PMATH 450/650 Solutions to the Exercises for Chapter 3

1: (a) Let $f:[a,b]\to\mathbf{R}$ be continuous. Show that $||f||_\infty=\max\limits_{a\le x\le b}|f(x)|$.

Solution: Let $m=\max\limits_{a\le x\le b}|f(x)|$. Then $|f|^{-1}(m,\infty]=\emptyset$ so $\lambda\big(|f|^{-1}(m,\infty]\big)=0$ and hence $||f||_\infty\le m$. Since $|f|$ is continuous in $[a,b]$ we can choose $c\in[a,b]$ such that $|f(c)|=m$. Let $y<m$. Since $|f|$ is continuous at c with $|f(c)|=m$ we can choose δ with $0<\delta<b-a$ so that, for all $x\in[a,b]$, if $|x-c|<\delta$ then $\big||f(x)|-m\big|<(m-y)$. For $x\in[a,b]$ with $|x-c|<\delta$ we have $-m+y<|f(x)|-m$ and so $|f(x)|>y$. It follows that $\big(c-\delta\,,\,c+\delta\big)\subseteq|f|^{-1}(y,\infty]$ and hence $\lambda\big(|f|^{-1}(y,\infty]\big)\ge\lambda\big(\big(c-\delta\,,\,c+\delta\big)\cap[a,b]\big)\ge\delta>0$. Since $\lambda\big(|f|^{-1}(y,\infty]\big)>0$ for all $y<m$ it follows, from the definition of $||f||_\infty$, that $||f||_\infty\ge m$.

(b) Let $A\subseteq\mathbf{R}$ be measurable. Suppose that $f_n\to f$ in $L_\infty(A)$. Show that there exists $B\subseteq A$ with $\lambda(B)=0$ such that $f_n\to f$ uniformly in $A\setminus B$.

Solution: For $n\in\mathbf{Z}^+$, let $B_n=\big\{x\in A\,\big|\,\big|f_n(x)-f(x)\big|>||f_n-f||_\infty\big\}$. From Assignment 2 we know that $\lambda(B_n)=0$ for all $n\in\mathbf{Z}^+$. Let $B=\bigcup\limits_{n=1}^\infty B_n$. Then $\lambda(B)\le\sum\limits_{n=1}^\infty\lambda(B_n)=0$. Since $f_n\to f$ in $L_\infty(A)$ we have $||f_n-f||_\infty\to0$ as $n\to\infty$, so given $\epsilon>0$ we can choose $m\in\mathbf{Z}^+$ such that for all $n\ge m$ we have $||f_n-f||_\infty<\epsilon$. Then for all $n\ge m$ and for all $x\in A\setminus B$ we have $\big|f_n(x)-f(x)\big|\le||f_n-f||_\infty<\epsilon$. Thus $f_n\to f$ uniformly in $A\setminus B$.

2: (a) Show that if $1\le p<q\le\infty$ then $\ell_p\subsetneqq\ell_q$.

Solution: Let $1\le p<q\le\infty$. Let $x\in\ell_p$. Since $\sum\limits_{n=1}^\infty|x_n|^p<\infty$ it follows that $|x_n|^p\to0$ as $n\to\infty$ and hence $|x_n|\to0$ as $n\to\infty$. Since $|x_n|\to0$ as $n\to\infty$, we can choose $m\in\mathbf{Z}^+$ so that for all $n\ge m$ we have $|x_n|\le1$. Since $|x_n|\le1$ for all $n\ge m$ it follows that the sequence $\big\{|x_n|\big\}$ is bounded so we have $||x||_\infty<\infty$ and hence $x\in\ell_\infty$. This shows that $\ell_p\subseteq\ell_q$ in the case that $q=\infty$. Now suppose that $q<\infty$. For all $n\ge m$, since $|x_n|\le1$ and $p<q$ we have $|x_n|^q\le|x_n|^p$. Since $|x_n|^q\le|x_n|^p$ for all $n\ge m$ and $\sum|x_n|^p$ converges, it follows that $\sum|x_n|^q$ converges (by the Comparison Test). Thus $x\in\ell_q$ and so $\ell_p\subseteq\ell_q$, as required.

