Chapter 1. Lebesgue Measure

1.1 Definition: When I is equal to any one of the bounded intervals (a,b), $[a,b)$, $(a,b]$ or $[a,b]$, where $a,b\in\mathbb{R}$ with $a\leq b$, we define the **size** of I to be $|I|=b-a$. When I is equal to any of the unbounded intervals $(-\infty,a)$, $(-\infty,a]$, (a,∞), $[a,\infty)$ or $(-\infty,\infty)$, where $a\in\mathbb{R}$, we define the size of I to be $|I|=\infty$.

1.2 Definition: For a bounded set $A\subseteq\mathbb{R}$, the **Jordan outer content** of A is

$$c^*(A)=\inf\left\{\sum_{k=1}^{n}|I_k|\,\middle|\,n\in\mathbb{Z}^+,\text{ each } I_k \text{ is a bounded open interval and } A\subseteq\bigcup_{k=1}^{n}I_k\right\}.$$

1.3 Theorem: *(Properties of Jordan Outer Content) Let $A,B\subseteq\mathbb{R}$ be bounded.*
(1) If $a\in\mathbb{R}$ then $c^*(a+A)=c^*(A)$.
(2) If $r\in\mathbb{R}$ then $c^*(rA)=|r|\,c^*(A)$.
(3) If $A\subseteq B$ then $c^*(A)\leq c^*(B)$.
(4) If A is a finite set then $c^*(A)=0$.
(5) If A is a bounded interval then $c^*(A)=|A|$.
(6) We have $c^*(\overline{A})=c^*(A)$.
(7) (Finite Subadditivity) We have $c^*(A\cup B)\leq c^*(A)+c^*(B)$.

Proof: The proof is left as an exercise.

1.4 Exercise: Show that when $A\subseteq\mathbb{R}$ and I and J are bounded intervals with $A\subseteq I\subseteq J$ we have $|I|-c^*(I\setminus A)=|J|-c^*(J\setminus A)$.

1.5 Definition: For a bounded set $A\subseteq\mathbb{R}$, we say that A has (a well-defined) **Jordan content** when

$$c^*(A)=|I|-c^*(I\setminus A)$$

where I is any interval which contains A and, in this case, we define the **Jordan content** of A to be $c(A)=c^*(A)$.

1.6 Exercise: Show that $\mathbb{Q}\cap[0,1]$ does not have a well-defined Jordan content.

1.7 Theorem: *(Properties of Content) Let $A,B\subseteq\mathbb{R}$ be bounded.*
(1) If $a\in\mathbb{R}$ then $a+A$ has Jordan content if and only if A does.
(2) If $0\neq r\in\mathbb{R}$ then rA has Jordan content if and only if A does.
(3) If $c^*(A)=0$ then A has Jordan content.
(4) Every bounded interval has Jordan content.
(5) The set A has Jordan content if and only if $c^*(\overline{A}\setminus A^o)=0$.
(6) If A and B have Jordan content then so do $A\cup B$, $A\cap B$ and $A\setminus B$.
(7) If A and B have Jordan content and $A\cap B=\emptyset$ then $c(A\cup B)=c(A)+c(B)$.

Proof: The proof is left as an exercise.

1.8 Definition: For a set $A \subseteq \mathbb{R}$, the (Lebesgue) **outer measure** of A is

$$\lambda^*(A) = \inf\left\{ \sum_{n=1}^{\infty} |I_n| \,\Big|\, \text{each } I_n \text{ is a bounded open interval and } A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

1.9 Theorem: *(Properties of Outer Measure) Let $A, B \subseteq \mathbb{R}$ and let $A_k \subseteq \mathbb{R}$ for $k \in \mathbb{Z}^+$.*
(1) (Translation) If $a \in \mathbb{R}$ then $\lambda^*(a + A) = \lambda^*(A)$.
(2) (Scaling) If $0 \neq r \in \mathbb{R}$ then $\lambda^*(rA) = |r|\lambda^*(A)$.
(3) (Inclusion) If $A \subseteq B$ then $\lambda^*(A) \leq \lambda^*(B)$.
(4) If A is finite or countable then $\lambda^*(A) = 0$.
(5) If I is an interval then $\lambda^*(I) = |I|$.
(6) (Countable Subadditivity) We have $\lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(A_n)$.

Proof: We leave the proofs of parts (1), (2) and (3) as an exercise. We prove Part (4) in the case that A is countable. Let $A = \{a_1, a_2, a_3, \cdots\}$. Let $\epsilon > 0$. For each $n \in \mathbb{Z}^+$, let $I_n = \left(a_n - \frac{\epsilon}{2^n}, a_n + \frac{\epsilon}{2^n}\right)$. Then $A \subseteq \bigcup_{n=1}^{\infty} I_n$ so we have $\lambda^*(A) \leq \sum_{n=1}^{\infty} |I_n| = 2\epsilon$. Since $0 \leq \lambda^*(A) < 2\epsilon$ for every $\epsilon > 0$, it follows that $\lambda^*(A) = 0$.

