Chapter 2. Lebesque Integration

2.1 Definition: When $f : [a, b] \to \mathbb{R}$ is bounded and $X = (x_0, x_1, \cdots, x_\ell)$ is a partition of $[a, b]$, which means that $a = x_0 < x_1 < \cdots < x_\ell = b$, and $I_k = [x_{k-1}, x_k]$ is the k^{th} subinterval of X, the **upper and lower Riemann sums** for f on X are given by

$$U(f, X) = \sum_{k=1}^{\ell} M(I_k)\,|I_k|$$
$$L(f, X) = \sum_{k=1}^{\ell} m(I_k)\,|I_k|$$

where $M(I_k) = \sup\left\{f(t)\,\middle|\,t\in I_k\right\}$ and $m(I_k) = \inf\left\{f(t)\,\middle|\,t\in I_k\right\}$, and we define the **upper and lower Riemann integrals** of f on $[a, b]$ to be

$$U(f) = \inf\left\{U(f, X)\,\middle|\,X \text{ is a partition of } [a, b]\right\}$$
$$L(f) = \sup\left\{L(f, X)\,\middle|\,X \text{ is a partition of } [a, b]\right\},$$

We say that a function $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** on $[a, b]$ when f is bounded and $U(f) = L(f)$. In this case, we define the **Riemann integral** of f on $[a, b]$ to be

$$\int_a^b f = \int_a^b f(x)\,dx = U(f) = L(f).$$

2.2 Theorem: *(An Equivalent Definition of Integrability) Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if f has the property that for every $\epsilon > 0$ there exists a partition X of $[a, b]$ such that $U(f, X) - L(f, X) < \epsilon$.*

Proof: We omit the proof (this was likely proven in a previous course).

2.3 Theorem: *(Properties of the Riemann Integral) Let $f, g : [a, b] \to \mathbb{R}$ be bounded.*

(1) If f and g are Riemann integrable on $[a, b]$ and $f \le g$ then $\int_a^b f \le \int_a^b g$.
(2) If f and g are Riemann integrable on $[a, b]$ and $c \in \mathbb{R}$ then the functions cf and $f + g$ are Riemann integrable on $[a, b]$ and $\int_a^b (cf) = c\int_a^b f$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
(3) If $c \in (a, b)$ then f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable both on $[a, c]$ and on $[c, b]$ and, in this case, $\int_a^b f = \int_a^c f + \int_c^b f$.
(4) If $f(x) = g(x)$ for all but finitely many $x \in [a, b]$ then f is Riemann integrable on $[a, b]$ if and only if g is Riemann integrable on $[a, b]$ and, in this case, $\int_a^b f = \int_a^b g$.
(5) If f is monotonic then f is Riemann integrable.
(6) If f is continuous then f is Riemann integrable.

Proof: We omit the proof.

2.4 Theorem: *(The Fundamental Theorem of Calculus) Let $f, g : [a, b] \to \mathbb{R}$. Suppose that g is differentiable with $g' = f$ in $[a, b]$ and that f is Riemann integrable on $[a, b]$. Then*

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

Proof: We omit the proof.

2.5 Theorem: *(Lebesgue's Characterization of Riemann Integrability) Let $f:[a,b]\to\mathbb{R}$. Then f is Riemann integrable on $[a,b]$ if and only if f is bounded and the set of all points in $[a,b]$ at which f is discontinuous has measure zero.*

Proof: For an interval I with $I\cap[a,b]\neq\emptyset$, define the **oscillation** of f on $[a,b]$ to be

$$\Omega(I)=\sup\big\{f(x)-f(y)\,\big|\,x,y\in I\cap[a,b]\big\}=M(I)-m(I)$$

where $M(I)=\sup\big\{f(x)\,\big|\,x\in I\cap[a,b]\big\}$ and $m(I)=\inf\big\{f(x)\,\big|\,x\in I\cap[a,b]\big\}$. Note that for $x\in[a,b]$, $\Omega(x-h,x+h)$ is increasing with h, and define the **oscillation** of f at x to be

$$\omega(x)=\lim_{h\to 0^+}\Omega(x-h,x+h).$$

Verify that f is continuous at x if and only if $\omega(x)=0$, and so the set of points at which f is discontinuous is

$$D=\big\{x\in[a,b]\big|\,\omega(x)>0\big\}=\bigcup_{n=1}^{\infty}D_n \text{ where } D_n=\big\{x\in[a,b]\,\big|\,\omega(x)\geq\tfrac{1}{n}\big\}.$$

We claim that each set D_n is closed (hence compact). Suppose, for a contradiction, that D_n is not closed. Choose a sequence $(x_k)_{k\geq 1}$ in D_n with $x_n\to x$ but $x\notin D_n$. Since $x\notin D_n$ we have $\omega(x)<\frac{1}{n}$, that is $\lim\limits_{h\to 0^+}\Omega(x-h,x_h)<\frac{1}{n}$ so we can choose $h>0$ such that $\Omega(x-h,x+h)<\frac{1}{n}$. Since $x_k\to x$, we can choose k so that $\|x_k-x\|<\frac{h}{2}$ and then we have $\big(x_k-\frac{h}{2},x_k+\frac{h}{2}\big)\subseteq\big(x-h,x+h\big)$, and hence $\omega(x_k)\leq\Omega\big(x_k-\frac{h}{2},x_k+\frac{h}{2}\big)\leq\Omega(x-k,x+h)<\frac{1}{n}$. But this means that $x_k\notin D_n$, giving the desired contradiction. Thus each D_n is closed. Since $D=\bigcup_{n=1}^{\infty}D_n$, with each D_n closed, and $D_1\subseteq D_2\subseteq D_3\cdots$, it follows that D is measurable (indeed $D\in\mathcal{F}_\delta$) with $\lambda(D)=\lim\limits_{n\to\infty}\lambda(D_n)$.

