Chapter 6. Homotopy of Paths, and The Fundamental Group

Homotopy and The Fundamental Group

6.1 Definition: Let X be a topological space and let $\alpha, \beta : [0,1] \to X$ be paths from a to b in X. An (endpoint-fixing) **homotopy** from α to β in X is a continuous map $F : [0,1] \times [0,1] \to X$ such that $F(0,t) = \alpha(t)$ and $F(1,t) = \beta(t)$ for all $t \in [0,1]$, and $F(s,0) = a$ and $F(s,1) = b$ for all $\in [0,1]$. Note that, in this case, for each $s \in [0,1]$ the map $f_s : [0,1] \to X$ given by $f_s(t) = F(s,t)$ is a path from a to b in X. We say that α is **homotopic** to β (or that α is **homotopy-equivalent** to β) in X, and we write $\alpha \sim \beta$ in X, when there exists a homotopy from α to β in X.

6.2 Theorem: *Let X be a topological space, and let $a, b \in X$. Then homotopy-equivalence is an equivalence relation on the set of all paths from a to b in X.*

Proof: Let α, β, γ be paths from a to b in X. Note that $\alpha \sim \alpha$: indeed the map $F : [0, l1] \times [0,1] \to X$ given by $F(s,t) = \alpha(t)$ is a homotopy from α to α in X. Note that if $\alpha \sim \beta$ then $\beta \sim \alpha$: indeed if F is a homotopy from α to β in X then the map $G : [0,1] \times [0,1] \to X$ given by $G(s,t) = F(1-s,t)$ is a homotopy from β to α in X. Finally note that if $\alpha \sim \beta$ in X and $\beta \sim \gamma$ in X then $\alpha \sim \gamma$ in X: indeed, if F is a homotopy from α to β in X and G is a homotopy from β to γ in X, then the map $H : [0,1] \times [0,1] \to X$ given by

$$H(s,t) = \left\{ \begin{array}{cl} F(2s,t) & \text{, if } 0 \le s \le \frac{1}{2} \\ G(2s-1,t) & \text{, if } \frac{1}{2} \le s \le 1 \end{array} \right\}$$

is a homotopy from α to γ in X.

6.3 Notation: Given a topological space X and a point $a \in X$, we denote the set of homotopy-equivalence classes of loops at a in X by $\pi_1(X,a)$, that is

$$\begin{aligned} \pi_1(X,a) &= \big\{ [\alpha] \,\big|\, \alpha \text{ is a loop at } a \text{ in } X \big\} \text{ , where} \\ [\alpha] &= \big\{ \beta \,\big|\, \beta \text{ is a loop at } a \text{ in } X \text{ with } \beta \sim \alpha \text{ in } X \big\}. \end{aligned}$$

6.4 Notation: Let X be a topological space. When $a \in X$, we write κ_a to denote the **constant loop** at a given by

$$\kappa_a(t) = a$$

for all $t \in [0,1]$. When α is a path from a to b in X, we write α^{-1} to denote the **inverse path** from b to a in X given by

$$\alpha^{-1}(t) = \alpha(1-t)\,.$$

When α is a path from a to b in X and β is a path from b to c in X, we write $\alpha\beta$ to denote the **product path** from a to c in X given by

$$(\alpha\beta)(t) = \left\{ \begin{array}{cl} \alpha(2t) & \text{, if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{, if } \frac{1}{2} \le t \le 1 \end{array} \right\}$$

6.5 Theorem: *Let X be a topological space*

(1) When α and β are paths from a to b in X, if $\alpha \sim \beta$ in X then $\alpha^{-1} \sim \beta^{-1}$ in X.
(2) When α and β are paths from a to b in X, and γ and δ are paths from b to c in X, if $\alpha \sim \gamma$ in X and $\beta \sim \delta$ in X then $\alpha\gamma \sim \beta\delta$ in X.

(3) When α is a path from a to b in X, we have $\kappa_a\alpha \sim \alpha$ in X and $\alpha\,\kappa_b \sim \alpha$ in X.
(4) When α is a path from a to b in X, we have $\alpha\,\alpha^{-1} \sim \kappa_a$ in X and $\alpha^{-1}\alpha \sim \kappa_b$ in X.
(5) When α is a path from a to b in X and β is a path from b to c in X, and γ is a path from c to d in X, we have $(\alpha\beta)\gamma \sim \alpha(\beta\gamma)$ in X.