Let us prove Part (5). When I is a degenerate interval (so I is empty or has only one point) we know, from Part (4), that $\lambda^*(I) = 0$. Suppose that I is a nondegenerate bounded interval, say I is equal to one of the intervals (a, b), $[a, b)$, $(a, b]$ or $[a, b]$ where $a < b$. Let $\epsilon > 0$, let $I_1 = (a - \epsilon, b + \epsilon)$ and let $I_n = \emptyset$ for $n \geq 2$. Then $I \subseteq \bigcup_{n=1}^{\infty} I_n$ so we have $\lambda^*(I) \leq \sum_{n=1}^{\infty} |I_n| = b - a + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, it follows that $\lambda^*(I) \leq b - a$. It remains to show that $\lambda^*(I) \geq b - a$. Let $I_1, I_2, I_3, \cdots$ be any bounded open intervals such that $I \subseteq \bigcup_{n=1}^{\infty} I_n$. Let $0 < \epsilon < \frac{b-a}{2}$ and consider the compact interval $K = [a + \epsilon, b - \epsilon] \subset I$. Note that $\mathcal{U} = \{I_1, I_2, I_3, \cdots\}$ is an open cover of K. Choose a finite subset $\mathcal{V} \subseteq \mathcal{U}$ so that $K \subseteq \bigcup_{J \in \mathcal{V}} J$. Choose $J_1 = (a_1, b_1) \in \mathcal{V}$ so that $a_1 < a - \epsilon < b_1$. If $b_1 \leq b - \epsilon$ then choose $J_2 = (a_2, b_2) \in \mathcal{V}$ so that $a_2 < b_1 < b_2$. If $b_2 \leq b - \epsilon$ then choose $J_3 = (a_3, b_3) \in \mathcal{V}$ so that $a_3 < b_2 < b_3$. Continue this procedure until we have chosen $J_\ell = (a_\ell, b_\ell) \in \mathcal{V}$ with $b_\ell > b - \epsilon$, and note that $K \subseteq J_1 \cup J_2 \cup \cdots \cup J_\ell$ and $\{J_1, J_2, \cdots, J_\ell\} \subseteq \mathcal{V} \subseteq \mathcal{U}$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} |I_n| \geq \sum_{n=1}^{\ell} |J_n| &= (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_\ell - a_\ell) \\
&> \big(a_2 - (a + \epsilon)\big) + (a_3 - a_2) + (a_4 - a_3) + \cdots + (a_\ell - a_{\ell-1}) + \big((b - \epsilon) - a_\ell\big) \\
&= b - a - 2\epsilon.
\end{aligned}$$

Since ϵ was arbitrarily small it follows that $\sum_{n=1}^{\infty} |I_n| \geq b - a$. Since this is true for all bounded open intervals $I_1, I_2, I_3, \cdots$ which cover I, it follows that $\lambda^*(I) \geq b - a$, as required.

When I is an unbounded interval, we must have $\lambda^*(I) = \infty$ because for every $R > 0$ we can choose a bounded interval $J \subseteq I$ with $|J| > R$ and then we have $\lambda^*(I) \geq \lambda^*(J) > R$.

To prove Part (6), let $A_1, A_2, A_3, \cdots \subseteq \mathbb{R}$. If $\lambda^*(A_\ell) = \infty$ for some ℓ, then we have $\sum_{k=1}^{\infty} \lambda^*(A_k) = \infty$ and hence $\lambda^*\Big(\bigcup_{k=1}^{\infty} A_k\Big) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$. Suppose $\lambda^*(A_k) < \infty$ for all k. Let $\epsilon > 0$. For each $n \in \mathbb{Z}^+$, choose open bounded intervals $I_{n,1}, I_{n,2}, I_{n,3}, \cdots$ so that $A_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k}$ and $\sum_{k=1}^{\infty} |I_{n.k}| \leq \lambda^*(A_n) + \frac{\epsilon}{2^n}$. Then we have $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}$ so that

$$\lambda^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n,k=1}^{\infty} |I_{n,k}| \leq \sum_{n=1}^{\infty} \big(\lambda^*(A_n) + \tfrac{\epsilon}{2^n}\big) = \sum_{n=1}^{\infty} \lambda^*(A_n) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\lambda^*\Big(\bigcup_{n=1}^{\infty} A_k\Big) \leq \sum_{n=1}^{\infty} \lambda^*(A_n)$, as required.

1.10 Definition: For $A \subseteq \mathbb{R}$, we say that A is (Lebesgue) **measurable** when for every set $X \subseteq \mathbb{R}$ we have

$$\lambda^*(X) = \lambda^*(X \cap A) + \lambda^*(X \setminus A).$$

When A is measurable, we define the (Lebesgue) **measure** of A to be $\lambda(A) = \lambda^*(A)$. We let $\mathcal{M}$ denote the set of all measurable subsets of $\mathbb{R}$.

1.11 Note: For any sets $A, X \subseteq \mathbb{R}$, we have $X = (X \cap A) \cup (X \setminus A)$ and so (by subadditivity) $\lambda^*(X) \leq \lambda^*(X \cap A) + \lambda^*(X \setminus A)$. Thus a set $A \subseteq \mathbb{R}$ is measurable if and only if for every set $X \subseteq \mathbb{R}$ we have

$$\lambda^*(X) \geq \lambda^*(X \cap A) + \lambda^*(X \setminus A).$$

1.12 Theorem: *(Properties of Measure) Let $A, B, A_k \subseteq \mathbb{R}$ for $k \in \mathbb{Z}^+$.*
(1) If $a \in \mathbb{R}$ then A is measurable if and only if $a + A$ is measurable.
(2) If $0 \neq r \in \mathbb{R}$ then A is measurable if and only if rA is measurable.
(3) $\emptyset$ and $\mathbb{R}$ are measurable.
(4) If $\lambda^*(A) = 0$ then A is measurable.
(5) If A is measurable then so is $A^c = \mathbb{R} \setminus A$.
(6) If A and B are measurable then so are $A \cup B$, $A \cap B$ and $A \setminus B$.
(7) Every interval is measurable.
(8) If the sets A_k are measurable then so are $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$.
(9) If the sets A_k are measurable and disjoint then $\lambda\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \lambda(A_k)$.
(10) Let A_k be measurable for $k \geq 1$. If $A_k \subseteq A_{k+1}$ for all k, then $\lambda\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \lambda(A_n)$. If $A_k \supseteq A_{k+1}$ for all k, and $\lambda(A_m)$ is finite for some $m \in \mathbb{Z}^+$, then $\lambda\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \lambda(A_n)$.

Proof: We leave the proofs of Parts (1) and (2) as an exercise. To prove Part (3), note that $\emptyset$ and $\mathbb{R}$ are measurable because for every set $X \subseteq \mathbb{R}$ we have

$$\lambda^*(X \cap \emptyset) + \lambda^*(X \setminus \emptyset) = \lambda^*(\emptyset) + \lambda^*(X) = \lambda^*(X), \text{ and}$$
$$\lambda^*(X \cap \mathbb{R}) + \lambda^*(X \setminus \mathbb{R}) = \lambda^*(X) + \lambda^*(\emptyset) = \lambda^*(X).$$

To prove Part (4), let $A \subseteq \mathbb{R}$ and suppose that $\lambda^*(A) = 0$. Let $X \subseteq \mathbb{R}$. Since $X \cap A \subseteq A$ and $X \setminus A \subseteq X$ we have

$$\lambda^*(X \cap A) + \lambda^*(X \setminus A) \leq \lambda^*(A) + \lambda^*(X) = \lambda^*(X).$$

