PMATH 450/650 Solutions to the Exercises for Chapter 4

1: (a) Let P_2 denote the space of polynomials of degree at most 2 with coefficients in $\mathbf{R}$ using the inner product given by $\langle f,g\rangle = f(0)g(0)+f(1)g(1)+f(2)g(2)$. Find the orthonormal basis for P_2 which is obtained by applying the Gram-Schmidt Procedure to the basis $\{1,x,x^2\}$.

Solution: Write $p_0=1$, $p_1=x$ and $p_2=x^2$. We take

$$\begin{aligned}
q_0 &= p_0 = 1\\
q_1 &= p_1 - \frac{\langle p_1,q_0\rangle}{|q_0|^2}\,q_0 = x - \frac{\langle x,1\rangle}{|1|^2}\,1 = x - \frac{0\cdot 1+1\cdot 1+2\cdot 1}{1^2+1^2+1^2}\,1 = x-1\\
q_2 &= p_2 - \frac{\langle p_2,q_0\rangle}{|q_0|^2}\,q_0 - \frac{\langle p_2,q_1\rangle}{|q_1|^2}\,q_1 = x^2 - \frac{\langle x^2,1\rangle}{|1|^2}\,1 - \frac{\langle x^2,x-1\rangle}{|x-1|^2}\,(x-1)\\
&= x^2 - \frac{0\cdot 1+1\cdot 1+4\cdot 1}{1^2+1^2+1^2}\,1 - \frac{(0)(-1)+(1)(-0)+(4)(1)}{(-1)^2+(0)^2+(1)^2}\,(x-1)\\
&= x^2 - \tfrac{5}{3} - 2(x-1) = x^2-2x+\tfrac{1}{3}\,.
\end{aligned}$$

Note that $|q_0|^2 = 1^2+1^2+1^2=3$, $|q_1|^2=(-1)^2+(0)^2+(1)^2=2$ and $|q_2|^2 = \left(\tfrac13\right)^2+\left(-\tfrac23\right)^2+\left(\tfrac13\right)=\tfrac69=\tfrac23$, and so normalizing yields the orthonormal basis $\{r_0,r_1,r_2\}$ with

$$r_0=\frac{q_0}{|q_0|}=\tfrac{1}{\sqrt3}\ ,\quad r_1=\frac{q_1}{|q_1|}=\tfrac{1}{\sqrt2}(x-1)\ ,\quad r_2=\frac{q_2}{|q_2|}=\sqrt{\tfrac32}\left(x^2-2x+\tfrac13\right).$$

(b) Let $\mathbf{R}^\infty$ denote the space of sequences $x=(x_1,x_2,x_3,\cdots)$ with each $x_k\in\mathbf{R}$ such that $x_k=0$ for all but finitely many indices k, using the inner product given by $\langle x,y\rangle=\sum\limits_{k=1}^\infty x_ky_k$. Let U be the subspace $U=\left\{x\in\mathbf{R}^\infty\big|\sum\limits_{k=1}^\infty x_k=0\right\}$. Find the orthonormal basis for U which is obtained by applying the Gram-Schmidt Procedure to the basis $\{u_1,u_2,u_3,\cdots\}$ where $u_k=e_k-e_{k+1}$.

Solution: Let $\{v_1,v_2,v_3,\cdots\}$ be the orthogonal basis obtained by applying the Gram-Schmidt Procedure to $\{u_1,u_2,u_3,\cdots\}$, and let $\{w_1,w_2,w_3\cdots\}$ be the orthonormal basis obtained by letting $w_n=\frac{v_n}{||v_n||}$. Let

$$s_n=\sum_{k=1}^n e_k=(1,1,\cdots,1,0,0,\cdots).$$

We claim that

$$v_n=\tfrac1n\,s_n-e_{n+1}=\left(\tfrac1n,\tfrac1n,\cdots,\tfrac1n,-1,0,0,\cdots\right)\ \text{ for all }\ n\ge1.$$

