PMATH 450/650 Solutions to the Exercises for Chapter 1

1: (a) Show that for all sets $A\subseteq\mathbf{R}$ we have $c^*\big(\overline{A}\big)=c^*(A)$.

Solution: Since $A\subseteq\overline{A}$ it is clear that $c^*(A)\leq c^*(\overline{A})$. Let $\epsilon>0$. Choose bounded open intervals $I_1,I_2,\cdots,I_n$ so that $A\subseteq\bigcup\limits_{k=1}^n I_k$ and $\sum\limits_{k=1}^n|I_k|\leq c^*(A)+\epsilon$. For each index k, say $I_k=(a_k,b_k)$ and let $J_k=\big(a_k-\frac{\epsilon}{2n},b_k+\frac{\epsilon}{2n}\big)$ so that $\overline{I}_k\subseteq J_k$ and $|J_k|=|I_k|+\frac{\epsilon}{n}$. Then $\overline{A}\subseteq\bigcup\limits_{k=1}^n J_k$ so we have

$$c^*\big(\overline{A}\big)\leq\sum_{k=1}^n|J_k|=\sum_{k=1}^n\big(|I_k|+\tfrac{\epsilon}{n}\big)=\sum_{k=1}^n|I_k|+\epsilon\leq c^*(A)+2\epsilon.$$

Since ϵ was arbitrarily small, it follows that $c^*\big(\overline{A}\big)\leq c^*(A)$, as required.

(b) Show that for every compact set $A\subseteq\mathbf{R}$ we have $c^*(A)=\lambda^*(A)$.

Solution: Let $A\subseteq\mathbf{R}$ be bounded. Let $\epsilon>0$. Choose bounded open intervals $I_1,I_2,\cdots,I_n$ so that $A\subseteq\bigcup\limits_{k=1}^n A_k$ and $\sum\limits_{k=1}^n|I_k|\leq c^*(A)+\epsilon$. Let $I_k=\emptyset$ for $k>n$. Then $A\subseteq\sum\limits_{k=1}^\infty I_k$ so $\lambda^*(A)\leq\sum\limits_{k=1}^\infty|I_k|=\sum\limits_{k=1}^n|I_k|\leq c^*(A)+\epsilon$. Since $\epsilon>0$ was arbitrary, it follows that $\lambda^*(A)\leq c^*(A)$.

Now let $A\subseteq\mathbf{R}$ be compact. Let $\epsilon>0$. Choose bounded open intervals $I_1,I_2,I_3,\cdots$ so that $A\subseteq\bigcup\limits_{k=1}^\infty I_k$ and $\lambda^*(A)\leq\sum\limits_{k=1}^\infty|I_k|+\epsilon$. Since A is compact, we can choose finitely many indices $k_1<k_2<\cdots<k_n$ such that $A\subseteq\bigcup\limits_{i=1}^n I_{k_i}$. It follows that $c^*(A)\leq\sum\limits_{i=1}^n|I_{k_i}|\leq\sum\limits_{k=1}^\infty|I_k|\leq\lambda^*(A)+\epsilon$. Since $\epsilon>0$ was arbitrary, it follows that $c^*(A)\leq\lambda^*(A)$.

2: (a) Let $A\subseteq\mathbf{R}$ with $\lambda^*(A)>0$. Show that there is a bounded open interval I such that $\lambda^*(A\cap I)>\frac{1}{2}|I|$.