Part (5) holds because if $A \subseteq \mathbb{R}$ is measurable and $X \subseteq \mathbb{R}$ then, since $X \cap A^c = X \setminus A$ and $X \setminus A^c = X \cap A$, we have

$$\lambda^*(X \cap A^c) + \lambda^*(X \setminus A^c) = \lambda^*(X \setminus A) + \lambda^*(X \cap A) = \lambda^*(X).$$

To prove Part (6), suppose that A and B are measurable and let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned}\lambda^*(X) &= \lambda^*(X \cap A) + \lambda^*(X \setminus A) \text{ , since } A \text{ is measurable} \\ &= \lambda^*(X \cap A) + \lambda^*\big((X \setminus A) \cap B\big) + \lambda^*\big((X \setminus A) \setminus B\big) \text{ , since } B \text{ is measurable} \\ &= \lambda^*(X \cap A) + \lambda^*\big((X \setminus A) \cap B\big) + \lambda^*\big(X \setminus (A \cup B)\big) \\ &\geq \lambda^*\big(X \cap (A \cup B)\big) + \lambda^*\big(X \setminus (A \cup B)\big) \text{ , by subadditivity}\end{aligned}$$

since $(X \cap A) \cup \big((X \setminus A) \cap B\big) = X \cap (A \cup B)$. This shows that $A \cup B$ is measurable. Using Part (5), it follows that $A \cap B$ is measurable because $A \cap B = (A^c \cup B^c)^c$ and hence that $A \setminus B$ is measurable because $A \setminus B = A \cap B^c$.

Let us prove Part (7) in the case of a nonempty bounded open interval. Let $I = (a, b)$ where $a < b$. Let $X \subseteq \mathbb{R}$. Verify that when $\lambda^*(X) = \infty$ then we also have $\lambda^*(X \setminus A) = \infty$ so that, in this case, $\lambda^*(X) = \infty = \lambda^*(X \cap A) + \lambda^*(X \setminus A)$. Suppose $\lambda^*(X) < \infty$, and let $\epsilon > 0$. Choose open bounded intervals $I_1, I_2, I_3, \cdots$ so that $X \subseteq \bigcup\limits_{n=1}^{\infty} I_n$ and $\sum\limits_{n=1}^{\infty} |I_n| < \lambda^*(X) + \epsilon$. For $n \in \mathbb{Z}^*$, let $J_n = I_n \cap (a, b)$, $K_n = I_n \cap (-\infty, a)$ and $L_n = I_n \cap (b, \infty)$. Then $X \cap I \subseteq \bigcup\limits_{n=1}^{\infty} J_n$ so that $\lambda^*(X \cap I) \leq \sum\limits_{n=1}^{\infty} |J_n|$ and $X \setminus I \subseteq (a-\epsilon, a+\epsilon) \cup (b-\epsilon, b+\epsilon) \cup \bigcup\limits_{n=1}^{\infty} K_n \cup \bigcup\limits_{n=1}^{\infty} L_n$ so that $\lambda^*(X \setminus I) \leq 4\epsilon + \sum\limits_{n=1}^{\infty} |K_n| + \sum\limits_{n=1}^{\infty} |L_n|$ and so we have

$$\lambda^*(X \cap I) + \lambda^*(X \setminus I) \leq 4\epsilon + \sum_{n=1}^{\infty} \big(|I_n| + |J_n| + |K_n|\big) = 4\epsilon + \sum_{n=1}^{\infty} |I_n| < \lambda^*(X) + 5\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\lambda^*(X \cap I) + \lambda^*(X \setminus I) \leq \lambda^*(X)$, Since $X \subseteq \mathbb{R}$ was arbitrary, we see that I is measurable.

Before proving Parts (8) and (9) we remark that for $A, B \subseteq \mathbb{R}$, if A is measurable and $A \cap B = \emptyset$ then for all $X \subseteq \mathbb{R}$ we have

$$\begin{aligned}\lambda^*\big(X \cap (A \cup B)\big) &= \lambda^*\big((X \cap (A \cup B)) \cap A\big) + \lambda^*\big((X \cap (A \cup B)) \setminus A\big) \\ &= \lambda^*(X \cap A) + \lambda^*(X \cap B)\end{aligned}$$

It follows, inductively, that if $A_1, A_2, \cdots, A_n \subseteq \mathbb{R}$ are measurable and disjoint then for all $X \subseteq \mathbb{R}$ we have

$$\lambda^*\big(X \cap \bigcup_{k=1}^{n} A_k\big) = \sum_{k=1}^{n} \lambda^*(X \cap A_k).$$

Now let $A_1, A_2, A_3, \cdots \subseteq \mathbb{R}$ be measurable and disjoint and let $X \subseteq \mathbb{R}$. For all $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\sum_{k=1}^{n} \lambda^*(X \cap A_k) &= \lambda^*\big(X \cap \bigcup_{k=1}^{n} A_k\big) \quad \text{, by the above remark,} \\ &\leq \lambda^*\big(X \cap \bigcup_{k=1}^{\infty} A_k\big) \quad \text{, since } X \cap \bigcup_{k=1}^{n} A_k \subseteq X \cap \bigcup_{k=1}^{\infty} A_k, \\ &= \lambda^*\big(\bigcup_{k=1}^{\infty} (X \cap A_k)\big) \text{ , since } X \cap \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (X \cap A_k), \\ &\leq \sum_{k=1}^{\infty} \lambda^*(X \cap A_k) \quad \text{, by subadditivity.}\end{aligned}$$

Taking the limit as n tends to infinity gives

$$\lambda^*\big(X \cap \bigcup_{k=1}^{\infty} A_k\big) = \sum_{k=1}^{\infty} \lambda^*(X \cap A_k).$$

The special case $X = \mathbb{R}$ gives the formula $\lambda^*\big(\bigcup_{k=1}^{\infty} A_k\big) = \sum_{k=1}^{\infty} \lambda^*(A_k)$ for Part (9). For all $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\lambda^*(X) &= \lambda^*\big(X \cap \bigcup_{k=1}^{n} A_k\big) + \lambda^*\big(X \setminus \bigcup_{k=1}^{n} A_k\big)\\ &= \sum_{k=1}^{n} \lambda^*(X \cap A_k) + \lambda^*\big(X \setminus \bigcup_{k=1}^{n} A_k\big)\\ &\geq \sum_{k=1}^{n} \lambda^*(X \cap A_k) + \lambda^*\big(X \setminus \bigcup_{k=1}^{\infty} A_k\big)\end{aligned}$$