Let f be bounded and suppose that $\lambda(D)>0$. Since $\lambda(D)=\lim\limits_{n\to\infty}D_n$ we can choose $n\geq 1$ such that $\lambda(D_n)>0$, say $\lambda(D_n)=m>0$. Let $X=(x_0,x_1,\cdots,x_\ell)$ be any partition of $[a,b]$, and let $I_k=[x_{k-1},x_k]$. Note that if $I_k\cap D_n\neq\emptyset$ then for $x\in I_k\cap D_n$ we have $\omega(x)\geq\frac{1}{n}$ and hence $M(I_k)-m(I_k)=\Omega(I_k)\geq\omega(x)\geq\frac{1}{n}$. Also note that $D_n\subseteq\bigcup\limits_{k\in K}I_k$ where $K=\big\{k\,\big|\,I_k\cap D_n\neq\emptyset\big\}$, and so we have

$$\begin{aligned}U(f,X)-L(f,X)&=\sum_{k=1}^{\ell}(M(I_k)-m(I_k))|I_k|\geq\sum_{k\in K}(M(I_k)-m(I_k))|I_k|\\&\geq\frac{1}{n}\sum_{k\in K}|I_k|\geq\frac{1}{n}\lambda(D)=\frac{m}{n}.\end{aligned}$$

Since $U(f,X)-L(f,X)>\frac{m}{n}$ for every partition X of $[a,b]$, it follows (from Theorem 2.2) that f is not Riemann integral on $[a,b]$.

Now suppose that f is bounded and that $\lambda(D)=0$. Let $\epsilon>0$ and choose $n\in\mathbb{Z}^+$ such that $\frac{(M-m)+(b-a)}{n}<\epsilon$ where $M=\sup\big\{f(x)\,\big|\,x\in[a,b]\big\}$ and $m=\inf\big\{f(x)\,\big|\,x\in[a,b]\big\}$. Since $\lambda(D)=0$ and $D_n\subseteq D$, we also have $\lambda(D_n)=0$. Choose disjoint open intervals $I_1,I_2,I_3,\cdots$ such that $D_n\subseteq\bigcup_{k=1}^{\infty}I_k$ and $\sum_{k=1}^{\infty}|I_k|<\frac{1}{n}$. Since D_n is compact, we can choose $\ell\in\mathbb{Z}^+$ such that $D_n\subseteq\bigcup_{k=1}^{\ell}I_k$. Let $J_k=I_k\cap[a,b]$, and note that $D_n\subseteq\bigcup_{k=1}^{\ell}J_j$ and $\sum_{k=1}^{\ell}|J_k|<\frac{1}{n}$. Let $E=[a,b]\setminus\bigcup_{k=1}^{\ell}J_k$ and note that E is a finite union of disjoint closed intervals in $[a,b]$ (so E is compact), and E is disjoint from D_n.

We claim that we can choose $\delta>0$ such that for every nonempty interval L in E with $|L|<\delta$ we have $\Omega(L)<\frac{1}{n}$. For each $x\in E$ we have $x\notin D_n$ so that $\omega(x)<\frac{1}{n}$, and so we can choose $h_x>0$ such that $\Omega(x-h_x,x+h_x)<\frac{1}{n}$. Since E is compact, we can choose $x_1,x_2,\cdots,x_r\in E$ such that $E\subseteq\bigcup_{k=1}^r\left(x_k-\frac{h_{x_k}}{2},x_k+\frac{h_{x_k}}{2}\right)$. Let $\delta=\min\left\{\frac{h_{x_1}}{2},\cdots,\frac{h_{x_r}}{2}\right\}$. Then for every nonempty interval L in E with $|L|<\delta$, we can choose an index k such that $L\cap\left(x_k-\frac{h_{x_k}}{2},x_k+\frac{h_{x_k}}{2}\right)\neq\emptyset$ and then, since $|L|<\delta\leq\frac{h_{x_k}}{2}$, we have $L\subseteq(x_k-h_k,x_k+h_k)$ so that $\Omega(L)\leq\Omega(x_k-h_k,x_k+h_k)<\frac{1}{n}$, as claimed.

Let X be a partition of $[a,b]$ which includes all the endpoints of the intervals $J_1,\cdots,J_\ell$ along with some additional endpoints chosen from E so that the subintervals of X include all the closed intervals $\overline{J}_1,\cdots,\overline{J}_\ell$ along with additional closed intervals $L_1,\cdots,L_m$ in E with each $|L_k|<\delta$. Then we have

$$\begin{aligned}U(f,X)-L(f,X)&=\sum_{k=1}^{\ell}\left(M(\overline{J}_k)-m(\overline{J}_k)\right)|J_k|+\sum_{k=1}^{m}\left(M(L_k)-m(L_k)\right)|L_k|\\&\leq(M-m)\sum_{k=1}^{\ell}|J_k|+\frac{1}{n}\sum_{k=1}^{m}|L_k|\leq\frac{M-m}{n}+\frac{b-a}{n}<\epsilon.\end{aligned}$$

Thus (by Theorem 2.2) f is Riemann integrable.

2.6 Example: The function $f:[0,1]\to[0,1]$ defined by $f(x)=1$ when $x\in\mathbb{Q}$ and $f(x)=0$ when $x\notin\mathbb{Q}$ is discontinuous everywhere in $[0,1]$, and is not Riemann integrable.

2.7 Example: The function $f:[0,1]\to[0,1]$ given by $f\left(\frac{a}{b}\right)=\frac{1}{b}$ when $a,b\in\mathbb{Z}$ with $0\leq a\leq b$ and $\gcd(a,b)=1$, and $f(x)=0$ when $x\notin\mathbb{Q}$, is discontinuous at all rational points, and is Riemann integrable.

2.8 Example: Define $s:\mathbb{R}\to[0,1]$ by $s(x)=0$ for $x\leq 0$ and $s(x)=1$ for $x>0$. Let $\mathbb{Q}\cap[0,1]=\{a_1,a_2,a_3,\cdots\}$ and define $f:[0,1]\to[0,1]$ by $f(x)=\sum_{k=1}^{\infty}\frac{s(x-a_k)}{2^k}$. Then f is increasing with jump discontinuities at all rational points, and f is Riemann integrable.

