PMATH 450/650 Exercises for Chapter 2

1: Let $f_n : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be measurable functions for $n \in \mathbf{Z}^+$, and let $a \in \mathbf{R}$.

(a) Show that $\left\{ x \in A \,\middle|\, \liminf\limits_{n\to\infty} f_n(x) > a \right\} = \bigcup\limits_{m=1}^{\infty} \bigcup\limits_{\ell=1}^{\infty} \bigcap\limits_{n=\ell}^{\infty} \left\{ x \in A \middle| f_n(x) \geq a + \frac{1}{m} \right\}$.

(b) Show that the set $\left\{ x \in A \,\middle|\, \{f_n\} \text{ converges} \right\}$ is measurable.

2: In this problem, you are asked to prove parts of several of the theorems from Chapter 2 in the Lecture Notes. Your proofs should not make use of any theorems from Chapter 2 (you may use theorems from Chapter 1). Let $A = B \cup C$ where $B, C \subseteq \mathbf{R}$ are disjoint and measurable, let $f : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be a function, and let g and h be the restrictions of f to B and to C respectively.

(a) Show that f is measurable if and only if g and h are both measurable.

(b) Suppose that f is a nonnegative simple function. Show that $\displaystyle\int_A f = \int_B g + \int_C h$.

(c) Suppose that f is a nonnegative measurable function. Show that $\displaystyle\int_A f = \int_B g + \int_C h$.

(d) Suppose that f is an integrable function. Show that g and h are integrable and $\displaystyle\int_A f = \int_B g + \int_C h$.

3: Let $f : A \subseteq \mathbf{R} \to [0, \infty]$ be a nonnegative measurable function.

(a) Show that if $0 < a$ then

$$\lambda\Big(f^{-1}\big((a,\infty]\big)\Big) \leq \frac{1}{a}\int_A f\,.$$

(b) Show that if $\displaystyle\int_A f = 0$ then $f = 0$ a.e. in A.

(c) Show that if $0 < a < \lambda(A) < \infty$ and $f(x) > 0$ for all $x \in A$ then

$$\inf\left\{ \int_B f \,\middle|\, B \subseteq A \text{ is measurable with } \lambda(B) \geq a \right\} > 0\,.$$

4: (a) Let $f_n : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be measurable and let $g_n : A \subseteq \mathbf{R} \to [0, \infty]$ be nonnegative and measurable with $|f_n| \leq g_n$ for all $n \in \mathbf{Z}^+$. Suppose that $\lim\limits_{n\to\infty} f_n$ exists, $\lim\limits_{n\to\infty} g_n$ exists, and $\displaystyle\lim_{n\to\infty}\int_A g_n = \int_A \lim_{n\to\infty} g_n < \infty$. Show that $\displaystyle\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty}\int_A f_n$.

(b) Let $f_n : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be integrable, suppose $\lim\limits_{n\to\infty} f_n$ exists and is integrable, and let $f = \lim\limits_{n\to\infty} f_n$. Show that $\displaystyle\lim_{n\to\infty}\int_A |f_n - f| = 0$ if and only if $\displaystyle\lim_{n\to\infty}\int_A |f_n| = \int_A |f|$.

5: Let $A \subseteq \mathbf{R}$ be measurable with $\lambda(A) < \infty$ and let Let $f : A \to \mathbf{R}$ be bounded. Define the upper and lower Lebesgue integrals of f on A to be

$$U(f) = \inf\left\{ \int_A s \,\middle|\, s \text{ is a simple function on } A \text{ with } s \geq f \right\} \text{ and}$$
$$L(f) = \sup\left\{ \int_A s \,\middle|\, s \text{ is a simple function on } A \text{ with } s \leq f \right\}.$$

(a) Show that f is measurable if and only if $U(f) = L(f)$ and, in this case, $\displaystyle\int_A f = U(f) = L(f)$.

(b) When $A = [a, b]$, show that if f is Riemann integrable then f is Lebesgue integrable and the two kinds of integral agree.