Note that $\ell_p\ne\ell_\infty$ because, for example, for the constant sequence x, given by $x_n=1$ for all $n\in\mathbf{Z}^+$, we have $x\in\ell_\infty$ but $x\notin\ell_p$. Also note that when $p<q<\infty$ we have $\ell_p\ne\ell_q$ because, for example, for the sequence x given by $x_n=\frac{1}{n^{1/p}}$ we have $x\in\ell_q$ but $x\notin\ell_p$.

(b) Show that ℓ_p is separable for $1\le p<\infty$ but that ℓ_∞ is not.

Solution: Suppose that $1\le p<\infty$. We claim that ℓ_p is separable, indeed we claim that $\mathbf{Q}^\infty$ is dense in ℓ_p where $\mathbf{Q}^\infty$ denotes the set of sequences of rational numbers whose terms are eventually zero. Let $x=(x_1,x_2,\cdots)\in\ell_p$. Let $\epsilon>0$. Since $\sum\limits_{k=1}^\infty|x_k|^p<\infty$ we can choose $m\in\mathbf{Z}^+$ so that $\sum\limits_{k=m+1}^\infty|x_k|^p<\frac{\epsilon^p}{2}$. For each index k with $1\le k\le m$, choose $r_k\in\mathbf{Q}$ such that $|r_k-x_k|^p<\frac{\epsilon^p}{2m}$ and let $r_k=0$ for $k>m$. Then we have $||x-r||_p^p=\sum\limits_{k=1}^\infty|x_k-r_k|^p=\sum\limits_{k=1}^m|x_k-r_k|^p+\sum\limits_{k=m+1}^\infty|x_k|^p<m\,\frac{\epsilon^p}{2m}+\frac{\epsilon^p}{2}=\epsilon^p$ and so $||x-r||_p<\epsilon$. This shows that $\mathbf{Q}^\infty$ is dense in ℓ_p, as claimed. Finally, note that $\mathbf{Q}^\infty$ is countable because $\mathbf{Q}^\infty=\bigcup\limits_{n=1}^\infty\mathbf{Q}^n$.

We claim that ℓ_∞ is not separable. Let $S\subseteq\ell_\infty$ be any dense subset. Let 2^ω denote the set of binary sequences. Note that $2^\omega\subseteq\ell_\infty$. For each $\alpha\in2^\omega$, choose $s_\alpha\in S$ with $||s_\alpha-\alpha||_\infty<\frac12$. By the Triangle Inequality, when $\alpha\ne\beta$ we have $1=||\alpha-\beta||_\infty\le||\alpha-s_\alpha||_\infty+||s_\alpha-s_\beta||_\infty+||s_\beta-\beta||_\infty<||s_\alpha-s_\beta||_\infty+1$ so that $||s_\alpha-s_\beta||_\infty>0$ and hence $s_\alpha\ne s_\beta$. It follows that S is uncountable, indeed the map $F:2^\omega\to S$ given by $F(\alpha)=s_\alpha$ is injective, so we have $|S|\ge\big|2^\omega\big|=2^{\aleph_0}$.

3: (a) Show that $L_\infty(0,1) \neq \bigcap_{1<p<\infty} L_p(0,1)$ and that $L_1(0,1) \neq \bigcup_{1<p<\infty} L_p(0,1)$.

Solution: By Theorem 3.23, we know that $L_\infty(0,1) \subseteq L_p(0,1)$ for all $1 < p < \infty$ so $L_\infty(0,1) \subseteq \bigcap_{1<p<\infty} L_p(0,1)$. To see that $L_\infty(0,1) \neq \bigcap_{1<p<\infty} L_p(0,1)$ consider the map $f:(0,1) \to \mathbf{R}$ given by $f(x) = \ln x$. Note that $f \notin L_\infty(0,1)$ because for $a > 0$ we have $\lambda\big(|f|^{-1}(a,\infty]\big) = \lambda(0, e^{-a}) = e^{-a} > 0$. We claim that $f \in \bigcap_{1<p<\infty} L_p(0,1)$. We have $\int_0^1 |\ln x|\, dx = \int_0^1 -\ln x\; dx = \Big[x - x\ln x\Big]_0^1 = 1$, and for $n \in \mathbf{Z}^+$, integration by Parts gives

$$\int_0^1 |\ln x|^{n+1} dx = \int_0^1 (-1)^{n+1} (\ln x)^{n+1} dx = \Big[(-1)^{n+1} x \ln x\Big]_0^1 + \int_0^1 (-1)^n (n+1)(\ln x)^n dx = (n+1)\int_0^1 |\ln x|^n dx$$

so, by induction, it follows that $\int_0^1 |\ln x|\, dx = n!$ for all $n \in \mathbf{Z}^+$ and hence $f \in L_n(0,1)$. Given $1 < p < \infty$ we can choose $n \in \mathbf{Z}^+$ with $p \leq n$ and then $f \in L_n(0,1) \subseteq L_p(0,1)$, hence $f \in \bigcap_{1<p<\infty} L_p(0,1)$, as claimed.

