PMATH 450/650 Exercises for Chapter 5

1: Let $a_n \in \mathbf{C}$ for $n \in \mathbf{Z}$ and let $s_\ell(x) = \sum_{n=-\ell}^{\ell} a_n e^{inx}$. Let $f \in L_1(T)$ and let $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$.

(a) Show that if $f \in L_\infty(T)$ and $\lim\limits_{\ell\to\infty} s_\ell = f$ in $L_\infty(T)$ then $a_n = \hat{f}(n)$ for all $n \in \mathbf{Z}$.

(b) Show that if $f \in L_1(T)$ and $\lim\limits_{\ell\to\infty} s_\ell = f$ in $L_1(T)$ then $a_n = \hat{f}(n)$ for all $n \in \mathbf{Z}$.

(c) Let $1 < p < \infty$. Show that if $f \in L_p(T)$ and $\lim\limits_{\ell\to\infty} s_\ell = f$ in $L_p(T)$ then $a_n = \hat{f}(n)$ for all $n \in \mathbf{Z}$.

2: Let $f \in L_1(T)$.

(a) Use Integration by Parts to show that if $f \in \mathcal{C}^1$ then $\left|\hat{f}(n)\right| \leq \frac{M}{|n|}$ for all $n \in \mathbf{Z}$ where $M = \max\limits_{-\pi\leq x\leq\pi} \left|f'(x)\right|$.

(b) Use induction to show that if $f \in \mathcal{C}^k$ then $\left|\hat{f}(n)\right| \leq \frac{M}{(2\pi)^{k-1}|n|^k}$ for all $n \in \mathbf{Z}$ where $M = \max\limits_{-\pi\leq x\leq\pi} \left|f^{(k)}(x)\right|$.

(c) Show that if $f \in \mathcal{C}^2$ then $\lim\limits_{\ell\to\infty} s_\ell = f$ in $L_\infty(T)$.

3: Let $f : \mathbf{R} \to \mathbf{R}$ be the 2π-periodic function with $f(x) = x^3 - \pi^2 x$ for $-\pi \leq x \leq \pi$.

(a) Find the coefficients of the real Fourier series for f.

(b) Show that $\lim\limits_{\ell\to\infty} s_\ell(f) = f$ in $L_\infty(T)$.

(c) By evaluating at $x = \frac{\pi}{2}$, evaluate $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$.

4: Let $f : \mathbf{R} \to \mathbf{R}$ be the 2π-periodic function with $f(x) = \begin{cases} 1 \text{ if } 0 < x < \pi, \\ -1 \text{ if } -\pi < x < 0, \\ 0 \text{ if } x = 0, \pm\pi. \end{cases}$

(a) Find the coefficients of the real Fourier series for f.

(b) By recognizing $s_{2\ell}(f)\left(\frac{\pi}{2\ell}\right)$ as a Riemann sum, show that $\lim\limits_{\ell\to\infty} s_{2\ell}(f)\left(\frac{\pi}{2\ell}\right) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$.

(c) Using a computer to approximate the value of $\dfrac{2}{\pi}\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$, show that $\liminf\limits_{\ell\to\infty} ||s_\ell(f) - f||_\infty > 0.17$.

(d) (Optional) Show that $\left\{s_\ell(f)(x)\right\}$ converges for all x.