ all have measure zero. Let E be the union of all the sets B_n and $C_{k,l}$. Since E is a countable union of sets of measure zero, we have $\lambda(E)=0$. Given $\epsilon>0$, since $(f_n)_{n\geq 1}$ is Cauchy in $L_\infty(A)$ we can choose $m\in\mathbb{Z}^+$ so that for all $k,l\geq m$ we have $\|f_k-f_l\|_\infty \leq \epsilon$. Then for all $k,l\geq m$ we have $|f_k(x)-f_l(x)| \leq \|f_k-f_l\|_\infty \leq \epsilon$ for all $x\in A\backslash E$. It follows, by the Cauchy criterion for uniform convergence, that the sequence (f_n) converges uniformly in $A\setminus E$. Define $f: A\to\mathbb{R}$ by

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{, if } x\in A\setminus E \\ 0 & \text{, if } x\in E. \end{cases}$$

We claim that $f\in L_\infty(A)$ and that $\lim\limits_{n\to\infty} f_n = f$ in $L_\infty(A)$. Given $\epsilon>0$, since (f_n) converges uniformly to f in $A\setminus E$, we can choose $m\in\mathbb{Z}^+$ so that for all $n\geq m$ we have $|f_n(x)-f(x)|\leq\epsilon$ for all $x\in A\backslash E$ hence $\|f_n-f\|_\infty\leq\epsilon$ since $\lambda(E)=0$. This shows that for all $\epsilon>0$ there exists $m\in\mathbb{Z}^+$ such that for all $n\geq m$ we have $\|f-f_n\|_\infty\leq\epsilon$. Taking $\epsilon=1$ and choosing m as above, we have $\|f_m-f\|_\infty\leq 1$ so by Minkowski's Inequality $\|f\|_\infty \leq \|f-f_m\|_\infty + \|f_m\|_\infty \leq 1+\|f_m\|_\infty$ and so $f\in L_\infty(A)$.

3.22 Theorem: *Let $1\leq p<q\leq\infty$ and let $A\subseteq\mathbb{R}$ be measurable. Then*
(1) $\ell_p\subseteq\ell_q$, and
(2) if $\lambda(A)<\infty$ then $L_q(A)\subseteq L_p(A)$.

Proof: We leave the proof of Part (1) as an exercise. To prove Part (2), suppose that $\lambda(A)<\infty$. Consider first the case that $q<\infty$. Let $f\in L_q(A)$. Then by Hölder's Inequality, for any $u,v>1$ with $\frac{1}{u}+\frac{1}{v}=1$ we have

$$\|f\|_p^p = \int_A |f|^p = \Big\||f|^p\Big\|_1 \leq \Big\||f|^p\Big\|_u \Big\|1\Big\|_v = \left(\int_A |f|^{pu}\right)^{1/u} \lambda(A)^{1/v}.$$

Choose $u=\frac{q}{p}$ and, to get $\frac{1}{v}=1-\frac{1}{u}=1-\frac{p}{q}=\frac{q-p}{q}$, choose $v=\frac{q}{q-p}$. Then

$$\|f\|_p^p \leq \left(\int_A |f|^q\right)^{p/q} \lambda(A)^{(q-p)/q} = \|f\|_q^p\, \lambda(A)^{(q-p)/q}$$

so that $\|f\|_p \leq \|f\|_q\, \lambda(A)^{\frac{1}{p}-\frac{1}{q}}$. Thus $\|f\|_p<\infty$ so $f\in L_p(A)$.

Now consider the case that $q=\infty$. Let $f\in L_\infty(A)$. Let $B=\big\{x\in A\big| |f(x)|\leq\|f\|_\infty\big\}$ and $C=\big\{x\in A\big| |f(x)|>\|f\|_\infty\big\}$. By Lemma 3.15 we have $\lambda(C)=0$, so

$$\|f\|_p^p = \int_A |f|^p = \int_B |f|^p \leq \int_B \|f\|_\infty^p = \|f\|_\infty^p\,\lambda(B) = \|f\|_\infty^p\,\lambda(A)$$

so that $\|f\|_p\leq\|f\|_\infty\,\lambda(A)^{1/p}$. Thus $\|f\|_p<\infty$ so $f\in L_p(A)$.