Taking the limit as n tends to infinity gives

$$\begin{aligned}\lambda^*(X) &\geq \sum_{k=1}^{\infty} \lambda^*(X \cap A_k) + \lambda^*\big(X \setminus \bigcup_{k=1}^{\infty} A_k\big)\\ &= \lambda^*\big(X \cap \bigcup_{k=1}^{\infty} A_k\big) + \lambda^*\big(X \setminus \bigcup_{k=1}^{\infty} A_k\big)\end{aligned}$$

so that $\bigcup_{k=1}^{\infty} A_k$ is measurable, proving Part (8) in the case that the sets A_k are disjoint. To complete the proof of Part (8) in the case that $A_1, A_2, A_3, \cdots \subseteq \mathbb{R}$ are measurable (but not necessarily disjoint) simply note that

$$\bigcup_{k=1}^{\infty} A_k = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2)) \cup (A_4 \setminus (A_1 \cup A_2 \cup A_3)) \cup \cdots$$

which is a countable union of disjoint measurable sets.

At this stage, we recall that we only proved Part (7) in the case of a bounded open interval. We note that every interval can be obtained from bounded open intervals by performing complements and countable unions or intersections, and so every interval is measurable.

To prove the first statement of Part (10), suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Let $B_1 = A_1$ and $B_k = A_k \setminus A_{k-1}$ for $k \geq 2$, Then the sets B_k are measurable and disjoint and we have $A_n = \bigcup_{k=1}^{n} B_k$ for all $n \in \mathbb{Z}^+$ and also $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Thus

$$\lambda\big(\bigcup_{n=1}^{\infty} A_n\big) = \lambda\big(\bigcup_{n=1}^{\infty} B_n\big) = \sum_{n=1}^{\infty} \lambda(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda(B_k) = \lim_{n\to\infty} \lambda\big(\bigcup_{k=1}^{n} B_k\big) = \lim_{n\to\infty} \lambda(A_n).$$

Finally, note that the second statement of Part 10 follows from the first, by taking complements in A_m.

1.13 Theorem: *All open and closed sets in $\mathbb{R}$ are measurable.*

Proof: Recall that every set in $\mathbb{R}^n$ (or any metric or topological space) is equal to the disjoint union of its connected components, and recall that the connected components of an open set are all open. Note that the set of connected components of an open set in $\mathbb{R}^n$ is at most countable because we can choose an element of $\mathbb{Q}^n$ inside each of the open connected components. Also recall that the connected sets in $\mathbb{R}$ are the intervals in $\mathbb{R}$. It follows that every nonempty open set in $\mathbb{R}$ is equal to the finite or countable disjoint union of its connected components, each of which is a nonempty open interval. Thus every open set in $\mathbb{R}$ is measurable. We also remark that when the connected components of the nonempty open set $U \subseteq \mathbb{R}$ are the disjoint open intervals $I_1, I_2, I_3, \cdots$ we have $\lambda(U) = \sum_{k \geq 1} |I_k|$. Closed sets are also measurable because every closed set is the complement of an open set.

1.14 Corollary: *For $A \subseteq \mathbb{R}$ we have*

$$\lambda^*(A) = \inf\left\{\lambda(U) \,\middle|\, U \subseteq \mathbb{R} \text{ is open with } A \subseteq U\right\}.$$

1.15 Example: The (standard) **Cantor set** is the set $C \subseteq [0,1]$ constructed as follows. Let $C_0 = [0,1]$. Let I_1 be the open middle third of C_0, that is let $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, and let $C_1 = A_0 \setminus U_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Let I_2 and I_3 be the open middle thirds of the two component intervals of C_1, that is let $I_2 = \left(\frac{1}{9}, \frac{2}{9}\right)$ and $I_3 = \left(\frac{7}{9}, \frac{8}{9}\right)$, and let $C_2 = C_1 \setminus (I_2 \cup I_3)$. Having constructed the set C_k, which is the disjoint union of 2^k closed intervals each of length $\frac{1}{3^k}$, let $I_{2^k}, I_{2^k+1}, \cdots, I_{2^{k+1}-1}$ be the open middle thirds of these 2^k component intervals and let $C_{k+1} = C_k \setminus (I_{2^k}, I_{2^k+1}, \cdots, I_{2^{k+1}-1})$. Finally, we let

$$C = \bigcap_{k=1}^{\infty} C_k.$$

Since $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, and since each C_k is the disjoint union of 2^k closed intervals each of size $\frac{1}{3^k}$ so that $\lambda(C_k) = \left(\frac{2}{3}\right)^k$, we have

$$\lambda(C) = \lim_{k \to \infty} \lambda(C_k) = 0.$$

Note that C_k is the set of all numbers $x \in [0,1]$ which can be written in base 3 such that the the first k digits of x are not equal to 1, and so C is the set of all numbers $x \in [0,1]$ which can be written in base 3 with none of the digits of x equal to 1, and it follows that the cardinality of C is $|C| = 2^{\aleph_0}$.

1.16 Example: We can construct a (generalized) **Cantor set** $C \subseteq [0,1]$, having any desired value for the measure $\lambda(C) < 1$ as follows. Let $0 \leq m < 1$. Choose a sequence of positive real numbers $a_1, a_2, \cdots$ with $\sum_{k=1}^{\infty} a_k = 1 - m$. Let $C_0 = [0,1]$ and note that $\lambda(C_0) = 1$. Choose an open interval $I_1 \subseteq C_0$ with $\lambda(I_1) = a_1$ such that $C_0 \setminus I_1$ is the disjoint union of two nondegenerate closed intervals each of measure less than $\frac{1}{2}$. Let $C_1 = C_0 \setminus I_1$ and note that $\lambda(C_1) = 1 - a_1$. Having constructed the set C_k, which is the disjoint union of 2^k nondegenerate closed intervals each of measure less than $\frac{1}{2^k}$ and having total measure $\lambda(C_k) = 1 - (a_1 + a_2 + \cdots + a_k)$, we choose 2^k open intervals $I_{2^k}, I_{2^k+1}, \cdots, I_{2^{k+1}-1}$ which are contained in each of the 2^k component intervals of C_k so that the set $C_{k+1} = C_k \setminus (I_{2^k} \cup \cdots \cup I_{2^{k+1}-1})$ is the disjoint union of 2^{k+1} non-degenerate closed intervals each of measure less than $\frac{1}{2^{k+1}}$ and having total measure $\lambda(C_{k+1}) = 1 - (a_1 + \cdots + a_{k+1})$. Finally, we let $C = \bigcap_{k=1}^{\infty} C_k$ and note that $\lambda(C) = \lim_{k \to \infty} \lambda(C_k) = 1 - \sum_{k=1}^{\infty} a_k = m$.