We have $v_1=u_1=e_1-e_2=s_1-e_2$, so the claim holds when $n=1$. Let $n\ge2$ and suppose the claim holds for all $\ell<n$. For $\ell<n$ we have

$$\langle u_n,v_\ell\rangle=\left\langle e_n-e_{n+1}\,,\,\tfrac1\ell\sum_{k=1}^\ell e_k-e_{\ell+1}\right\rangle=\langle e_n,-e_{\ell+1}\rangle=\begin{cases}-1 \text{ if } \ell=n-1,\\ \ \ 0 \text{ if } \ell<n-1.\end{cases}$$

and we have $|v_\ell|^2=\ell\cdot\frac{1}{\ell^2}+1=\frac{\ell+1}{\ell}$ so that in particular $|v_{n-1}|^2=\frac{n}{n-1}$. Thus

$$\begin{aligned}
v_n&=u_n-\sum_{\ell=1}^{n-1}\frac{\langle u_n,v_\ell\rangle}{|v_\ell|^2}\,v_\ell=u_n-\frac{\langle u_n,v_{n-1}\rangle}{|v_{n-1}|^2}\,v_{n-1}=u_n+\tfrac{n-1}{n}\,v_{n-1}\\
&=e_n-e_{n+1}+\tfrac{n-1}{n}\left(\tfrac{1}{n-1}s_{n-1}-e_n\right)=e_n-e_{n+1}+\tfrac1n s_{n-1}-\tfrac{n-1}{n}e_n\\
&=\tfrac1n s_{n-1}+\tfrac1n e_n-e_{n+1}=\tfrac1n s_n-e_{n+1}.
\end{aligned}$$

By Induction, $v_n=\frac1n s_n-e_{n+1}$ and $|v_n|^2=\frac{n+1}{n}$ for all $n\ge1$. Normalizing gives

$$w_n=\frac{v_n}{|v_n|}=\sqrt{\tfrac{n}{n+1}}\left(\tfrac1n\,s_n-e_{n+1}\right)=\left(\sqrt{\tfrac{1}{n^2+n}},,\cdots,\sqrt{\tfrac{1}{n^2+n}},-\sqrt{\tfrac{n}{n+1}},0,0,\cdots\right).$$

2: Use the Cauchy-Schwarz Inequality to solve each of the following problems, involving real-valued functions.

(a) Let $f \in L_2[0,\infty)$. Show that $\lim_{n\to\infty} \int_{[n,n+1]} f = 0$.

Solution: Since $f \in L_2[0,\infty)$, we have $\sum_{n=0}^{\infty} \int_{[n,n+1)} |f|^2 = \int_{[0,\infty)} |f|^2 < \infty$ hence $\lim_{n\to\infty} \int_{[n,n+1)} |f|^2 = 0$. Since $[0,1] \setminus [0,1) = \{1\}$ which has measure zero, we have $\int_{[n,n+1]} |f|^2 = \int_{[n,n+1)} |f|^2$, and so $\lim_{n\to\infty} \int_{[n,n+1]} |f|^2 = 0$. By the Cauchy-Schwartz Inequality in $L_2[n,n+1]$, we have

$$\int_{[n,n+1]} f = \langle f, 1\rangle \leq ||f||_2^2\, ||1||_2^2 = \left(\int_{[n,n+1]} |f|^2\right)^{1/2} \left(\int_{[n,n+1]} 1^2\right)^{1/2} = \left(\int_{[n,n+1]} |f|^2\right)^{1/2} \longrightarrow 0 \text{ as } n \to \infty.$$

(b) Let $f \in L_2[0,1]$ be nonnegative with $\int_{[0,1]} f^2 = \int_{[0,1]} f^3 = \int_{[0,1]} f^4 < \infty$. Show that there exists a measurable set $A \subseteq [0,1]$ such that $f = \chi_A$ a.e. in $[0,1]$.

Solution: In $L_2[0,1]$ we have

$$\langle f, f^2\rangle = \int_{[0,1]} f^3 \text{ and}$$

$$||f||_2 ||f^2||_2 = \left(\int_{[0,1]} f^2\right)^{1/2} \left(\int_{[0,1]} f^4\right)^{1/2} = \left(\int_{[0,1]} f^3\right)^{1/2} \left(\int_{[0,1]} f^3\right)^{1/2} = \int_{[0,1]} f^3.$$