2.9 Example: Given a Cantor set $C=[0,1]\setminus U$, where $U=\bigcup_{k=1}^{\infty}I_k$ with the sets I_k being the disjoint open intervals from Example 1.17, we can construct a corresponding **Cantor function** $f:[0,1]\to[0,1]$ with $f(x)=\frac{1}{2}$ on I_1, $f(x)=\frac{1}{4}$ on I_2, $f(x)=\frac{3}{4}$ on I_3, $f(x)=\frac{1}{8}$ on I_4, $f(x)=\frac{3}{8}$ on I_5, $f(x)=\frac{5}{8}$ on I_6, $f(x)=\frac{7}{8}$ on I_7 and so on, and then extending f to make it continuous on all of $[0,1]$. Then f is continuous and nondecreasing with $f'(x)=0$ for all $x\in U$.

2.10 Example: When $C=[0,1]\setminus U$ is a Cantor set and $f:[0,1]\to[0,1]$ is the corresponding Cantor function (as in the previous example), the function $g:[0,1]\to[0,2]$ given by $g(x)=x+f(x)$ is a homeomorphism. Note that g sends each component interval of U to an interval of the same size, so that we have $\lambda\big(g(U)\big)=\lambda(U)$.

In the case that C is the standard Cantor set we have $\lambda\big(g(U)\big)=\lambda(U)=1$. It follows that $\lambda\big(g(C)\big)=2-\lambda(U)=1$, so g sends a set of measure zero to a set of measure 1. Also note that if we choose a nonmeasurable set $B\subseteq g(C)$ and let $A=g^{-1}(B)$, then $A\subseteq C$ so that A is a measurable set with measure zero, but g sends A to the nonmeasurable set $g(A)=B$.

2.11 Example: Given a Cantor set $C = [0,1] \backslash U$ where U is the disjoint union $U = \bigcup_{k=1}^{\infty} I_k$, choose intervals $J_k \subsetneq I_k$ so that J_k has the same centre as I_k with $|J_k| = \frac{1}{2}|I_k|$, then choose continuous functions $f_k : [0,1] \to [0,1]$ such that $f(x) = 0$ outside J_k and $f(x) = 1$ at the midpoint of J_k and then let $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for all $x \in [0,1]$. Then f is continuous in U and discontinuous in C. When $\lambda(C) > 0$, f is not Riemann integrable. If we define $g(x) = \sum_{k=1}^{\infty} \int_0^x f_k(t)\, dt$ then g is differentiable with $g' = f$ in $[a,b]$.

2.12 Example: Let $\mathbb{Q} \cap [0,1] = \{a_1, a_2, \cdots\}$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \sum_{k=1}^{\infty} \frac{(x-a_k)^{1/3}}{2^k}$. Then f is increasing with $f'(x) = \sum_{k=1}^{\infty} \frac{(x-a_k)^{-2/3}}{3 \cdot 2^k}$ when $x \notin \mathbb{Q}$ and $f'(x) = \infty$ when $x \in \mathbb{Q}$. Verify that $f'(x) \geq \frac{1}{3}$ for all x. The map f sends the interval $[0,1]$ homeomorphically to an interval $[a,b]$ and the inverse map $g : [a,b] \to [0,1]$ is increasing and differentiabe with $g'(x) = 0$ for all $x \in \mathbb{Q}$ and $g'(x) \leq 3$ for all x. Note that g' cannot be Riemann integrable because if it was then we would have $\int_a^b g' = g(b) - g(a) = 1$ but, because $g'(x) = 0$ for all $x \in \mathbb{Q}$, all of the lower Riemann sums are zero.

2.13 Definition: For $E \subseteq A \subseteq \mathbb{R}$, the **characteristic function** for E on A is the function $\chi_E : A \to \{0,1\}$ given by

$$\chi_E(x) = \begin{cases} 1 \text{ , if } x \in E, \\ 0 \text{ , if } x \notin E. \end{cases}$$

For $a, b \in \mathbb{R}$ with $a < b$, a **step function** on $[a,b]$ is a function $s : [a,b] \to \mathbb{R}$ of the form

$$s = \sum_{k=1}^{n} c_k \chi_{I_k}$$

where $n \in \mathbb{Z}^+$, each $c_k \in \mathbb{R}$, and the sets I_k are disjoint intervals with $\bigcup_{k=1}^{n} I_k = [a,b]$. The numbers c_k and the intervals I_k are uniquely determined from s if we require that I_{k-1} is to the left of I_k and $c_{k-1} \neq c_k$ for $1 < k \leq n$, and then we have $I_k = s^{-1}(c_k)$.

2.14 Theorem: *For the step function on $[a,b]$ given by $s = \sum_{k=1}^{n} c_k \chi_{I_k}$, we have*

$$\int_a^b s = \int_a^b s(x)\, dx = \sum_{k=1}^{n} c_k |I_k|.$$

For a bounded function $f : [a,b] \to \mathbb{R}$ we have

$$U(f) = \inf \left\{ \int_a^b s \,\middle|\, s \text{ is a step function on } [a,b] \text{ with } s \geq f \right\},$$

$$L(f) = \sup \left\{ \int_a^b s \,\middle|\, s \text{ is a step function on } [a,b] \text{ with } s \leq f \right\}.$$

We say that f is **Riemann integrable** *on $[a,b]$ when $U(f) = L(f)$, and in this case we define the* **Riemann integral** *of f on $[a,b]$ to be*

$$\int_a^b f = \int_a^b f(x)\, dx = U(f) = L(f).$$

2.15 Definition: *We shall find it useful on occasion to allow our functions to take the values* $\pm\infty$ *so we use the set of* **extended real numbers** $[-\infty,\infty]=\mathbb{R}\cup\{-\infty,\infty\}$. *In* $[-\infty,\infty]$, *the open balls are the open intervals* $B(-\infty,r)=\left(-\infty,-\frac{1}{r}\right)$, $B(\infty,r)=\left(\frac{1}{r},\infty\right)$ *and* $B(a,r)=(a-r,a+r)$ *with* $a\in\mathbb{R}$. *For* $A\subseteq[-\infty,\infty]$, *we say that* A *is* **open** *in* $[-\infty,\infty]$ *when for every* $a\in A$ *there exists* $r>0$ *such that* $B(a,r)\subseteq A$. *Verify that every open set in* $[-\infty,\infty]$ *is a finite or countable union of disjoint open intervals, where each open inerval is of one of the forms* $\emptyset$, (a,b), $(-\infty,a)$, (a,∞), $(-\infty,\infty)$, $[-\infty,a)$, $(a,\infty]$ *or* $[-\infty,\infty]$ *where* $a,b\in\mathbb{R}$. *We also use (partially-defined) addition and multiplication operations on* $[-\infty,\infty]$, *as usual, leaving certain sums and products undefined. We do not define the expressions* $\infty+(-\infty)$, $-\infty+\infty$, $0\cdot(\pm\infty)$ *and* $(\pm\infty)\cdot 0$.