By Theorem 3.23 we know tthat $L_p(0,1) \subseteq L_1(0,1)$ for all $1 < p < \infty$, and so $\bigcup_{1<p<\infty} L_p(0,1) \subseteq L_1(0,1)$. To see that $L_1(0,1) \neq \bigcup_{1<p<\infty} L_p(0,1)$, consider the map $f:(0,1) \to \mathbf{R}$ given by $f(x) = \frac{1}{x(\ln x)^2}$ for $0 < x < \frac{1}{e}$ and $f(x) = 0$ for $\frac{1}{e} \leq x < 1$. Note that $f \in L_1(0,1)$ because letting $u = \ln x$ gives

$$\int_0^1 |f(x)|\, dx = \int_0^{1/e} \frac{dx}{x(\ln x)^2}\, dx = \int_{u=-\infty}^{-1} \frac{du}{u} = \Big[-\frac{1}{u^2}\Big]_{-\infty}^{-1} = 1.$$

We claim that $f \notin \bigcup_{1<p<\infty} L_p(0,1)$. Let $1 < p < \infty$. Since $p - 1 > 0$ we have $\lim_{x \to 0^+} x^{p-1}(\ln x)^{2p} = 0$ (if you do not know this fact then you should prove it using l'Hôpital's Rule) so we can choose $\delta > 0$ with $\delta \leq \frac{1}{e}$ so that $x^{p-1}(\ln x)^{2p} \leq 1$ for all $x \in (0,\delta)$. Then we have $\frac{1}{x^p(\ln x)^{2p}} \geq \frac{1}{x}$ for all $x \in (0,\delta)$ and so

$$\int_0^1 |f(x)|^p dx = \int_0^{1/e} \frac{dx}{x^p(\ln x)^{2p}} \geq \int_0^\delta \frac{dx}{x^p(\ln x)^{2p}} \geq \int_0^\delta \frac{dx}{x} = \infty.$$

This shows that $f \notin L_p(0,1)$ for all $1 < p < \infty$ and so $f \notin \bigcup_{1<p<\infty} L_p(0,1)$, as claimed.

(b) Let $A \subseteq \mathbf{R}$ be measurable with $\lambda(A) < \infty$ and let $f \in L_\infty(A)$. Show that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

Solution: We know, from the proof of Theorem 3.23, that $f \in L_p(A)$ with $||f||_p \leq \lambda(A)^{1/p}||f||_\infty$. It follows that

$$\limsup_{p\to\infty} ||f||_p \leq \lim_{p\to\infty} \lambda(A)^{1/p}||f||_\infty = ||f||_\infty.$$

Let $\epsilon > 0$. Let $B = \big\{x \in A \,\big|\, |f(x)| > ||f||_\infty - \epsilon\big\}$. Note that $\lambda(B) > 0$ by the definition of $||f||_\infty$. We have

$$||f||_p^p = \int_A |f|^p \geq \int_B |f|^p \geq \int_B \big(||f||_\infty - \epsilon\big)^p = \big(||f||_\infty - \epsilon\big)^p \lambda(B)$$

and so $||f||_p \geq \big(||f||_\infty - \epsilon\big)\lambda(B)^{1/p}$. Thus

$$\liminf_{p\to\infty} ||f||_p \geq \lim_{p\to\infty} \big(||f||_\infty - \epsilon\big)\lambda(B)^{1/p} = ||f||_\infty - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\liminf_{p\to\infty} ||f||_p \geq ||f||_\infty$. Since $\limsup_{p\to\infty} ||f||_p \leq ||f||_\infty$ and $\liminf_{p\to\infty} ||f||_p \geq ||f||_\infty$, we have $\lim_{p\to\infty} ||f||_p = ||f||_\infty$, as required.