PMATH 450/650 Solutions to the Exercises for Chapter 5

1: Let $a_n \in \mathbf{C}$ for $n \in \mathbf{Z}$ and let $s_\ell(x) = \sum_{n=-\ell}^{\ell} a_n e^{inx}$. Let $f \in L_1(T)$ and let $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$.

(a) Show that if $f \in L_\infty(T)$ and $\lim_{\ell\to\infty} s_\ell = f$ in $L_\infty(T)$ then $a_n = \hat{f}(n)$ for all $n \in \mathbf{Z}$.

Solution: Suppose $f \in L_\infty(T)$ and $\lim_{\ell\to\infty} .s_\ell = f$. Let $n \in \mathbf{Z}$. Let $\epsilon > 0$. Choose $\ell \geq |n| \in \mathbf{Z}^+$ so that $||s_\ell - f||_\infty \leq 2\pi\epsilon$. By Lemma 3.15 we know that $|s_\ell(t) - f(t)| \leq ||s_\ell - f||_\infty$ a.e. in T and so

$$\begin{aligned} |a_n - \hat{f}(n)| &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} s_\ell(t)e^{-int}\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \right| = \frac{1}{2\pi}\left| \int_{-\pi}^{\pi} (s_\ell - f)e^{-int}\,dt \right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|s_\ell(t) - f(t)\big|\,dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} ||s_\ell - f||_\infty dt = \tfrac{1}{2\pi}\,||s_\ell - f||_\infty \leq \epsilon. \end{aligned}$$

(b) Show that if $f \in L_1(T)$ and $\lim_{\ell\to\infty} s_\ell = f$ in $L_1(T)$ then $a_n = \hat{f}(n)$ for all $n \in \mathbf{Z}$.

Solution: Suppose that $f \in L_1(T)$ and $\lim_{\ell\to\infty} s_\ell = f$ in $L_1(T)$. Let $n \in \mathbf{Z}$. Let $\epsilon > 0$. Choose $\ell \geq |n| \in \mathbf{Z}^+$ so that $||s_\ell - f||_1 \leq 2\pi\epsilon$. Then

$$\begin{aligned} |a_n - \hat{f}(n)| &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} s_\ell(t)e^{-int}\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \right| = \frac{1}{2\pi}\left| \int_{-\pi}^{\pi} (s_\ell - f)e^{-int}\,dt \right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|s_\ell(t) - f(t)\big|\,dt = \tfrac{1}{2\pi}\,||s_\ell - f||_1 \leq \epsilon. \end{aligned}$$

(c) Let $1 < p < \infty$. Show that if $f \in L_p(T)$ and $\lim_{\ell\to\infty} s_\ell = f$ in $L_p(T)$ then $a_n = \hat{f}(n)$ for all $n \in \mathbf{Z}$.

Solution: Suppose that $f \in L_p(T)$ and $\lim_{\ell\to\infty} s_\ell = f$ in $L_p(T)$. Since $s_\ell - f \in L_p(T)$, by Theorem 3.23 we have $s_\ell - f \in L_1(T)$ with $||s_\ell - f||_1 \leq (2\pi)^{1-\frac{1}{p}}||s_\ell - f||_p$. Let $n \in \mathbf{Z}$. Let $\epsilon > 0$. Choose $\ell \geq |n| \in \mathbf{Z}^+$ so that $||s_\ell - f||_p \leq (2\pi)^{1/p}\,\epsilon$. Then $||s_\ell - p||_1 \leq (2\pi)^{1-\frac{1}{p}}||s_\ell - f||_p \leq 2\pi\epsilon$ and so, as in Part (b), we have $|a_n - \hat{f}(n)| \leq \epsilon$.

2: Let $f \in L_1(T)$.

(a) Use Integration by Parts to show that if $f \in \mathcal{C}^1$ then $\left|\hat{f}(n)\right| \le \frac{M}{|n|}$ for all $n \in \mathbf{Z}$ where $M = \max\limits_{-\pi \le x \le \pi} \left|f'(x)\right|$.