2.16 Definition: *For* $f:A\subseteq\mathbb{R}\to B\subseteq[-\infty,\infty]$, *we say that* f *is* **measurable** *(in* A*) when* $f^{-1}(U)$ *is measurable for every open set* U *in* $[-\infty,\infty]$ *(or equivalently for every open set* U *in* B*). Note that in particular, in order for* f *to be measurable, the set* A *must be measurable because* $A=f^{-1}\big([-\infty,\infty]\big)$.

2.17 Note: *If* $f:A\subseteq\mathbb{R}\to B\subseteq[-\infty,\infty]$ *is measurable and* $\varphi:B\subseteq[-\infty,\infty]\to[-\infty,\infty]$ *is continuous, then the composite* $\varphi\circ f:A\subseteq\mathbb{R}\to[-\infty,\infty]$ *is measurable because, for every open set* U *in* $[-\infty,\infty]$, $\varphi^{-1}(U)$ *is open in* B *since* φ *is continuous, and hence the set* $(\varphi\circ f)^{-1}(U)=f^{-1}\big(\varphi^{-1}(U)\big)$ *is measurable since the function* f *is measurable.*

2.18 Theorem: *Let* $A\subseteq\mathbb{R}$ *be measurable and let* $f:A\to[-\infty,\infty]$, *Then*

$$\begin{aligned}
f \text{ is measurable } &\iff f^{-1}\big((a,\infty]\big) \text{ is measurable for all } a\in\mathbb{R}\\
&\iff f^{-1}\big([a,\infty]\big) \text{ is measurable for all } a\in\mathbb{R}\\
&\iff f^{-1}\big([-\infty,a)\big) \text{ is measurable for all } a\in\mathbb{R}\\
&\iff f^{-1}\big([-\infty,a]\big) \text{ is measurable for all } a\in\mathbb{R}
\end{aligned}$$

Proof: We shall prove the first equivalence (the others are similar). If f is measurable then $f^{-1}(U)$ is measurable for every open set $U\subseteq[-\infty,\infty]$ so, in particular, $f^{-1}\big((a,\infty]\big)$ is measurable for every $a\in\mathbb{R}$. Suppose, conversely, that $f^{-1}\big((a,\infty]\big)$ is measurable for every $a\in\mathbb{R}$. Then for every $a,b\in\mathbb{R}$ with $a<b$, each of the following sets is measurable.

$$\begin{aligned}
f^{-1}\big([-\infty,a]\big)&=\mathbb{R}\setminus f^{-1}\big((a,\infty]\big),\\
f^{-1}\big([-\infty,a)\big)&=f^{-1}\big([-\infty,a-1]\big)\cup f^{-1}\big((a-1,a),\\
f^{-1}(a,b)&=f^{-1}\big([-\infty,b)\big)\cap f^{-1}\big((a,\infty]\big).
\end{aligned}$$

Since every open U set in $[-\infty,\infty]$ is a finite or countable union of sets U_k, each of which is of one of the forms $[-\infty,a)$, (a,b), $(a,\infty]$, and because $f^{-1}\big(\bigcup_{k=1}^{\infty}U_k\big)=\bigcup_{k=1}^{\infty}f^{-1}(U_k)$, it follows that $f^{-1}(U)$ is measurable for every open set U in $[-\infty,\infty]$.

2.19 Theorem: *Let $E \subseteq A \subseteq \mathbb{R}$ with A measurable, and let $f : A \to [-\infty, \infty]$.*

(1) The function $\chi_E : A \to \{0,1\}$ is measurable if and only if the set E is measurable.
(2) If f is continuous then f is measurable.
(3) If f is monotonic then f is measurable.

Proof: To prove Part (1), note that if E is not measurable then neither is χ_E because $\chi_E{}^{-1}\big((0,2)\big) = E$, and if E is measurable then so is χ_E because for all sets U in $[-\infty,\infty]$, the set $f^{-1}(U)$ is equal to one of the measurable sets $\emptyset$, E, $A \setminus E$ or A.

To prove Part (2), suppose that f is continuous and let U be any open set in $[-\infty, \infty]$. Since f is continuous and U is open, the set $f^{-1}(U)$ is open in A. Since $f^{-1}(U)$ is open in A, we can choose an open set V in $\mathbb{R}$ such that $f^{-1}(U) = V \cap A$, which is measurable.

To prove Part (3), suppose that f is monotonic, say f is increasing. Let $a \in \mathbb{R}$. For all $x, y \in A$, if $x \in f^{-1}\big((a,\infty]\big)$ and $y \geq x$ then $f(y) \geq f(x) > a$ so that $y \in f^{-1}\big((a,\infty]\big)$. It follows that the set $f^{-1}\big((a,\infty]\big)$ must a set of one of the forms $\emptyset$, $A \cap (b,\infty]$, $A \cap [b,\infty]$ or A, and so $f^{-1}\big((a,\infty]\big)$ is measurable.