1.17 Theorem: *Let $\mathcal{M}$ be the set of all measurable subsets of $\mathbb{R}$. Then $|\mathcal{M}| = 2^{2^{\aleph_0}}$.*

Proof: Let C be the standard Cantor set. Because $\lambda(C) = 0$ it follows that every subset of C is measurable. Because $|C| = 2^{\aleph_0}$ we have

$$2^{2^{\aleph_0}} = \big|\{A|A \subseteq \mathbb{R}\}\big| \geq |\mathcal{M}| \geq \big|\{A|A \subseteq C\}\big| = 2^{2^{\aleph_0}}.$$

1.18 Theorem: *There exists a nonmeasurable set in $\mathbb{R}$.*

Proof: Define an equivalence relation on the set $[0,1]$ by defining $x \sim y$ when $y - x \in \mathbb{Q}$. Let C denote the set of equivalence classes. For each $c \in C$, choose an element $x_c \in c$ and let $A = \{x_c|c \in C\} \subseteq [0,1]$. We shall prove that the set A is not measurable. Let $\mathbb{Q} \cap [0,2] = \{a_1, a_2, a_3, \cdots\}$, with the a_k distinct. For each $k \in \mathbb{Z}^+$, let $A_k = a_k + A \subseteq [0,3]$. We claim that the sets A_k are disjoint. Let $k, \ell \in \mathbb{Z}^+$ and suppose that $A_k \cap A_\ell \neq \emptyset$. Choose $y \in A_k \cap A_\ell$, say $y = a_k + x_c = a_\ell + x_d$ where $c, d \in C$. Since $x_c - x_d = a_\ell - a_k \in \mathbb{Q}$ we have $x_c \sim x_d$ and hence $c = d$ (since we only chose one element from each class). Since $c = d$ we have $x_c = x_d$, hence $a_k = a_\ell$, and hence $k = \ell$. Thus the sets A_k are disjoint, as claimed. Next, we claim that $[1,2] \subseteq \bigcup_{k=1}^{\infty} A_k$. Let $y \in [1,2]$. Since $y - 1 \in [0,1]$ we have $y - 1 \in c$ for some $c \in C$. Since $y - 1 \in c$ we have $y - 1 - x_c \in \mathbb{Q}$ hence also $y - x_c \in \mathbb{Q}$. Since $y \in [1,2]$ and $x_c \in [0,1]$ we have $y - x_c \in [0,2]$. Since $y - x_c \in \mathbb{Q} \cap [0,2]$ we have $y - x_c = a_k$ for some $k \in \mathbb{Z}^+$ so that $y \in A_k$. This proves that $[1,2] \subseteq \bigcup_{k=1}^{\infty} A_k$.

Suppose, for a contradiction, that the set A is measurable. By translation, each of the sets $A_k = a_k + A$ is measurable with $\lambda(A_k) = \lambda(A)$. Since the sets A_k are disjoint and measurable, additivity gives

$$\lambda\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \lambda(A_k) = \sum_{k=1}^{\infty} \lambda(A) = \begin{cases} 0 & \text{, if } \lambda(A) = 0, \\ \infty & \text{, if } \lambda(A) > 0. \end{cases}$$

But since $[0,1] \subseteq \bigcup_{k=1}^{\infty} A_k \subseteq [0,3]$ we also have $1 \leq \lambda\Big(\bigcup_{k=1}^{\infty} A_k\Big) \leq 3$, giving the desired contradiction.

1.19 Notation: Let X be a set. For any set $\mathcal{C}$ of subsets of X we write

$$\mathcal{C}_\sigma = \Big\{ \bigcup_{k=1}^{\infty} A_k \Big| \text{ each } A_k \in \mathcal{C} \Big\} \text{ and } \mathcal{C}_\delta = \Big\{ \bigcap_{k=1}^{\infty} A_k \Big| \text{ each } A_k \in \mathcal{C} \Big\}.$$

Note that $\mathcal{C}_{\sigma\sigma} = \mathcal{C}_\sigma$ and $\mathcal{C}_{\delta\delta} = \mathcal{C}_\delta$.

1.20 Definition: Let X be a set. A **σ-algebra** in X is a set $\mathcal{C}$ of subsets of X such that
(1) $\emptyset \in \mathcal{C}$,
(2) if $A \in \mathcal{C}$ then $A^c = X \setminus A \in \mathcal{C}$, and
(3) if $A_1, A_2, A_3, \cdots \in \mathcal{C}$ then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$.

Note that when $\mathcal{C}$ is a σ-algebra in X we have $\mathcal{C}_\sigma = \mathcal{C}$ and $\mathcal{C}_\delta = \mathcal{C}$.

1.21 Notation: In a metric space (or topological space) X, we let $\mathcal{G} = \mathcal{G}(X)$ denote the set of all open sets in X and we let $\mathcal{F} = \mathcal{F}(X)$ denote the set of all closed subsets of X. Note that $\mathcal{G}_\sigma = \mathcal{G}$ and $\mathcal{F}_\delta = \mathcal{F}$.

1.22 Example: For any set X, the set $\{\emptyset, X\}$ and the set $\mathcal{P}(X)$ of all subsets of X are σ-algebras in X, The set $\mathcal{M} = \mathcal{M}(\mathbb{R})$ of all measurable sets in $\mathbb{R}$ is a σ-algebra in $\mathbb{R}$.