Solution: Suppose that $f \in \mathcal{C}^1$ and let $M = \max\limits_{-\pi \le t \le \pi} \left|f'(t)\right|$. Integration by Parts gives

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = \frac{1}{2\pi}\left(\left[\frac{i}{n}\,f(t)e^{-int}\right]_{-\pi}^{\pi} - \frac{i}{n}\int_{-\pi}^{\pi} f'(t)e^{-int}dt\right) = \frac{-i}{2\pi n}\int_{-\pi}^{\pi} f'(t)e^{-int}dt.$$

Thus we have $c_n(f) = \frac{-i}{2\pi n}c_n(f')$, and

$$\left|\hat{f}(n)\right| \le \frac{1}{2\pi|n|}\int_{-\pi}^{\pi}\left|f'(t)\right|dt \le \frac{1}{2\pi|n|}\,2\pi M = \frac{M}{|n|}.$$

(b) Use induction to show that if $f \in \mathcal{C}^k$ then $\left|\hat{f}(n)\right| \le \frac{M}{(2\pi)^{k-1}|n|^k}$ for all $n \in \mathbf{Z}$ where $M = \max\limits_{-\pi \le x \le \pi}\left|f^{(k)}(x)\right|$.

Solution: Let $f \in \mathcal{C}^k$ and let $M = \max\limits_{-\pi \le t \le \pi}\left|f^{(k)}(t)\right|$. In Part (a) we showed that $c_n(f) = \frac{-i}{2\pi n}\,c_n(f')$ and it follows, by induction, that $c_n(f) = \left(\frac{-i}{2\pi n}\right)^k c_n\big(f^{(k)}\big)$, hence

$$\left|c_n(f)\right| = \left|\frac{1}{(2\pi)^k|n|^k}\int_{-\pi}^{\pi} f^{(k)}(t)e^{-int}\,dt\right| \le \frac{1}{(2\pi)^k|n|^k}\int_{-\pi}^{\pi}\left|f^{(k)}(t)\right|dt \le \frac{1}{(2\pi)^k|n|^k}\,2\pi M = \frac{M}{(2\pi)^{k-1}|n|^k}.$$

(c) Show that if $f \in \mathcal{C}^2$ then $\lim\limits_{\ell\to\infty} s_\ell = f$ in $L_\infty(T)$.

Solution: Let $f \in \mathcal{C}^2$ and let $M = \max\limits_{-\pi \le t \le \pi}\left|f''(t)\right|$. By Part (b) we have $|c_n(f)| \le \frac{M}{2\pi n^2}$ for all $n \in \mathbf{Z}$. Since $s_\ell(f)(x) = \sum\limits_{n=-\ell}^{\ell} c_n(f)e^{inx}$ and $\left|c_n(f)e^{inx}\right| = \left|c_n(f)\right| \le \frac{M}{2\pi n^2}$, the Weierstrass M Test shows that the sequence $\left\{s_\ell(f)(x)\right\}$ converges uniformly in T $\big($to some function $g(x)\big)$. By Fejér's Theorem, we have $\lim\limits_{\ell\to\infty} s_\ell(f)(x) = \lim\limits_{m\to\infty}\sigma_m(f)(x) = f(x)$ for all $x \in T$.

3: Let $f:\mathbf{R}\to\mathbf{R}$ be the 2π-periodic function with $f(x)=x^3-\pi^2x$ for $-\pi\le x\le\pi$.

(a) Find the coefficients of the real Fourier series for f.

Solution: Since $f(x)$ is odd we have $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,dx=0$ and $a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\cos nx\;dx=0$ and $b_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\sin nx\;dx=\dfrac{2}{\pi}\displaystyle\int_0^{\pi}(x^3-\pi^2x)\sin nx\;dx$. Integration by Parts gives

$$\int_0^\pi x\,\sin nx\;dx=\Big[-\tfrac{1}{n}\,x\,\cos nx\Big]_0^\pi+\int_0^\pi\tfrac{1}{n}\cos nx\;dx=-\tfrac{1}{n}\,\pi\,\cos n\pi=-\tfrac{(-1)^n\pi}{n}.$$

and

$$\begin{aligned}\int_0^\pi x^3\sin nx\;dx&=\Big[-\tfrac{1}{n}x^3\cos nx\Big]_0^\pi+\int_0^\pi\tfrac{3}{n}x^2\cos nx\;dx\\&=-\tfrac{(-1)^n\pi^3}{n}+\Big[\tfrac{3}{n^2}x^2\sin nx\Big]_0^\pi-\int_0^\pi\tfrac{6}{n^2}x\sin nx\;dx\\&=-\tfrac{(-1)^n\pi^3}{n}+0+\tfrac{6}{n^2}\,\tfrac{(-1)^n\pi}{n}=(-1)^n\Big(\tfrac{6\pi}{n^3}-\tfrac{\pi^3}{n}\Big)\end{aligned}$$

and so

$$b_n=\frac{2}{\pi}\int_0^\pi\left(x^3-\pi^2x\right)\sin nx\;dx=\tfrac{2}{\pi}\Big((-1)^n\big(\tfrac{6\pi}{n^3}-\tfrac{\pi^3}{n}\big)+(-1)^n\tfrac{\pi^3}{n}\Big)=\tfrac{(-1)^n12}{n^3}.$$

(b) Show that $\lim\limits_{\ell\to\infty}s_\ell(f)=f$ in $L_\infty(T)$.