2.20 Definition: Given a function $f : A \subseteq \mathbb{R} \to [-\infty,\infty]$, we define $f^+ : A \to [-\infty,\infty]$ and $f^- : A \to [-\infty,\infty]$ by

$$f^+(x) = \begin{cases} f(x) & \text{, if } f(x) \geq 0, \\ 0 & \text{, if } f(x) \leq 0, \end{cases} \qquad f^-(x) = \begin{cases} 0 & \text{, if } f(x) \geq 0, \\ -f(x) & \text{, if } f(x) \leq 0. \end{cases}$$

2.21 Theorem: *(Operations on Measurable Functions) Let $f, g : A \subseteq \mathbb{R} \to [-\infty,\infty]$ be measurable functions, and let $c \in \mathbb{R}$. Then each of the following fuctions are measurable*

$$cf\,,\ f+g\,,\ fg\,,\ |f|\,,\ f^+\,,\ f^-$$

provided they are well-defined.

Proof: We give the proof in the case that $f, g : A \to \mathbb{R}$, and we leave it as an exercise to deal with the case in which f and g take infinite values. Suppose that $f, g : A \to \mathbb{R}$. The function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = cx$ is continuous and so the the function $cf = \varphi \circ f$ is measurable, as in Note 2.17.

The function $f + g$ is measurable because for all $a \in \mathbb{R}$ we have

$$\begin{aligned} (f+g)^{-1}\big((a,\infty]\big) &= \big\{x \in A \big| f(x) + g(x) > a\big\} \\ &= \bigcup_{r \in \mathbb{Q}} \big\{f(x) > r \text{ and } g(x) > a - r\big\} \\ &= \bigcup_{r \in \mathbb{Q}} \big(f^{-1}\big(r,\infty]\big) \cap g^{-1}\big(a-r,\infty]\big)\big), \end{aligned}$$

which is measurable.

The function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = x^2$ is continuous so, as in Note 2.17, for every measurable function $h : A \to \mathbb{R}$, the function $h^2 = \varphi \circ h$ is also measurable. It follows that the function $fg = \frac{1}{4}\big((f+g)^2 - (f-g)^2\big)$ is measurable.

The function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = |x|$ is continuous so, as in Note 2.17, the function $|f| = \varphi \circ f$ is measurable, hence so are the functions $f^+ = \frac{1}{2}\big(|f| + f\big)$ and $f^- = \frac{1}{2}\big(|f| - f\big)$.

2.22 Theorem: *(Decomposition) Let* $A = \bigcup_{k=1}^{\infty} A_k$ *where the sets* A_k *are disjoint measurable sets in* $\mathbb{R}$*, and let* $f : A \to [-\infty, \infty]$*. Then* f *is measurable (in* A*) if and only if the restriction of* f *to each of the sets* A_k *is measurable (in* A_k*).*

Proof: Let $f_k : A_k \to [-\infty, \infty]$ be the restriction of f to A_k. For $U \subseteq [-\infty, \infty]$ open, since $f_k^{-1}(U) = f^{-1}(U) \cap A_k$ it follows that if f is measurable then so is each f_k, and since $f^{-1}(U) = \bigcup_{k=1}^{\infty} f_k^{-1}(U)$ it follows that if each f_k is measurable then so is f.

2.23 Theorem: *(Limits of Measurable Functions) Let* $f_n : A \subseteq \mathbb{R} \to [-\infty, \infty]$ *be measurable for each* $n \in \mathbb{Z}^+$*. Then each of the following functions are well-defined and measurable:*

$$\sup\{f_n \big| n \in \mathbb{Z}^+\}\,,\ \inf\{f_n \big| n \in \mathbb{Z}^+\}\,,\ \limsup_{n\to\infty}\{f_n\}\,,\ \liminf_{n\to\infty}\{f_n\}.$$

Proof: Let $g = \sup\{f_n | n \in \mathbb{Z}^+\}$. For $x \in A$ and $a \in \mathbb{R}$ we have

$$\begin{aligned} x \in g^{-1}\big((a, \infty]\big) &\iff g(x) > a \iff \sup\{f_n | n \in \mathbb{Z}^+\} > a \\ &\iff f_n(x) > a \text{ for some } n \in \mathbb{Z}^+ \iff x \in \bigcup_{n=1}^{\infty} {f_n}^{-1}\big((a, \infty]\big). \end{aligned}$$

Thus for all $a \in \mathbb{R}$ we have $g^{-1}\big((a, \infty]\big) = \bigcup\limits_{n=1}^{\infty} {f_n}^{-1}\big((a, \infty]\big)$, which is measurable. Similarly, when $h = \inf\{f_n \big| n \in \mathbb{Z}^+\}$ and $a \in \mathbb{R}$ we have $h^{-1}\big([a, \infty]\big) = \bigcap\limits_{n=1}^{\infty} {f_n}^{-1}\big([a, \infty]\big)$, which is measurable. Also, we have

$$\begin{aligned} \limsup_{n\to\infty} f_n &= \inf\Big\{ \sup\{f_n | n \geq 1\}, \sup\{f_n | n \geq 2\}, \sup\{f_n | n \geq 3\}, \cdots \Big\} \text{ and} \\ \liminf_{n\to\infty} f_n &= \sup\Big\{ \inf\{f_n | n \geq 1\}, \inf\{f_n | n \geq 2\}, \inf\{f_n | n \geq 3\}, \cdots \Big\}. \end{aligned}$$

It follows that $\limsup\limits_{n\to\infty} f_n$ and $\liminf\limits_{n\to\infty} f_n$ are measurable.

2.24 Definition: Let $A \subseteq \mathbb{R}$ be measurable. We say that a property or statement holds for **almost every** (written a.e.) $x \in A$, or **almost everywhere** (written a.e.) in A, when the property or statement holds for every $x \in A \setminus E$ for some set $E \subseteq A$ with $\lambda(E) = 0$. For example, for functions $f, g : A \to [-\infty, \infty]$, we say that $f(x) = g(x)$ for a.e. $x \in A$ (or $f = g$ a.e. in A) when $f(x) = g(x)$ for every $x \in A \setminus E$ for some set $E \subseteq A$ with $\lambda(E) = 0$.

2.25 Theorem: *Let* $A \subseteq \mathbb{R}$ *be measurable and let* $f, g : A \to [-\infty, \infty]$*.*

(1) If $\lambda(A) = 0$ *then* f *is measurable.*

(2) If $f = g$ *a.e. in* A *then* f *is measurable if and only if* g *is measurable.*

Proof: The proof is left as an exercise.