1.23 Note: Note that given any set $\mathcal{C}$ of subsets of a set X there exists a unique smallest σ-algebra in X which contains $\mathcal{C}$, namely the intersection of all σ-algebras in X which contain $\mathcal{C}$.

1.24 Definition: For a metric space (or topological space) X, the **Borel** σ-algebra of subsets of X, denoted by $\mathcal{B} = \mathcal{B}(X)$, is the smallest σ-algebra in X which contains $\mathcal{G}$ (hence also $\mathcal{F}$). The elements of $\mathcal{B}$ are called **Borel** sets. Note that $\mathcal{B}$ contains all of the sets $\mathcal{G}, \mathcal{G}_\delta, \mathcal{G}_{\delta\sigma}, \mathcal{G}_{\sigma\delta\sigma}, \cdots$ and all of the sets $\mathcal{F}, \mathcal{F}_\sigma, \mathcal{F}_{\sigma\delta}, \mathcal{F}_{\sigma\delta\sigma}, \cdots$.

1.25 Exercise: Show that, in $\mathbb{R}$, we have $\mathcal{F} \subseteq \mathcal{G}_\delta$ or, equivalently, that $\mathcal{G} \subseteq \mathcal{F}_\sigma$,

1.26 Theorem: *All Borel sets in $\mathbb{R}$ are measurable.*

Proof: The set $\mathcal{M}$ of all measurable subsets of $\mathbb{R}$ is a σ-algebra which contains $\mathcal{G}$, and the Borel σ-algebra $\mathcal{B}$ is the intersection of all σ-algebra in which contain $\mathcal{G}$, so we have $\mathcal{B} \subseteq \mathcal{M}$.

1.27 Remark: It can be shown, using transfinite induction, that in $\mathbb{R}$ we have $|\mathcal{B}| = 2^{\aleph_0}$. Since $|\mathcal{B}| < |\mathcal{M}|$, it follows that there exist measurable sets which are not Borel.

1.28 Theorem: *For every set $A \subseteq \mathbb{R}$ there exists a set $B \in \mathcal{G}_\delta$ with $A \subseteq B$ such that $\lambda(B) = \lambda^*(A)$.*

Proof: Let $A \subseteq \mathbb{R}$. If $\lambda^*(A) = \infty$ then we can choose $B = \mathbb{R}$. Suppose that $\lambda^*(A) < \infty$. For each $n \in \mathbb{Z}^+$, choose bounded open intervals $I_{n,1}, I_{n,2}, I_{n,3}, \cdots$ such that $A \subseteq \bigcup\limits_{k=1}^{\infty} I_{n,k}$ and $\sum\limits_{k=1}^{\infty} |I_{n,k}| \leq \lambda^*(A) + \frac{1}{n}$, then let $U_n = \bigcup\limits_{k=1}^{\infty} I_{n,k}$. Note that for each $n \in \mathbb{Z}^+$ the set U_n is open with $A \subseteq U_n$, and we have $\lambda(U_n) \leq \sum\limits_{k=1}^{\infty} |I_{n,k}| \leq \lambda^*(A) + \frac{1}{n}$. Let $B = \bigcap\limits_{n=1}^{\infty} U_n$ and note that $B \in \mathcal{G}_\delta$. Since $A \subseteq U_n$ for all $n \in \mathbb{Z}^+$, we have $A \subseteq \bigcap\limits_{n=1}^{\infty} U_n$, that is $A \subseteq B$, and hence $\lambda^*(A) \leq \lambda(B)$. For every $n \in \mathbb{Z}^+$ we have $B \subseteq U_n$ so that $\lambda(B) \leq \lambda(U_n) \leq \lambda^*(A) + \frac{1}{n}$, and it follows that $\lambda(B) \leq \lambda^*(A)$. Thus $\lambda(B) = \lambda^*(A)$, as required.

1.29 Theorem: *Let $A \subseteq \mathbb{R}$. Then the following statements are equivalent.*
(1) A is measurable.
(2) For every $\epsilon > 0$ there exists an open set U with $A \subseteq U \subseteq \mathbb{R}$ such that $\lambda^*(U \setminus A) < \epsilon$.
(3) There exists a set $B \in \mathcal{G}_\delta$ with $A \subseteq B \subseteq \mathbb{R}$ such that $\lambda^*(B \setminus A) = 0$.
(4) For every $\epsilon > 0$ there exists a closed set $K \subseteq A$ such that $\lambda^*(A \setminus K) < \epsilon$.
(5) There exists a set $C \in \mathcal{F}_\sigma$ with $C \subseteq A$ such that $\lambda^*(A \setminus C) = 0$.

Proof: We prove that (1) is equivalent to (3) and leave proofs of other equivalences as an exercise. To show that (3) implies (1), suppose that there exists a set $B \in \mathcal{G}_\delta$ with $A \subseteq B$ such that $\lambda^*(B \setminus A) = 0$. Since $\lambda^*(B \setminus A) = 0$ we know that $B \setminus A$ is measurable, and hence the set $A = B \setminus (B \setminus A)$ is also measurable.

Suppose, conversely, that A is measurable. If $\lambda(A)<\infty$ then, by Theorem 1.28, we can choose $B\in\mathcal{G}_\delta$ with $A\subseteq B$ such that $\lambda(B)=\lambda(A)$, and then $\lambda(B\setminus A)=\lambda(B)-\lambda(A)=0$, as required. Suppose that $\lambda(A)=\infty$. Let $A_0=A\cap\mathbb{Z}$ and let $B_0=A_0$. Note that B_0 is closed, hence $B_0\in\mathcal{G}_\delta$. Enumerate the intervals $(m,m+1)$ by letting $I_{2k-1}=(k-1,k)$ and $I_{2k}=(-k,-k+1)$ for $k\geq 1$. For $n\geq 1$, let $A_n=A\cap I_n$. Using Theorem 1.28, we can choose $E_n\in\mathcal{G}_\delta$ with $A_n\subseteq E_n$ and $\lambda(A_n)=\lambda(E_n)$. Let $B_n=E_n\cap I_n$ so that $B_n\in\mathcal{G}_\delta$ with $A_n\subseteq B_n\subseteq I_n$ and $\lambda(A_n)=\lambda(B_n)$. Note that $\lambda(B_n\setminus A_n)=\lambda(B_n)-\lambda(A_n)=0$. Let $B=\bigcup_{n=0}^\infty B_n$. Then we have $\lambda(B\setminus A)=\lambda\big(\bigcup_{n=0}^\infty(B_n\setminus A_n)\big)=\sum_{n=0}^\infty\lambda(B_n\setminus A_n)=0$.