Solution: Suppose, for a contradiction, that for every bounded open interval I we have $\lambda^*(A\cap I)\leq\frac{1}{2}|I|$. Let $A_k=A\cap[k,k+1]$ for each $k\in\mathbf{Z}$. Note that $\lambda^*(A_k)>0$ for some $k\in\mathbf{Z}$ because if we had $\lambda^*(A_k)=0$ for all $k\in\mathbf{Z}$ then we would have $\lambda^*(A)=\lambda^*\big(\bigcup\limits_{k\in\mathbf{Z}}A_k\big)\leq\sum\limits_{k\in\mathbf{Z}}\lambda^*(A_k)=0$. Choose $k\in\mathbf{Z}$ so that $\lambda^*(A_k)>0$ and let $B=A_k$. Choose bounded open intervals $I_1,I_2,I_3,\cdots$ so that $B\subseteq\bigcup\limits_{k=1}^\infty I_k$ and $\sum\limits_{k=1}^\infty|I_k|\leq\frac{3}{2}\lambda^*(B)$. Then

$$\lambda^*(B)=\lambda^*\big(\bigcup_{k=1}^\infty(B\cap I_k)\big)\leq\sum_{k=1}^\infty\lambda^*(B\cap I_k)\leq\sum_{k=1}^\infty\lambda^*(A\cap I_k)\leq\sum_{k=1}^\infty\tfrac{1}{2}|I_k|\leq\tfrac{1}{2}\cdot\tfrac{3}{2}\lambda^*(B)=\tfrac{3}{4}\lambda^*(B),$$

which gives the desired contradiction.

(b) Let $A\subseteq\mathbf{R}$ be bounded. Show that there is a set $B\subseteq A$ with $\lambda^*(B)=\frac{1}{2}\lambda^*(A)$.

Solution: Choose $a,b\in\mathbf{R}$ so that $A\subseteq[a,b]$. Define $f:[a,b]\to\mathbf{R}$ by $f(x)=\lambda^*\big(A\cap[a,x]\big)$. Note that $f(a)=\lambda^*\big(A\cap\{a\}\big)\leq\lambda^*(\{a\})=0$ and $f(b)=\lambda^*\big(A\cap[a,b]\big)=\lambda^*(A)$. For $a\leq x\leq y\leq b$ we have $f(y)=\lambda^*\big(A\cap[a,y]\big)\leq\lambda^*\big(A\cap[a,x]\big)=f(x)$ and we have

$$\begin{aligned}f(y)=\lambda^*\big(A\cap[a,y]\big)&=\lambda^*\Big(\big(A\cap[a,x]\big)\cup\big(A\cap(x,y]\big)\Big)\leq\lambda^*\big(A\cap[a,x]\big)+\lambda^*\big(A\cap(x,y]\big)\\&\leq\lambda^*\big(A\cap[a,x]\big)+\lambda^*\big((x,y]\big)=f(x)+(y-x).\end{aligned}$$

Thus $0\leq f(y)-f(x)\leq y-x$ for all x,y with $a\leq x\leq y\leq b$, and so f is continuous on $[a,b]$. Since $f:[a,b]\to\mathbf{R}$ is continuous with $f(a)=0$ and $f(b)=\lambda^*(A)$, we can choose $x\in(a,b)$ such that $f(x)=\frac{1}{2}\lambda^*(A)$. Thus we can take $B=A\cap[a,x]$ to get $\lambda^*(B)=\lambda^*\big(A\cap[a,x]\big)=f(x)=\frac{1}{2}\lambda^*(A)$.

3: (a) Let $A \subseteq \mathbf{R}$ be measurable with $\lambda(A) > 0$. Show that there exists a nonmeasurable set $B \subseteq A$.

Solution: As in Problem 2(a), choose $k \in \mathbf{Z}$ so that for $S = A \cap [k, k+1]$ we have $\lambda(S) > 0$. Define an equivalence relation on S by defining $x \sim y \iff y - x \in \mathbf{Q}$. Let C be the set of equivalence classes. For each class $c \in C$ choose an element $x_c \in c$. Note that $x_c \in c \subseteq S = A \cap [k, k+1]$. Let $B = \{x_c | c \in C\} \subseteq A \cap [k, k+1]$. We claim that B is not measurable.