Solution: Since $s_\ell(f)(x)=\sum\limits_{n=1}^{\ell}\frac{(-1)^n12}{n^3}\sin nx$ and $\left|\frac{(-1)^n12}{n^3}\sin nx\right|\le\frac{12}{n^3}$, it follows from the Weierstrass M Test that $\big\{s_\ell(f)(x)\big\}$ converges uniformly in T (to some function g), and by Fejér's Theorem we have $\lim\limits_{\ell\to\infty}s_\ell(f)(x)=\lim\limits_{m\to\infty}\sigma_m(f)(x)=f(x)$ for all $x\in T$.

(c) By evaluating at $x=\frac{\pi}{2}$, evaluate $\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$.

Solution: Since $f(x)=x^3-\pi^2x$ for $-\pi\le x\le\pi$, we have $f\big(\frac{\pi}{2}\big)=\big(\frac{\pi}{2}\big)^3-\pi^2\big(\frac{\pi}{2}\big)=-\frac{3\pi^3}{8}$. On the other hand, since $f(x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin nx$, and since when $n=2k$ we have $\sin\frac{n\pi}{2}=0$ and when $n=2k+1$ we have $\sin\frac{n\pi}{2}=(-1)^k$, we have $f\big(\frac{\pi}{2}\big)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin\frac{n\pi}{2}=\sum\limits_{k=0}^{\infty}\frac{(-1)^{2k+1}12}{(2k+1)^3}(-1)^k=-\sum\limits_{k=0}^{\infty}\frac{(-1)^k12}{(2k+1)^3}$. Thus

$$\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+1)^3}=-\tfrac{1}{12}\,f\big(\tfrac{\pi}{2}\big)=\tfrac{1}{12}\cdot\tfrac{3\pi^3}{8}=\tfrac{\pi^3}{32}.$$

4: Let $f:\mathbf{R}\to\mathbf{R}$ be the 2π-periodic function with $f(x)=\begin{cases}1 \text{ if } 0<x<\pi,\\ -1 \text{ if } -\pi<x<0,\\ 0 \text{ if } x=0,\pm\pi.\end{cases}$

(a) Find the coefficients of the real Fourier series for f.

Solution: Since $f(x)$ is odd we have $a_0=0$ and $a_n=0$ for all $n\in\mathbf{Z}^+$ and we have

$$b_n=\tfrac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx=\tfrac{2}{\pi}\int_0^\pi \sin nx\,dx=\tfrac{2}{\pi}\Big[-\tfrac{1}{n}\cos nx\Big]_0^\pi=-\tfrac{2}{\pi n}\big((-1)^n-1\big)=\begin{cases}\frac{4}{n\pi} \text{ if } n \text{ is odd.}\\ 0 \text{ if } n \text{ is even.}\end{cases}$$

(b) By recognizing $s_{2\ell}(f)\big(\frac{\pi}{2\ell}\big)$ as a Riemann sum, show that $\lim\limits_{\ell\to\infty}s_{2\ell}(f)\big(\frac{\pi}{2\ell}\big)=\dfrac{2}{\pi}\displaystyle\int_0^\pi\frac{\sin x}{x}\,dx.$

Solution: When we partition the interval $[0,\pi]$ into ℓ equal-sized subintervals, the endpoints of the subintervals are $x_k=\frac{\pi k}{\ell}$ and the midpoints of the subintervals are $m_k=\frac{x_k+x_{k-1}}{2}=\frac{(2k-1)\pi}{2\ell}$. The Riemann sum for $\displaystyle\int_0^\pi\frac{\sin x}{x}\,dx$ using the midpoints of this partition is

$$R_\ell=\sum_{k=1}^{\ell}\frac{\sin m_k}{m_k}(x_k-x_{k-1})=\sum_{k=1}^{\ell}\frac{\sin\frac{(2k-1)\pi}{2\ell}}{\frac{(2k-1)\pi}{2\ell}}\cdot\frac{\pi}{\ell}=2\sum_{k=1}^{\ell}\frac{\sin\frac{(2k-1)\pi}{2\ell}}{(2k-1)}$$