It remains to show that $B\in\mathcal{G}_\delta$. For each $n\geq 1$, since $B_n\in\mathcal{G}_\delta$ we can write $B_n=\bigcap_{k=1}^\infty V_{n,k}$ where each $V_{n,k}$ is open. Since $B_n\subseteq I_n$ we also have $B_n=\bigcap_{k=1}^\infty U_{n,k}$ where $U_{n,k}=V_{n,k}\cap I_n$. Since $U_{n,k}\subseteq I_n$ and the sets I_n are disjoint, it follows (as you can verify) that $\bigcup_{n=1}^\infty\bigcap_{k=1}^\infty U_{n,k}=\bigcap_{k=1}^\infty\bigcup_{n=1}^\infty U_{n,k}$ and hence $\bigcup_{n=1}^\infty B_n\in\mathcal{G}_\delta$. Since $B_0\in\mathcal{G}_\delta$ and $\bigcup_{n=1}^\infty B_n\in\mathcal{G}_\delta$, we also have $B=B_0\cup\bigcup_{n=1}^\infty B_n\in\mathcal{G}_\delta$, as required.

1.30 Theorem: *Let $A,B\subseteq\mathbb{R}$. Suppose that $A\subseteq B$ and B is measurable with $\lambda(B)<\infty$. Then A is measurable if and only if $\lambda(B)=\lambda^*(A)+\lambda^*(B\setminus A)$.*

Proof: If A is measurable then for all $X\subseteq\mathbb{R}$ we have $\lambda^*(X)=\lambda^*(X\cap A)+\lambda^*(X\setminus A)$ so that in particular (taking $X=B$) we have $\lambda^*(B)=\lambda^*(A)+\lambda^*(B\setminus A)$.

Suppose that $\lambda(B)=\lambda^*(A)+\lambda^*(B\setminus A)$, and let $X\subseteq\mathbb{R}$. By Theorem 1.28, we can choose $E\in\mathcal{G}_\delta$ with $X\cap B\subseteq E$ such that $\lambda(E)=\lambda^*(X\cap B)$. Let $C=E\cap B$ and note that C is measurable with $X\cap B\subseteq C\subseteq B$. Since $X\cap B\subseteq C$ we have $\lambda^*(X\cap B)\leq\lambda(C)$ and since $C\subseteq E$ we have $\lambda(C)\leq\lambda(E)=\lambda^*(X\cap B)$, and so $\lambda(C)=\lambda^*(X\cap B)$.

We claim that $\lambda(C)=\lambda^*(C\cap A)+\lambda^*(C\setminus A)$. Note that

$$\begin{aligned}\lambda(B)&=\lambda^*(A)+\lambda^*(B\backslash A)\text{ , as assumed in the statement of the theorem}\\&=\lambda^*(A\cap C)+\lambda^*(A\backslash C)+\lambda^*((B\backslash A)\cap C)+\lambda^*((B\backslash A)\backslash C)\text{, since } C \text{ is measurable}\\&=\lambda^*(A\cap C)+\lambda^*(A\backslash C)+\lambda^*(C\backslash A)+\lambda^*((B\backslash A)\backslash C)\text{ , since } (B\backslash A)\cap C=C\backslash A\\&=\lambda^*(C\cap A)+\lambda^*(C\backslash A)+\lambda^*(A\backslash C)+\lambda^*((B\backslash A)\backslash C)\text{ , by reordering terms}\\&\geq\lambda^*(C\cap A)+\lambda^*(C\backslash A)+\lambda(B\backslash C)\text{ , since } (A\backslash C)\cup((B\backslash A)\backslash C)=B\backslash C\\&\geq\lambda(C)+\lambda(B\backslash C)\text{ , since } (C\cap A)\cup(C\backslash A)=C\\&=\lambda(B)\text{ , since } B \text{ is the disjoint union } B=C\cup(B\backslash C).\end{aligned}$$

Since the first and last terms above are equal, it follows that all terms must be equal, so in particular we have $\lambda^*(C\cap A)+\lambda^*(C\setminus A)+\lambda(B\setminus C)=\lambda^*(C)+\lambda(B\setminus C)$ and hence (since $\lambda(B\setminus C)\leq\lambda(B)<\infty$) we have $\lambda^*(C\cap A)+\lambda^*(C\setminus A)=\lambda^*(C)$, as claimed.

Finally, note that

$$\begin{aligned}\lambda^*(X)&=\lambda^*(X\cap B)+\lambda^*(X\setminus B)\text{ , since } B \text{ is measurable}\\&=\lambda^*(C)+\lambda^*(X\setminus B)\text{ , since } \lambda^*(X\cap B)=\lambda^*(C)\\&=\lambda^*(C\cap A)+\lambda^*(C\setminus A)+\lambda^*(X\setminus B)\text{ , by the above claim}\\&\geq\lambda^*((X\cap B)\cap A)+\lambda^*((X\cap B)\setminus A)+\lambda^*(X\setminus B)\text{ , since } X\cap B\subseteq C,\\&=\lambda^*(X\cap A)+\lambda^*((X\cap B)\setminus A)+\lambda^*(X\setminus B)\text{ , since } (X\cap B)\cap A=X\cap A\\&\geq\lambda^*(X\cap A)+\lambda^*(X\setminus A)\text{ , since } ((X\cap B)\setminus A)\cup(X\setminus B)=X\setminus A\end{aligned}$$

so that A is measurable, as required.

1.31 Definition: Let X be a metric space and let $A \subseteq X$. We say A is **dense** (in X) when for every nonempty open ball $B \subseteq X$ we have $B \cap A \neq \emptyset$, or equivalently when $\overline{A} = X$. We say A is **nowhere dense** (in X) when for every nonempty open ball $B \subseteq \mathbb{R}$ there exists a nonempty open ball $C \subseteq B$ with $C \cap A = \emptyset$, or equivalently when $\overline{A}^{\,o} = \emptyset$.