Since $\langle f, f^2\rangle = ||f||_2 ||f^2||_2$ it follows, from the Cauchy-Schwartz Inequality, that $\{f, f^2\}$ is linearly dependent in $L_2[0,1]$. Thus either $f = 0$ (and $f^2 = 0$) in $L_2[0,1]$, or $f \neq 0$ and $f^2 = cf$ in $L_2[0,1]$ for some $0 \neq c \in \mathbf{R}$. If $f = 0$ in $L_2[0,1]$ then $f = \chi_\emptyset$ a.e. in $[0,1]$. Suppose that $f \neq 0$ and $f^2 = cf$ in $L_2[0,1]$ with $0 \neq c \in \mathbf{R}$. Since $\int_{[0,1]} f^2 = \int_{[0,1]} f^3 = \int_{[0,1]} cf^2 = c \int_{[0,1]} f^2$ and $\int_{[0,1]} f^2 = ||f||_2^2 \neq 0$, we must have $c = 1$. Thus we have $f^2 = f$ in $L_2[0,1]$, and so $f(x)^2 = f(x)$, hence $f(x) \in \{0,1\}$, for a.e. $x \in [0,1]$. If $A = \{x \in [0,1] \big| f(x) = 1\}$ and $B = \{x \in [0,1] \big| f(x) = 0\}$ and $C = [0,1] \setminus (A \cup B)$ then $\lambda(C) = 0$ and $f(x) = \chi_A(x)$ for all $x \in [0,1] \setminus C$.

3: Let $1 \leq p \leq \infty$.

(a) Show that if $p \neq 2$ then there does not exist an inner product on ℓ_p such that $||x||_p^2 = \langle x, x\rangle$ for all $x \in \ell_p$.

Solution: Suppose that there exists such an inner product. By the Parallelogram Law, for all $x, y \in \ell_p$ we have $||x+y||_p^2 + ||x-y||_p^2 = 2\big(||x||_p^2 + ||y||_p^2\big)$. Note that $p \neq \infty$ because $||e_1+e_2||_\infty^2 = ||e_1-e_2||_\infty^2 = 1+1 = 2$ while $2\big(||e_1||_\infty^2 + ||e_2||_\infty^2\big) = 2(1+1) = 4$. When $1 \leq p < \infty$,

$$||e_1+e_2||_p^2 + ||e_1-e_2||_p^2 = 2\big(||e_1||_p^2 + ||e_2||_p^2\big) \implies 2^{2/p} + 2^{2/p} = 2(1+1) \implies 2^{2/p} = 2^1 \implies \tfrac{2}{p} = 1 \implies p = 2.$$

(b) Let $A \subseteq \mathbf{R}$ be measurable with $\lambda(A) > 0$. Show that if $p \neq 2$ then there does not exist an inner product on $L_p(A)$ such that $||f||_p^2 = \langle f, f\rangle$ for all $f \in L_p(A)$.

Solution: Let $A_n = A \cap [n-1, n)$ for $n \in \mathbf{Z}$. Since $\lambda(A) > 0$ we can choose $n \in \mathbf{Z}$ so that $\lambda(A_n) > 0$. By our solution to Problem 2(b) on Assignment 1, we can choose $a \in (n, n+1)$ such that for $B = A_n \cap [n, a)$ and $C = A_n \cap [a, n+1)$ we have $\lambda(B) = \lambda(C) = \frac{1}{2}\lambda(A_n)$. Let $L = \frac{1}{2}\lambda(A_n) = \lambda(B) = \lambda(C)$. Let $f, g : A \to \mathbf{R}$ be given by $f = \chi_B$ and $g = \chi_C$, and note that $|f+g| = |f-g| = \chi_{B\cup C}$. Suppose that there exists an inner product on $L_p(A)$ which induces the p-norm. Then the Parallelogram Law must hold. Note that $p \neq \infty$ because $||f+g||_\infty^2 + ||f-g||_\infty^2 = 1+1 = 2$ while $2\big(||f||_\infty^2 + ||g||_\infty^2\big) = 2(1+1) = 4$. When $1 \leq p < \infty$ we have

$$\begin{aligned} ||f+g||_p^2 + ||f-g||_p^2 = 2\big(||f||_p^2 + ||g||_p^2\big) &\implies (2L)^{2/p} + (2L)^{2/p} = 2(L^{2/p} + L^{2/p}) \\ &\implies (2L)^{2/p} = 2\,L^{2/p} \implies 2^{2/p} = 2 \implies p = 2. \end{aligned}$$

4: (a) Let $B = \{x \in \ell_2 \,|\, ||x||_2 \le 1\}$. Show that B is not compact.

Solution: Let $E = \{e_1, e_2, e_3, \cdots\}$, let $U_0 = \ell_2 \setminus E$, let $U_n = B(e_n, 1) = \{x \in \ell_2 \big| ||x - e_n||_2 < 1\}$ for $n \in \mathbf{Z}^+$. and let $\mathcal{U} = \{U_0, U_1, U_2, \cdots\}$. Note that E is closed (because for all $k \ne \ell$ we have $||e_k - e_\ell||_2 = \sqrt{2}$, so every Cauchy sequence in E is eventually constant) and so U_0 is open, and so $\mathcal{U}$ is an open cover of B. But $\mathcal{U}$ has no finite subcover, indeed $\mathcal{U}$ has no proper subcover, because the point $0 \in B$ only lies in the set U_0 and for each $k \in \mathbf{Z}^+$, the point $e_k \in B$ only lies in the set U_k (when $n \in \mathbf{Z}^+$ with $n \ne k$ we have $||e_k - e_n||_2 = \sqrt{2}$ so $e_k \notin B(e_n, 1) = U_n$).