3.23 Theorem: *Let $1 \leq p < q < r \leq \infty$ and let $A \subseteq \mathbb{R}$ be measurable. Then*
(1) $\ell_p \cap \ell_r \subseteq \ell_q \subseteq \ell_p + \ell_r$, and
(2) $L_p(A) \cap L_r(A) \subseteq L_q(A) \subseteq L_p(A) + L_r(A)$.

Proof: Part (1) follows as an immediate corollary of Theorem 3.22. Let us prove Part (2). First we claim that $L_q(A) \subseteq L_p(A) + L_r(A)$. Let $f \in L_q(A)$. Let $B = \big\{x \in A \big| |f(x)| > 1\big\}$ and let $C = \big\{x \in A \big| |f(x)| \leq 1\big\}$. Let $g = f \cdot \chi_{_B}$ and $h = f \cdot \chi_{_C}$ so that $f = g + h$. Note that $g \in L_p(A)$ because

$$\|g\|_p^p = \int_A |g|^p = \int_B |f|^p \leq \int_B |f|^q \leq \int_A |f|^q = \|f\|_q^q < \infty,$$

note that $h \in L_\infty(A)$ because $|h(x)| \leq 1$ for all $x \in A$ so that $\|h\|_\infty \leq 1$, and note that when $r < \infty$ we have $h \in L_r(A)$ because

$$\|h\|_r^r = \int_A |h|^r = \int_C |f|^r \leq \int_C |f|^q \leq \int_A |f|^q = \|f\|_q^q < \infty.$$

Thus we have $L_q(A) \subseteq L_p(A) + L_r(A)$ as claimed.

Next we claim that $L_p(A) \cap L_r(A) \subseteq L_q(A)$. Let $f \in L_p(A) \cap L_r(A)$. Suppose first that $r < \infty$. Note that for any $0 < k, l \in \mathbb{R}$ with $k + l = q$ and for any $1 < u, v \in \mathbb{R}$ with $\frac{1}{u} + \frac{1}{v} = 1$, Hölder's Inequality gives

$$\|f\|_q^q = \int_A |f|^q \leq \big\||f|^k\|_u \big\||f|^l\|_v = \left(\int_A |f|^{ku}\right)^{1/u} \left(\int_A |f|^{lv}\right)^{1/v}.$$

We solve the equations $k + l = q$, $\frac{1}{u} + \frac{1}{v} = 1$, $ku = p$ and $lv = r$ to get

$$k = \tfrac{p(r-q)}{r-p}\ ,\ l = \tfrac{r(q-p)}{r-p}\ ,\ u = \tfrac{r-p}{r-q} \text{ and } v = \tfrac{r-p}{q-p}$$

and note that since $1 \leq p < q < r < \infty$ we have $k, l > 0$ and $1 < u, v < \infty$. Thus

$$\|f\|_q^q \leq \left(\int_A |f|^{ku}\right)^{1/u} \left(\int_A |f|^{lv}\right)^{1/v} = \left(\int_A |f|^p\right)^{k/p} \left(\int_A |f|^r\right)^{l/r} = \|f\|_p^k \|f\|_r^l < \infty.$$

When $r = \infty$, we let $B = \big\{x \in A \big| |f(x)| > \|f\|_\infty\big\}$ and $C = \big\{x \in A \big| |f(x)| \leq \|f\|_\infty\big\}$, and then by Lemma 3.15 we have $\lambda(B) = 0$, and so

$$\|f\|_q^q = \int_A |f|^q = \int_C |f|^q = \int_C |f|^p |f|^{q-p} \leq \|f\|_\infty^{q-p} \int_C |f|^p \leq \|f\|_p^p \|f\|_\infty^{q-p} < \infty.$$

This proves that $L_p(A) \cap L_r(A) \subseteq L_q(A)$ as claimed.

3.24 Definition: A metrix space is called **separable** when it contains a countable dense subset.

3.25 Theorem: *Let $1 \leq p < \infty$ and let $a < b$.*
(1) ℓ_p is separable but ℓ_∞ is not.
(2) $L_p\big([a,b]\big)$ is separable but $L_\infty\big([a,b]\big)$ is not.