2.26 Definition: Let $A \subseteq \mathbb{R}$. A **simple function** on A is a function $s : A \to \mathbb{R}$ of the form

$$s = \sum_{k=1}^{n} c_k \chi_{A_k}$$

where $n \in \mathbb{Z}^+$, each $c_k \in \mathbb{R}$, and the sets A_k are disjoint measurable sets with $\bigcup_{k=1}^{n} A_k = A$. The numbers c_k and sets A_k are uniquely determined from the function s if we require that $c_1 < c_2 < \cdots < c_n$, and then we have $A_k = s^{-1}(c_k)$.

2.27 Definition: For the nonnegative simple function $s : A \subseteq \mathbb{R} \to [0, \infty)$ given by $s = \sum_{k=1}^{n} c_k \chi_{A_k}$, the (Lebesgue) **integral** of s on A is defined to be

$$\int_A s(x)\,dx = \int_A s = \int_A s\;d\lambda = \sum_{k=1}^{n} c_k \lambda(A_k).$$

Note that the value of the integral does not depend on whether or not the numbers c_k are distinct because if $c_k = c_l$ then $c_k\lambda(A_k) + c_l\lambda(A_l) = c_k\big(\lambda(A_k) + \lambda(A_l)\big) = c_k\lambda(A_k \cup A_l)$.

2.28 Theorem: *(Properties of Integration for Non-negative Simple Functions)*
Let $r, s : A \subseteq \mathbb{R} \to [0, \infty)$ be nonnegative simple functions, and let $c \in \mathbb{R}$.

(1) If $r \le s$ then $\int_A r \le \int_A s$.

(2) We have $\int_A (cs) = c\int_A s$ and $\int_A (r + s) = \int_A r + \int_A s$.

(3) If $A = B \cup C$, where B and C are disjoint and measurable, then $\int_A s = \int_B s + \int_C s$.

(4) If $B \subseteq A$ is measurable then $\int_B s = \int_A s \cdot \chi_B$.

(5) If $\lambda(A) = 0$ then $\int_A s = 0$.

(6) If $r = s$ a.e. in A then $\int_A r = \int_A s$, and if $\int_A r = 0$ then $r = 0$ a.e. in A.

Proof: We shall prove Parts (1) and (2) and leave the proofs of the remaining parts as an exercise. Let $r = \sum_{k=1}^{n} a_k \chi_{A_k}$ and $s = \sum_{l=1}^{m} b_l \chi_{B_l}$ and let $C_{k,l} = A_k \cap B_l$. Note that the sets $C_{k,l}$ are disjoint with $\bigcup_{k=1}^{n} C_{k,l} = \bigcup_{k=1}^{n} (A_k \cap B_l) = \big(\bigcup_{k=1}^{n} A_k\big) \cap B_l = A \cap B_l = B_l$ and it follows that $\sum_{k=1}^{n} \chi_{C_{k,l}} = \chi_{B_l}$ and that $\sum_{k=1}^{n} \lambda(C_{k,l}) = \lambda(B_l)$. Similary, we have $\bigcup_{l=1}^{m} C_{k,l} = A_k$, $\sum_{l=1}^{n} \chi_{C_{k,l}} = \chi_{A_k}$ and $\sum_{l=1}^{m} \lambda(C_{k,l}) = \lambda(A_k)$.

To prove Part (1), suppose that $r \le s$. For all pairs (k, l) with $C_{k,l} \ne \emptyset$, we can choose $x \in C_{k,l}$ and then we have $a_k = r(x) \le s(s) = b_l$. It follows that

$$\begin{aligned}\int_A r &= \sum_{k=1}^{n} a_k\lambda(A_k) = \sum_{k=1}^{n} a_k \sum_{l=1}^{m} \lambda(C_{k,l}) = \sum_{k,l} a_k\lambda(C_{k,l}) = \sum_{k,l\,\ni\, C_{k,l}\ne\emptyset} a_k\lambda(C_{k,l}) \\ &\le \sum_{k,l\,\ni\, C_{k,l}\ne\emptyset} b_l\lambda(C_{k,l}) = \sum_{k,l} b_l\lambda(C_{k,l}) = \sum_{l=1}^{m} b_l \sum_{k=1}^{n} \lambda(C_{k,l}) = \sum_{l=1}^{m} b_l\lambda(B_l) = \int_A s\,.\end{aligned}$$

The first formula in Part (2) is clear. Let us prove the second formula. We have

$$r+s=\sum_{k=1}^{n}a_k\chi_{A_k}+\sum_{l=1}^{m}b_l\chi_{B_l}=\sum_{k=1}^{n}a_k\sum_{l=1}^{m}\chi_{C_{k,l}}+\sum_{l=1}^{m}b_l\sum_{k=1}^{n}\chi_{C_{k,l}}=\sum_{k,l}(a_k+b_l)\chi_{C_{k,l}}$$

and so

$$\begin{aligned}\int_A(r+s)&=\sum_{k,l}(a_k+b_l)\lambda(C_{k,l})=\sum_{k,l}a_k\lambda(C_{k,l})+\sum_{k,l}b_l\lambda(C_{k,l})\\&=\sum_{k=1}^{n}a_k\sum_{l=1}^{m}\lambda(C_{k,l})+\sum_{l=1}^{m}b_l\sum_{k=1}^{n}\lambda(C_{k,l})\\&=\sum_{k=1}^{n}a_k\lambda(A_k)+\sum_{l=1}^{m}b_l\lambda(B_l)=\int_A r+\int_A s\,.\end{aligned}$$

2.29 Note: Given any nonnegative measurable function $f:A\subseteq\mathbb{R}\to[0,\infty]$, we can construct an increasing sequence $\{s_n\}$ of nonnegative simple functions $s_n:A\to[0,\infty)$ with $s_n\to f$ pointwise in A as follows. For $n\in\mathbb{Z}^+$, we let

$$s_n(x)=\begin{cases}\frac{k-1}{2^n}\ ,\text{ if }\frac{k-1}{2^n}\le f(x)<\frac{k}{2^n}\text{ with }k\in\{1,2,\cdots,n2^n\},\\ \quad n\ \ ,\text{ if }f(x)\ge n,\end{cases}$$

that is $s_n=\sum_{k=1}^{n2^n}\frac{k-1}{2^n}\chi_{A_k}$ where $A_k=f^{-1}\left[\frac{k-1}{2^n},\frac{k}{2^n}\right)$ for $1\le k<n2^n$ and $A_{n2^n}=f^{-1}[n,\infty]$. We remark that if f is bounded then $s_n\to f$ uniformly in A.