(b) Let $r_k \ge 0$ for all $k \in \mathbf{Z}^+$, and let $S = \{x \in \ell_2 \,\big|\, |x_k| \le r_k \text{ for all } k \in \mathbf{Z}^+\}$. Show that S is compact if and only if $\sum\limits_{k=1}^{\infty} |r_k|^2$ converges in $\mathbf{R}$.

Solution: If $\sum\limits_{k=1}^{\infty} |r_k|^2 = \infty$. then S is unbounded because $s_n = \sum\limits_{k=1}^{n} r_k e_k \in S$ and $||s_n||_2 = \sum\limits_{k=1}^{n} |r_k|^2 \longrightarrow \infty$ as $n \to \infty$, and hence S is not compact. Suppose that $\sum\limits_{k=1}^{\infty} |r_k|^2 < \infty$. We claim that every sequence in S has a convergent subsequence whose limit lies in S. Let $\{x_n\}_{n\ge1}$ be a sequence in S, say $x_n = \sum\limits_{k=1}^{\infty} x_{n,k} e_k$ with $|x_{n,k}| \le r_k$ for all n, k. Since $x_{n,1} \in [-r_1, r_1]$ for all n, we can choose $m_1 < m_2 < m_3 < \cdots$ so that the sequence $\{x_{m_n,1}\}_{n\ge1}$ converges in $\mathbf{R}$, say to $c_1 \in [-r_1, r_1]$. Denote the subsequence $\{x_{m_n}\}_{n\ge1}$ of $\{x_n\}$ in ℓ_2 by $\{x^1_n\}$ so we have $x^1_n = \sum\limits_{k=1}^{\infty} x^1_{n,k} e_k$ with $x^1_{n,k} = x_{m_n,k}$. Note that $x^1_{n,k} \in [-r_k, r_k]$ for all n, k and $\lim\limits_{n\to\infty} x^1_{n,1} = c_1 \in [-r_1, r_1]$. Since $x^1_{n,2} \in [-r_2, r_2]$ for all n, we can re-choose $m_1 < m_2 < m_3 < \cdots$ so that the sequence $\{x^1_{m_n,2}\}_{n\ge1}$ converges in $\mathbf{R}$, say to $c_2 \in [-r_2, r_2]$. Denote the subsequence $\{x^1_{m_n}\}_{n\ge1}$ of $\{x^1_n\}$ in ℓ_2 by $\{x^2_n\}$ so we have $x^2_n = \sum\limits_{k=1}^{\infty} x^2_{n,k} e_k$ with $x^2_{n,k} = x^1_{m_n,k}$. Note that $x^2_{n,k} \in [-r_k, r_k]$ for all n, k and $\lim\limits_{n\to\infty} x^2_{n,1} = c_1$ and $\lim\limits_{n\to\infty} x^2_{n,2} = c_2$. Repeat this procedure to obtain successive subsequences $\{x^m_n\}_{n\ge1}$ in ℓ_2 for each $m \in \mathbf{Z}^+$ given by $x^m_n = \sum\limits_{k=1}^{\infty} x^m_{n,k} e_k$ with $|x^m_{n,k}| \le r_k$ for all m, n, k such that $\lim\limits_{n\to\infty} x^m_{n,k} = c_k \in [-r_k, r_k]$ in $\mathbf{R}$ for all $k \le m$. Let $\{y_n\}$ be the diagonal sequence $y_n = x^n_n = \sum\limits_{k=1}^{\infty} x^n_{n,k} e_k$, and note that $\{y_n\}$ is a subsequence of the original sequence $\{x_n\}$. We claim that $y \to c$ in ℓ_2 where $c = \sum\limits_{k=1}^{\infty} c_k e_k$. Let $\epsilon > 0$. Since $\sum\limits_{k=1}^{\infty} |r_k|^2 < \infty$, we can choose $m \in \mathbf{Z}^+$ so that $\sum\limits_{k=m+1}^{\infty} |r_k|^2 < \frac{\epsilon^2}{8}$. Since $\lim\limits_{n\to\infty} x^m_{n,k} = c_k$ for all $k \le m$, we can choose $N \in \mathbf{Z}^+$ with $N \ge m$ so that for all $n \ge N$ we have $\left|x^m_{n,k} - c_k\right| < \frac{\epsilon^2}{2m}$ for all $k \le m$. Note that when $m' \ge m$, $\{x^{m'}_n\}$ is a subsequence of $\{x^m_n\}$ so for each $n \in \mathbf{Z}^+$ we have $x^{m'}_n = x^m_{n'}$ for some $n' \ge n$. In particular, when $n \ge N$ we have $y_n = x^n_n = x^m_{n'}$ for some $n' \ge n$, and so