Let $\mathbf{Q} \cap [-1, 1] = \{a_1, a_2, a_3, \cdots\}$ with the elements a_k all distinct. For each $k \in \mathbf{Z}^+$, let $B_k = a_k + B$. Note that since $a_k \in [-1, 1]$ and $B \subseteq [k, k+1]$, we have $B_k = a_k + B \subseteq [k-1, k+2]$. We claim that the sets B_k are disjoint. Let $k, l \in \mathbf{Z}^+$ and suppose that $B_k \cap B_l \neq \emptyset$. Choose $y \in B_k \cap B_l$, say $y = a_k + x_c = a_l + x_d$ where $c, d \in C$. Then we have $x_c - x_d = a_l - a_k \in \mathbf{Q}$ hence $x_c \sim x_d$ hence $c = d$ hence $a_k = a_l$ and hence $k = l$. This proves that the sets B_k are disjoint, as claimed. We also claim that $S = A \cap [k, k+1] \subseteq \bigcup_{k=1}^{\infty} B_k \subseteq [k-1, k+2]$. It is clear that $\bigcup_{k=1}^{\infty} B_k \subseteq [k-1, k+2]$ because $B_k \subseteq [k-1, k+2]$ for all $k \in \mathbf{Z}^+$. Let $y \in S = A \cap [k, k+1]$. Then $y \in c$ for some equivalence class c, and then we have $y \sim x_c$ hence $y - x_c \in \mathbf{Q}$. Since $y \in [k, k+1]$ and $x_c \in [k, k+1]$ we have $y - x_c \in [-1, 1]$. Since $y - x_c \in \mathbf{Q} \cap [-1, 1]$ we have $y - c_c = a_k$ for some $k \in \mathbf{Z}^+$ and then $y = a_k + x_c \in a_k + B = B_k \subseteq \bigcup_{k=1}^{\infty} B_k$. This proves that $S \subseteq \bigcup_{k=1}^{\infty} B_k$, as claimed.

Suppose, for a contradiction, that B is measurable. Then the translated sets $B_k = a_k + B$ are measurable with $\lambda(B_k) = \lambda(B)$ for all $k \in \mathbf{Z}$. Since the sets B_k are disjoint, it follows that

$$\lambda\Big(\bigcup_{k=1}^{\infty} B_k\Big) = \sum_{k=1}^{\infty} \lambda(B_k) = \sum_{k=1}^{\infty} \lambda(B) = \begin{cases} 0 \text{ , if } \lambda(B) = 0, \\ \infty \text{ , if } \lambda(B) > 0 \end{cases}$$

But since $S \subseteq \bigcup_{k=1}^{\infty} B_k \subseteq [k-1, k+2]$ we have $0 < \lambda(S) \leq \lambda\Big(\bigcup_{k=1}^{\infty} B_k\Big) \leq 3$ giving the desired contradiction.

(b) Show that there exist disjoint sets $A, B \subseteq \mathbf{R}$ such that $\lambda^*(A \cup B) \neq \lambda^*(A) + \lambda^*(B)$.

Solution: Choose a nonmeasurable set $C \subseteq \mathbf{R}$. Choose a set $X \subseteq \mathbf{R}$ such that $\lambda^*(X) \neq \lambda^*(X \cap A) + \lambda^*(X \setminus A)$. Let $A = X \cap C$ and $B = X \setminus C$). Then A and B are disjoint with $A \cup B = X$ and so $\lambda^*(A \cup B) = \lambda^*(X) \neq \lambda^*(A) + \lambda^*(B)$.

4: (a) Show that $\mathcal{F} \subseteq \mathcal{G}_\delta$ (or, equivalently, that $\mathcal{G} \subseteq \mathcal{F}_\sigma$).