1.32 Example: The generalized Cantor sets are nowhere dense in $\mathbb{R}$.

1.33 Note: When $A \subseteq B \subseteq X$, note that if A is dense in X then so is B and, on the other hand, if B is nowhere dense in X then so is A.

1.34 Note: When $A, B \subseteq X$ with $B = A^c = X \setminus A$, note that A is nowhere dense $\iff$ $\overline{A}^{\,o} = \emptyset \iff \overline{B^o} = X \iff$ the interior of B is dense.

1.35 Definition: Let $A \subseteq X$. We say that A is **first category** (or that A is **meagre**) when A is equal to a countable union of nowhere dense sets. We say that A is **second category** when it is not first category. We say that A **residual** when A^c is first category.

1.36 Example: Every countable set in $\mathbb{R}$ is first category since if $A = \{a_1, a_2, a_3, \cdots\}$ then we have $A = \bigcup\limits_{k=1}^{\infty} \{a_k\}$. In particular $\mathbb{Q}$ is first category and $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ is residual.

1.37 Note: If $A \subseteq X$ is first category then so is every subset of A.

1.38 Note: If $A_1, A_2, A_3, \cdots \subseteq X$ are are all first category then so is $\bigcup\limits_{k=1}^{\infty} A_k$.

1.39 Theorem: *(The Baire Category Theorem) Let X be a complete metric space.*
(1) Every first category set in X has an empty interior.
(2) Every residual set in X is dense.
(3) Every countable union of closed sets with empty interiors in X has an empty interior.
(4) Every countable intersection of dense open sets in X is dense.

Proof: Parts (1) and (2) are equivalent by taking complements, and Parts (3) and (4) are special cases of Parts (1) and (2), so it suffices to prove Part (1). We sketch a proof.

Let $A \subseteq X$ be first category, say $A = \bigcup\limits_{n=1}^{\infty} C_n$ where each C_n is nowhere dense. Suppose, for a contradiction, that A has nonempty interior, and choose an open ball $B_0 = B(a_0, r_0)$ with $0 < r_0 < 1$ such that $\overline{B}_0 \subseteq A$. Since each C_n is nowhere dense, we can chose a nested sequence of open balls $B_n = B(a_n, r_n)$ with $0 < r_n < \frac{1}{2^n}$ such that $\overline{B}_n \subseteq B_{n-1}$ and $\overline{B}_n \cap C_n = \emptyset$. Because $r_n \to 0$, it folows that the sequence $\{a_n\}$ is Cauchy. Because X is complete, it follows that $\{a_n\}$ converges in X, say $a = \lim\limits_{n\to\infty} a_n$. Note that $a \in \overline{B}_n$ for all n since $a_k \in \overline{B}_n$ for all $k \geq n$. Since $a \in \overline{B}_0$ and $\overline{B}_0 \subseteq A$ we have $a \in A$. But since $a \in \overline{B}_n$for al $n \geq 1$, and $\overline{B}_n \cap C_n = \emptyset$, we have $a \notin C_n$ for all $n \geq 1$ hence $a \notin \bigcup\limits_{n=1}^{\infty} C_n$, that is $a \notin A$.

1.40 Example: Recall that $\mathbb{Q}$ is first category and $\mathbb{Q}^c$ is residual. The Baire Category Theorem shows that $\mathbb{Q}^c$ cannot be first category because if $\mathbb{Q}$ and $\mathbb{Q}^c$ were both first category then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would also be first category, but this is not possible since $\mathbb{R}$ does not have empty interior.

1.41 Exercise: For each $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for all $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}^+$ such that $f_n(x) \in \mathbb{Q}$. Prove that there exists $n \in \mathbb{Z}^+$ such that f_n is constant in some nondegenerate interval.

1.42 Exercise: Show that in $\mathbb{R}$ we have $\mathcal{F}_\sigma \neq \mathcal{G}_\delta$ and we have $G_\delta \neq \mathcal{G}_{\delta\sigma}$.

1.43 Remark: Note that each of the following sets $\mathcal{C}$ of subsets of $\mathbb{R}$

$$\mathcal{C} = \left\{A \subseteq \mathbb{R} \middle| A \text{ is finite or countable}\right\}$$
$$\mathcal{C} = \left\{A \subseteq \mathbb{R} \middle| \lambda(A) = 0\right\}$$
$$\mathcal{C} = \left\{A \subseteq \mathbb{R} \middle| A \text{ is first category}\right\}$$

has the following properties:
(1) if $A \subseteq B$ and $B \in \mathcal{C}$ then $A \in \mathcal{C}$,
(2) if $A_1, A_2, A_3, \cdots \in \mathcal{C}$ then $\bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{C}$, and
(3) if $A \in \mathcal{C}$ then $A^o = \emptyset$.
Because of this, it seems reasonable to consider the sets in $\mathcal{C}$ to be, in some sense, "small". The following theorem, then, states that every set in $\mathbb{R}$ is the union of two small sets.

1.44 Theorem: *Every subset of $\mathbb{R}$ is equal to the disjoint union of a set of measure zero and a set of first category.*

Proof: Let $\mathbb{Q} = \{a_1, a_2, a_3, \cdots\}$. For $k, \ell \in \mathbb{Z}^+$, let $I_{k,\ell} = \left(a_\ell - \frac{1}{2^{k+\ell}}\, , \, a_\ell + \frac{1}{2^{k+\ell}}\right)$ and for $k \in \mathbb{Z}^+$, let $U_k = \bigcup\limits_{\ell=1}^{\infty} I_{k,\ell}$. Note that $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ and for each $k \in \mathbb{Z}^+$ we have $\mathbb{Q} \subseteq U_k$ and $\lambda(U_k) \leq \sum\limits_{\ell=1}^{\infty} |I_{k,\ell}| = \frac{1}{2^{k-1}}$. Let $B = \bigcap\limits_{k=1}^{\infty} U_k$. Note that B is residual (it is a countable intersection of dense open sets) and we have $\lambda(B) = \lim\limits_{k \to \infty} \lambda(U_k) = 0$ since $\lambda(U_k) \leq \frac{1}{2^k}$ for all $k \in \mathbb{Z}^+$. Finally note that any set A is equal to the disjoint union $A = (A \cap B) \cup (A \cap B^c)$, and we have $\lambda(A \cap B) = 0$ and the set $A \cap B^c$ is first category.