Proof: We leave the proof of Part (1) as an exercise. We sketch a proof of Part (2) leaving the details as an exercise. To show that $L_p[a,b]$ is separable, we shall show that $\mathbb{Q}[x]$ is dense in $L_p[a,b]$ by showing that a given function $f \in L_p[a,b]$ can be approximated, arbitrarily closely in the p-norm, by a polynomial in $\mathbb{Q}[x]$. Since $f = f^+ - f^-$ it suffices to consider the case that f is nonnegative. By Note 2.28, together with the Monotone Convergence Theorem, we can approximate a given nonnegative function $f \in L_p[a,b]$, arbitrarily closely in the p-norm, using a nonnegative simple function since we can construct an increasing sequence of simple functions $s_n : [a,b] \to [0,\infty)$ with $s_n \to f$ pointwise on $[a,b]$. We can approximate a given nonnegative simple function $s : [a,b] \to [0,\infty)$, arbitrarily closely in the p-norm, using a nonnegative step function $r : [a,b] \to [0,\infty)$ because we can cover a measurable set $A \subseteq [a,b]$ by a disjoint union of intervals $J_k \subseteq [a,b]$ so that $\chi_{{}_A}$ is approximated by $\sum \chi_{{}_{J_k}}$. We can then approximate a given step function $r : [a,b] \to [0,\infty)$, arbitrarily closely in the p-norm, using a continuous function because for any interval J, the step function $\chi_{{}_J}$ can be approximated arbitrarily closely in the p-norm by a piecewise linear function. This shows that the set of continuous functions $C[a,b]$ is dense in $L_p[a,b]$, using the p-norm. On the other hand, using the ∞-norm (which agrees with the supremum norm for continuous functions), $\mathbb{Q}[x]$ is dense in $\mathbb{R}[x]$, and we know from the Stone-Weirstrass Theorem that $\mathbb{R}[x]$ is dense in $C[a,b]$. Since $\mathbb{Q}[x]$ is dense in $C[a,b]$ using the ∞-norm, it is also dense using the p-norm by the formula $\|f\|_p \leq (b-a)^{1/p}\|f\|_\infty$ which is obtained in the proof of Theorem 3.22.

We claim that $L_\infty[a,b]$ is not separable. Let S be any dense subset of $L_\infty[a,b]$. We must show that S is uncountable. For each $k \in \mathbb{N}$ let $x_k = b - \frac{b-a}{2^k}$ so that we have $a = x_0 < x_1 < x_2 < \cdots < b$. Let $\{0,1\}^\omega$ denote the set of binary sequences $\alpha = (\alpha_1, \alpha_2, \cdots)$ where each $\alpha_k \in \{0,1\}$. For each $\alpha \in \{0,1\}^\omega$, let $s_\alpha = \sum\limits_{k=1}^{\infty} \alpha_k \chi_{{}_{[x_{k-1},x_k)}}$ and note that when $\alpha \neq \beta$ we have $\|s_\alpha - s_\beta\|_\infty = 1$. Since S is dense in $L_\infty[a,b]$, for each $\alpha \in \{0,1\}^\omega$ we can choose $f_\alpha \in S$ such that $\|s_\alpha - f_\alpha\|_\infty < \frac{1}{2}$. Define $F : \{0,1\}^\omega \to S$ by $F(\alpha) = f_\alpha$. Note that F is injective because when $\alpha \neq \beta$ we have

$$1 = \|s_\alpha - s_\beta\|_\infty \leq \|s_\alpha - f_\alpha\|_\infty + \|f_\alpha - f_\beta\|_\infty + \|f_\beta - s_\beta\|_\infty < \tfrac{1}{2} + \|f_\alpha - f_\beta\|_\infty + \tfrac{1}{2}$$

so that $\|f_\alpha - f_\beta\|_\infty > 0$. Since F is injective we have $|S| \geq \big|\{0,1\}^\omega\big| = 2^{\aleph_0}$, and so S is uncountable, as required.

3.26 Remark: I may include a discussion of the complex-valued L_p spaces $L_p(A,\mathbb{C})$ later.