$$\begin{aligned} ||y_n - c||_2^2 &= \sum_{k=1}^{\infty} |y_{n,k} - c_k|^2 = \sum_{k=1}^{m} |y_{n,k} - c_k|^2 + \sum_{k=m+1}^{\infty} |y_{n,k} - c_k|^2 \\ &\le \sum_{k=1}^{m} |x^m_{n',k} - c_k| + \sum_{k=m+1}^{\infty} (2r_k)^2 \le m \cdot \tfrac{\epsilon^2}{2m} + 4 \cdot \tfrac{\epsilon^2}{8} = \epsilon^2 \end{aligned}$$

hence $||y_n - c||_2 < \epsilon$. Since every sequence in S has a subsequence which converges to an element in S, it follows that S is compact (recall that, in a metric space, sequential compactness is equivalent to compactness).

5: Let H be a separable Hilbert space over $\mathbf{C}$.

(a) Show that for every $u \in H$, the linear map $L : H \to \mathbf{C}$ given by $L(x) = \langle x, u \rangle$ is continuous.

Solution: Let $u \in H$ and let $L(x) = \langle x, u \rangle$. Given $\epsilon > 0$ choose $\delta = \dfrac{\epsilon}{||u|| + 1}$. Then for $x, y \in H$ with $||x - y|| < \delta$, the Cauchy-Schwarz Inequality gives

$$\big|L(x) - L(y)\big| = \big|\langle x, u \rangle - \langle y, u \rangle\big| = \big|\langle x - y, u \rangle\big| \leq ||x - y||\,||u|| < \frac{\epsilon\,||u||}{||u|| + 1} \leq \epsilon.$$

Thus $L(x)$ is continuous (and indeed uniformly continuous).

Remark: we used $\delta = \frac{\epsilon}{||u||+1}$ rather that $\delta = \frac{\epsilon}{||u||}$ in order to include the case in which $||u|| = 0$.

(b) Show that for every continuous linear map $L : H \to \mathbf{C}$ there exists a unique point $u \in H$ such that $L(x) = \langle x, u \rangle$ for all $x \in H$.

Solution: Let $L : H \to \mathbf{C}$ be a continuous linear map. If $L = 0$ then we can take $u = 0$ to get $L(x) = \langle x, u \rangle$ for all x. Suppose that $L \neq 0$. Let $U = \ker(L) = \big\{x \in H \big| L(x) = 0\big\}$. Since L is linear, U is a subspace of H, and since L is continuous, U is closed, and it follows that $H = U \oplus U^\perp$. Since $L \neq 0$ it follows that $U \neq H$ and so $U^\perp \neq \{0\}$. Choose $w \in H$ with $L(w) \neq 0$ and choose $v \in U^\perp$ with $||v|| = 1$. Let $x \in H$. For $y = L(x)v - L(v)x$ we have $L(y) = L(x)L(v) - L(v)L(x) = 0$ so that $y \in U$ hence $\langle y, v \rangle = 0$, and so

$$L(x) = L(x)||v||^2 = L(x)\langle v, v \rangle = \big\langle L(x)v, v \big\rangle = \big\langle y + L(v)x\,,\ v \big\rangle = L(v)\langle x, v \rangle = \big\langle x\,,\ \overline{L(v)}\,v \big\rangle.$$

Thus $L(x) = \langle x, u \rangle$ where $u = \overline{L(v)}\,v$.

To prove uniqueness, note that if $L(x) = \langle x, u \rangle = \langle x, u' \rangle$ for all $x \in H$ then $\big\langle x\,,\ u - u' \big\rangle = 0$ for all $x \in H$ so. in particular, $||u - u'||^2 = \big\langle u - u'\,,\ u - u' \big\rangle = 0$ and hence $u = u'$.