PMATH 450/650 Exercises for Chapter 4

1: (a) Let P_2 denote the space of polynomials of degree at most 2 with coefficients in $\mathbf{R}$ using the inner product given by $\langle f,g\rangle = f(0)g(0)+f(1)g(1)+f(2)g(2)$. Find the orthonormal basis for P_2 which is obtained by applying the Gram-Schmidt Procedure to the basis $\{1,x,x^2\}$.

(b) Let $\mathbf{R}^\infty$ denote the space of sequences $x=(x_1,x_2,x_3,\cdots)$ with each $x_k\in\mathbf{R}$ such that $x_k=0$ for all but finitely many indices k, using the inner product given by $\langle x,y\rangle = \sum\limits_{k=1}^{\infty} x_k y_k$. Let U be the subspace $U=\left\{x\in\mathbf{R}^\infty \,\middle|\, \sum\limits_{k=1}^{\infty} x_k = 0\right\}$. Find the orthonormal basis for U which is obtained by applying the Gram-Schmidt Procedure to the basis $\{u_1,u_2,u_3,\cdots\}$ where $u_k=e_k-e_{k+1}$.

2: Use the Cauchy-Schwarz Inequality to solve each of the following problems, involving real-valued functions.

(a) Let $f\in L_2[0,\infty)$. Show that $\lim\limits_{n\to\infty}\int_{[n,n+1]} f = 0$.

(b) Let $f\in L_2[0,1]$ be nonnegative with $\int_{[0,1]} f^2 = \int_{[0,1]} f^3 = \int_{[0,1]} f^4 < \infty$. Show that there exists a measurable set $A\subseteq[0,1]$ such that $f=\chi_A$ a.e. in $[0,1]$.

3: Let $1\le p\le\infty$.

(a) Show that if $p\ne 2$ then there does not exist an inner product on ℓ_p such that $||x||_p^2=\langle x,x\rangle$ for all $x\in\ell_p$.

(b) Let $A\subseteq\mathbf{R}$ be measurable with $\lambda(A)>0$. Show that if $p\ne 2$ then there does not exist an inner product on $L_p(A)$ such that $||f||_p^2=\langle f,f\rangle$ for all $f\in L_p(A)$.

4: (a) Let $B=\left\{x\in\ell_2 \,\middle|\, ||x||_2\le 1\right\}$. Show that B is not compact.

(b) Let $r_k\ge 0$ for all $k\in\mathbf{Z}^+$, and let $S=\left\{x\in\ell_2 \,\middle|\, |x_k|\le r_k \text{ for all } k\in\mathbf{Z}^+\right\}$. Show that S is compact if and only if $\sum\limits_{k=1}^{\infty}|r_k|^2$ converges in $\mathbf{R}$.

5: Let H be a separable Hilbert space over $\mathbf{C}$.

(a) Show that for every $u\in H$, the linear map $L:H\to\mathbf{C}$ given by $L(x)=\langle x,u\rangle$ is continuous.

(b) Show that for every continuous linear map $L:H\to\mathbf{C}$ there exists a unique point $u\in H$ such that $L(x)=\langle x,u\rangle$ for all $x\in H$.