2.30 Definition: For a nonnegative measurable function $f:A\subseteq\mathbb{R}\to[0,\infty]$, we define the (Lebesgue) **integral** of f on A to be

$$\int_A f(x)\,dx=\int_A f=\int_A f\,d\lambda=\sup\left\{\int_A s\,\middle|\,s\text{ is a simple function on }A\text{ with }0\le s\le f\right\}.$$

We say that $f:A\to[0,\infty]$ is (Lebesgue) **integrable** (on A) when $\displaystyle\int_A f<\infty$.

2.31 Theorem: *(Properties of Integration for Non-negative Measurable Functions) Let $f,g:A\subseteq\mathbb{R}\to[0,\infty]$ be non-negative measurable functions and let $c\in\mathbb{R}$. Then*

(1) If $f\le g$ on A then $\displaystyle\int_A f\le\int_A g$.

(2) We have $\displaystyle\int_A(cf)=c\int_A f$ and $\displaystyle\int_A(f+g)=\int_A f+\int_A g$.

(3) If $A=B\cup C$, where B and C are disjoint and measurable, then $\displaystyle\int_A f=\int_B f+\int_C f$.

(4) If $B\subseteq A$ is measurable then $\displaystyle\int_B f=\int_A f\cdot\chi_B$.

(5) If $\lambda(A)=0$ then $\displaystyle\int_A f=0$.

(6) If $f=g$ a.e. in A then $\displaystyle\int_A f=\int_A g$, and $\displaystyle\int_A f=0$ then $f=0$ a.e. in A.

Proof: All parts follow fairly easily from the analogous parts of Theorem 2.28 except for the second formula in Part (2). We shall return to the proof of this formula later.

2.32 Theorem: *(Fatou's Lemma) Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be nonnegative measurable functions for $n \in \mathbb{Z}^+$. Then*

$$\int_A \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_A f_n \,.$$

Proof: By the definition of the integral on the left, it suffices to prove that for every nonnegative simple function s on A with $s \le \liminf\limits_{n\to\infty} f_n$ we have $\int_A s \le \liminf\limits_{n\to\infty} \int_A f_n$. Let s be any nonnegative simple function on A with $s \le \liminf\limits_{n\to\infty} f_n$. Write $s = \sum\limits_{k=1}^m a_k \chi_{A_k}$. For all $x \in A_k$ we have $a_k = s(x) \le \liminf\limits_{n\to\infty} f_n(x)$, and it follows that for all $0 \le r < 1$ there exists $n \in \mathbb{Z}^+$ such that for all $l \ge n$ we have $f_l(x) \ge r a_k$. Let $0 \le r < 1$. For $k, n \in \mathbb{Z}^+$, let

$$B_{k,n} = \left\{ x \in A_k \middle| f_l(x) \ge r a_k \text{ for all } l \ge n \right\} = \bigcap_{l \ge n} {f_l}^{-1}[r a_k, \infty].$$

Note that each set $B_{k,n}$ is measurable with $B_{k,1} \subseteq B_{k,2} \subseteq B_{k,3} \subseteq \cdots$ and $\bigcup\limits_{n=1}^\infty B_{k,n} = A_k$. It follows that $\lambda(A_k) = \lim\limits_{n\to\infty} \lambda(B_{k,n})$. For all $x \in B_{k,n}$ we have $f_l(x) \ge r a_k$ for all $l \ge n$ so that, in particular, $f_n(x) \ge r a_k$. It follows that $f_n \ge \sum\limits_{k=1}^m r a_k \chi_{B_{k,n}}$ hence

$$\int_A f_n \ge \sum_{k=1}^m r a_k \lambda(B_{k,n}).$$

Taking the lim inf on both sides gives

$$\liminf_{n\to\infty} \int_A f_n \ge \lim_{n\to\infty} \sum_{k=1}^m r a_k \lambda(B_{k,n}) = \sum_{k=1}^m r a_k \lambda(A_k) = r \int_A s \,.$$

Since $0 \le r < 1$ was arbitrary, it follows that $\liminf\limits_{n\to\infty} \int_A f_n \ge \int_A s$, as required.

2.33 Corollary: *Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be nonnegative measurable functions for $n \in \mathbb{Z}^+$. Suppose that the pointwise limit $\lim\limits_{n\to\infty} f_n(x)$ exists with $f_n(x) \le \lim\limits_{n\to\infty} f_n(x)$ for all $x \in A$. Then*

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n \,.$$

Proof: For all $n \in \mathbb{Z}^+$, since $f_n \le \lim\limits_{n\to\infty} f_n$ we have $\int_A f_n \le \int_A \lim\limits_{n\to\infty} f_n$. Taking the lim sup gives

$$\limsup_{n\to\infty} \int_A f_n \le \int_A \lim_{n\to\infty} f_n \,.$$

By Fatou's Lemma, we also have

$$\int_A \lim_{n\to\infty} f_n = \int_A \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_A f_n \,.$$

2.34 Corollary: *(Lebesgue's Monotone Convergence Theorem) Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be nonnegative measurable functions such that $\{f_n(x)\}$ is increasing for every $x \in A$. Then*

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n \,.$$

Proof: This is a special case of the previous corollary.