By Part (a) we have

$$s_{2\ell}(f)(x)=s_{2\ell-1}(f)(x)=\sum_{\substack{n\text{ odd}\\1\le n\le 2\ell}}\tfrac{4}{n\pi}\sin nx=\tfrac{4}{\pi}\sum_{k=1}^{\ell}\frac{\sin(2k-1)x}{2k-1}$$

so, in particular,

$$s_{2\ell}(f)\big(\tfrac{\pi}{2\ell}\big)=\tfrac{4}{\pi}\sum_{k=1}^{\ell}\frac{\sin\frac{(2k-1)\pi}{2\ell}}{(2k-1)}=\tfrac{2}{\pi}R_\ell$$

Thus

$$\lim_{\ell\to\infty}s_{2\ell}(f)\big(\tfrac{\pi}{2\ell}\big)=\frac{2}{\pi}\lim_{\ell\to\infty}R_\ell=\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx.$$

(c) Using a computer to approximate the value of $\dfrac{2}{\pi}\displaystyle\int_0^\pi\frac{\sin x}{x}\,dx$, show that $\liminf\limits_{\ell\to\infty}||s_\ell(f)-f||_\infty>0.17$.

Solution: Using uniform convergence of power series (allowing term-by-term integration) and the Alternating Series Test, and then using a calculator, we have

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx&=\tfrac{2}{\pi}\int_0^\pi\Big(1-\tfrac{1}{3!}x^2+\tfrac{1}{5!}x^4-\tfrac{1}{7!}x^7+\cdots\Big)dx=\tfrac{2}{\pi}\Big[x-\tfrac{1}{3\cdot3!}x^3+\tfrac{1}{5\cdot5!}x^5-\tfrac{1}{7\cdot7!}x^7\cdots\Big]_0^\pi\\&=\Big(2-\tfrac{2\pi^3}{2\cdot3!}+\tfrac{2\pi^4}{5\cdot5!}-\tfrac{2\pi^6}{7\cdot7!}+\cdots\Big)>\Big(2-\tfrac{2\pi^2}{3\cdot3!}+\tfrac{2\pi^4}{5\cdot5!}-\tfrac{2\pi^6}{7\cdot7!}\Big)>1.1735737\end{aligned}$$

Choose $m\in\mathbf{Z}^+$ so that for $\ell\ge m$ we have $s_{2\ell}(f)\big(\frac{\pi}{2\ell}\big)-f\big(\frac{\pi}{2\ell}\big)>(1.173-1)=0.173$. Then for all $\ell\ge m$ we have $||s_{2\ell-1}(f)-f||_\infty=||s_{2\ell}(f)-f||_\infty\ge\big|s_{2\ell}(f)\big(\frac{\pi}{2\ell}\big)-f\big(\frac{\pi}{2\ell}\big)\big|>0.173$ and so $\liminf\limits_{\ell\to\infty}||s_\ell(f)-f||_\infty\ge0.173$.

(d) (Optional) Show that $\big\{s_\ell(f)(x)\big\}$ converges for all x.

Solution: When $x=k\pi$ with $k\in\mathbf{Z}$ we have $s_\ell(f)(x)=0$ for all x. Suppose that $x\ne k\pi$ for $k\in\mathbf{Z}$. Then

$$\begin{aligned}\sum_{k=0}^{n}\sin(2k+1)x&=\text{Im}\Big(\sum_{k=0}^{n}e^{i(2k+1)x}\Big)=\text{Im}\left(\frac{e^{ix}\big(e^{i(n+1)2x}-1\big)}{e^{i\,2x}-1}\right)\\&=\text{Im}\left(\frac{e^{ix}\cdot2i\,e^{i(n+1)x}\sin(n+1)x}{2i\,e^{ix}\sin x}\right)=\frac{\sin^2(n+1)x}{\sin x}\le\frac{1}{\sin x}.\end{aligned}$$

Since the partial sums $\sum\limits_{k=0}^{n}\sin(2k+1)x$ are bounded by $\frac{1}{\sin x}$ and the sequence $\big\{\frac{4}{\pi(2k+1)}\big\}$ is decreasing with limit 0, it follows from Dirichlet's Test for Convergence that the series $\sum\limits_{k=0}^{\infty}\frac{4}{\pi(2k+1)}\sin(2k+1)x$ converges.