Solution: Let $\emptyset \neq A \in \mathcal{F}$. Since A is closed, for each $x \in \mathbf{R}$ the function $g_x : A \to [0,\infty)$ given by $g_x = |x-a|$ attains its minimum value. Define $f : \mathbf{R} \to [0,\infty)$ by $f(x) = \text{dist}(x, A) = \min\big\{|x-a|\big|a \in A\big\}$ and note that $f(x) = 0 \iff x \in A$. We claim that f is continuous. Choose $a \in A$ so that $|x-a| = \text{dist}(x,A) = f(x)$. Then

$$f(y) = \text{dist}(y, A) \leq |y-a| = |y-x+x-a| \leq |y-x| + |x-a| \leq |y-x| + f(x)$$

so we have $f(y)-f(x) \leq |y-x|$. By symmetry, we also have $f(x)-f(y) \leq |x-y|$ so that $\big|f(y)-f(x)\big| \leq |y-x|$. Since $\big|f(y)-f(x)\big| \leq |y-x|$ for all $x, y \in \mathbf{R}$, it follows that f is continuous, as claimed. Since f is continuous, the set $\big\{x \in \mathbf{R}\big|f(x) < \frac{1}{n}\big\} = f^{-1}\big(\frac{1}{n}, \infty\big)$ is open for each $n \in \mathbf{Z}^+$. and so

$$A = \big\{x \in \mathbf{R}\big|f(x) = 0\big\} = \bigcap_{n=1}^{\infty} \big\{x \in \mathbf{R}\big|fx < \tfrac{1}{n}\big\} \in \mathcal{G}_\delta.$$

(b) Show that $\mathcal{F}_\sigma \neq \mathcal{G}_\delta$.

Solution: Recall that $\mathbf{Q} \in \mathcal{F}_\sigma$ $\Big($indeed if $\mathbf{Q} = \{a_1, a_2, \cdots\}$ then $\mathbf{Q} = \bigcup_{k=1}^{\infty}\{a_k\}\Big)$ and it follows (by taking the complement) that $\mathbf{Q}^c \in \mathcal{G}_\delta$. We claim that $\mathbf{Q}^c \notin \mathcal{F}_\sigma$ (and hence, by taking complements, $\mathbf{Q} \notin \mathcal{G}_\delta$). Suppose, for a contradiction, that $\mathbf{Q}^c \in \mathcal{F}_\sigma$. Let $\mathbf{Q} = \bigcup_{k=1}^{\infty} A_k$ where each A_k is a closed set (which is contained in $\mathbf{Q}$) and let $\mathbf{Q}^c = \bigcup_{k=1}^{\infty} B_k$ where each B_k is a closed set (which is contained in $\mathbf{Q}^c$). Then $\mathbf{R} = \mathbf{Q} \cup \mathbf{Q}^c = \bigcup_{n=1}^{\infty} C_n$ where $C_{2k} = A_k$ and $C_{2k-1} = B_k$. For each $n \in \mathbf{Z}^+$, when n is even C_n is contained in $\mathbf{Q}$ and when n is odd C_n is contained in $\mathbf{Q}^c$ and, in either case, it follows that C has an empty interior. Thus $\mathbf{R}$ is a countable union of closed sets with empty interiors, and so $\mathbf{R}$ is first category. We know this is impossible, by the Baire Category Theorem, and so we have obtained the desired contradiction.

(c) Show that $\mathcal{G}_\delta \neq \mathcal{G}_{\delta\sigma}$ (or, equivalently, that $\mathcal{F}_\sigma \neq \mathcal{F}_{\sigma\delta}$).

Solution: Since $\mathcal{F} \subseteq \mathcal{G}_\delta$, we have $\mathcal{F}_\sigma \subseteq \mathcal{G}_{\delta\sigma}$. Since $\mathbf{Q} \in \mathcal{F}_\sigma$ and $\mathcal{F}_\sigma \subseteq \mathcal{G}_{\delta\sigma}$, we have $\mathbf{Q} \in \mathcal{G}_{\delta\sigma}$. Since $\mathbf{Q}^c \notin \mathcal{F}_\sigma$ we have $\mathbf{Q} \notin \mathcal{G}_\delta$. Since $\mathbf{Q} \in \mathcal{G}_{\delta\sigma}$ but $\mathbf{Q} \notin \mathcal{G}_\delta$ it follows that $\mathcal{G}_\delta \neq \mathcal{G}_{\delta\sigma}$.