2.35 Note: We now return to the proof of the second formula in Part (2) of Theorem 2.30. We suppose that $f, g : A \subseteq \mathbb{R} \to [0, \infty]$ are nonnegative measurable functions, and we need to prove that

$$\int_A (f+g) = \int_A f + \int_A g.$$

Proof: Using the construction described in Note 2.28, choose increasing sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative simple functions on A such that $\lim\limits_{n\to\infty} r_n = f$ and $\lim\limits_{n\to\infty} s_n = g$. Then the sequence $\{r_n + s_n\}$ is also increasing with $\lim\limits_{n\to\infty}(r_n + s_n) = f + g$. By the Monotone Convergence Theorem, along with Part (2) of Theorem 2.27, we have

$$\begin{aligned}\int_A (f+g) &= \int_A \lim_{n\to\infty}(r_n+s_n) = \lim_{n\to\infty}\int_A (r_n+s_n) = \lim_{n\to\infty}\left(\int_A r_n + \int_A s_n\right)\\ &= \lim_{n\to\infty}\int_A r_n + \lim_{n\to\infty}\int_A s_n = \int_A \lim_{n\to\infty} r_n + \int_A \lim_{n\to\infty} s_n = \int_A f + \int_A g\,.\end{aligned}$$

2.36 Corollary: *Let $A \subseteq \mathbb{R}$ be measurable and let $\{f_n\}$ be a sequence of nonnegative measurable functions $f_n : A \to [0, \infty]$. Then*

$$\int_A \sum_{n-1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_A f_n\,.$$

Proof: This follows by applying Lebesgue's Monotone Convergence Theorem to the sequence of partial sums $S_n(x) = \sum\limits_{k=1}^{n} f_k(x)$.

2.37 Corollary: *Let $A = \bigcup\limits_{k=1}^{\infty} A_k$ where the sets A_n are measurable and disjoint, and let $f : A \to [0, \infty]$ be nonnegative and measurable. Then*

$$\int_A f = \sum_{n=1}^{\infty} \int_{A_n} f\,.$$

Proof: This follows from the above corollary using $f_n = f \cdot \chi_{A_n}$.

2.38 Remark: For a σ-algebra $\mathcal{C}$, a **measure** on $\mathcal{C}$ is a function $\mu : \mathcal{C} \to [0, \infty]$ such that
(1) $\mu(\emptyset) = 0$, and
(2) if $A_1, A_2, A_3, \cdots \in \mathcal{C}$ are disjoint then $\mu\Big(\bigcup\limits_{k=1}^{\infty} A_k\Big) = \sum\limits_{k=1}^{\infty} \mu(A_k)$.

When $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$, and $f : \mathbb{R} \to [0, \infty]$ is any nonnegative measurable function on $\mathbb{R}$, the above corollary shows that we can define a measure μ on $\mathcal{M}$ by

$$\mu(A) = \int_A f\,.$$

2.39 Definition: For a measurable function $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$, we say that f is (Lebesgue) **integrable** (on A) when the functions f^+ and f^- are both Lebesgue integrable on A and, in this case, we define the (Lebesgue) **integral** of f on A to be

$$\int_A f(x)\,dx = \int_A f = \int_A f \; d\lambda = \int_A f^+ - \int_A f^- .$$

In the case that $A = [a, b]$ we also write $\displaystyle\int_A f(x)\,dx$ as $\displaystyle\int_a^b f(x)\,dx$.

2.40 Note: For $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$, f is integrable if and only if $|f|$ is integrable.

2.41 Theorem: *(Integration) Let $f, g : A \subseteq \mathbb{R} \to [-\infty, \infty]$ be integrable and let $c \in \mathbb{R}$.*

(1) We have $\left| \displaystyle\int_A f \right| \leq \displaystyle\int_A |f|$.

(2) If $f \leq g$ then $\displaystyle\int_A f \leq \int_A g$.

(3) We have $\displaystyle\int_A (cf) = c \int_A f$ *and* $\displaystyle\int_A (f + g) = \int_A f + \int_A g$.

(4) If $A = B \cup C$ where B and C are disjoint and measurable then $\displaystyle\int_A f = \int_B f + \int_C f$.

(5) If $B \subseteq A$ is measurable then $\displaystyle\int_B f = \int_A f \cdot \chi_B$.

(6) If $\lambda(A) = 0$ then $\displaystyle\int_A f = 0$.

(7) If $f = g$ a.e. on A then $\displaystyle\int_A f = \int_A g$*, and if* $\displaystyle\int_A |f| = 0$ *then $f = 0$ a.e. in A.*

Proof: The proof is left as an exercise.

2.42 Theorem: *(Lebesgue's Dominated Convergence Theorem) Let $A \subseteq \mathbb{R}$ be a measurable set and let $f_n : A \to [-\infty, \infty]$ be measurable functions for $n \in \mathbb{Z}^+$. Suppose the pointwise limit $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in A$. Suppose there exists an integrable function $g : A \to [0, \infty]$ such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{Z}^+$, $x \in A$. Then*

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n \, .$$

Proof: Let $f = \lim\limits_{n\to\infty} f_n$. Note that since $-g \leq f_n \leq g$ for all n we gave $-g \leq f \leq g$ so that f is integrable. By Fatou's Lemma, applied to the function $g + f_n$, we have

$$\int_A g + \int_A \lim_{n\to\infty} f_n = \int_A \liminf_{n\to\infty} (g + f_n) \leq \liminf_{n\to\infty} \int_A (g + f_n) = \int_A g + \liminf_{n\to\infty} f_n.$$

It follows, since $\int_A g < \infty$, that

$$\liminf_{n\to\infty} \int_A f_n \geq \int_A \lim_{n\to\infty} f_n.$$

By Fatou's Lemma, applied to the function $g - f_n$, we have

$$\int_A g - \int_A \lim_{n\to\infty} f_n = \int_A \liminf_{n\to\infty} (g - f_n) \leq \liminf_{n\to\infty} \int_A (g - f_n) = \int_A g - \limsup_{n\to\infty} \int_A f_n.$$

It follows, since $\int_A g < \infty$, that

$$\limsup_{n\to\infty} \int_A f_n \leq \int_A \lim_{n\to\infty} f_n.$$

2.43 Theorem: *Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integral. Then f is also measurable and Lebesgue integrable, and the two kinds of integral are equal.*

Proof: I may include a proof later.

2.44 Remark: I may include a discussion of complex-valued functions $f : A \subseteq \mathbb{R} \to \mathbb{C}$ later.