Proof: We prove some of the statements, and leave the proofs of the remaining statements as an exercise. For Parts 3 and 4, let α be a path from a to b in X. Verify that the map $F : [0,1] \times [0,1] \to X$ given by

$$F(s,t) = \left\{ \begin{array}{cl} a & \text{if } 0 \le t \le \frac{1-s}{2} \\ \alpha\left(\frac{2t-(1-s)}{1+s}\right) & \text{if } \frac{1-s}{2} \le t \le 1 \end{array} \right\}$$

is a (basepoint-fixing) homotopy from $\kappa_a\alpha$ to α in X, and the map $G : [0,1] \times [0,1] \to X$ given by

$$G(s,t) = \left\{ \begin{array}{cl} \alpha(2t) & \text{if } 0 \le t \le \frac{1-s}{2} \\ \alpha(1-s) & \text{if } \frac{1-s}{2} \le t \le \frac{1+s}{2} \\ \alpha(2-2t) & \text{if } \frac{1+s}{2} \le t \le 1 \end{array} \right\}$$

is a (basepoint-fixing) homotopy from $\alpha\alpha^{-1}$ to κ_a in X. For Part 5, let α be a path from a to b in X, let β be a path from b to c in X, and let γ be a path from c to d in X. Verify that the map $H : [0,1] \times [0,1] \to X$ given by

$$H(s,t) = \left\{ \begin{array}{cl} \alpha\left(\frac{4t}{1+s}\right) & \text{if } 0 \le t \le \frac{1+s}{4} \\ \beta\big(4t-(1+s)\big) & \text{if } \frac{1+s}{4} \le t \le \frac{2+s}{4} \\ \gamma\left(\frac{4t-(2+s)}{2-s}\right) & \text{if } \frac{2+s}{4} \le t \le 1 \end{array} \right\}$$

is a (basepoint-fixing) homotopy from $(\alpha\beta)\gamma$ to $\alpha(\beta\gamma)$ in X.

6.6 Definition: Let X be a topological space and let $a \in X$. By the above theorem, the set $\pi_1(X,a)$, of homotopy-equivalence classes of loops at a in X, is a group under the operation given by $[\alpha][\beta] = [\alpha\beta]$, with identity element $e = [\kappa_a]$ and with the inverse given by $[\alpha]^{-1} = [\alpha^{-1}]$. This group $\pi_1(X,a)$ is called the **fundamental group** of X at a.

6.7 Example: When X is a convex set in a normed linear space and $a \in X$, we have $\pi_1(X,a) = \{e\}$ where $e = [\kappa_a]$. Indeed, for a loop α at a in X, the map $F : [0,1]\times[0,1] \to X$ given by $F(s,t) = \alpha(t) + s(a - \alpha(t))$ is a homotopy from α to κ_a in X.

Lifting Paths and Homotopies to Polar Coordinates

6.8 Theorem: *Let X be a compact metric space and let $\mathcal{S}$ be an open cover of X. Then there exists a number $\lambda > 0$, called a* **Lebesgue number** *for the open cover $\mathcal{S}$, such that for every $a \in X$ the ball $B(a, \lambda)$ is contained in one of the sets in $\mathcal{S}$.*

Proof: For each $p \in X$, choose $U_x \in \mathcal{S}$ such that $p \in U_p$, then choose $r_p > 0$ so that $B(p, 2r_p) \subseteq U_p$. Note that $\{B(p, r_p) \,|\, p \in X\}$ is an open cover of X and so, since X is compact, we can choose points $p_1, p_2, \cdots, p_n \in X$ such that $X = \bigcup_{k=1}^n B(p, r_{p_k})$. Let $\lambda = \min\{r_{p_1}, r_{p_2}, \cdots, r_{p_n}\}$. We claim that for every $a \in X$, $B(a, \lambda)$ is contained in one of the sets in $\mathcal{S}$. Let $a \in X$. Since $X = \bigcup_{k=1}^n B(p_k, r_{p_k})$, we can choose an index k such that $a \in B(p_k, r_{p_k})$. Then for all $x \in B(a, \lambda)$ we have $d(x, p_k) \leq d(x, a) + d(a, p_k) < \lambda + r_{p_k} \leq 2\, r_{p_k}$ so that $B(x, a) \subseteq B(p_k, 2r_{p_k}) \subseteq U_{p_k}$, as required.

6.9 Theorem: *(Lifting Paths and Homotopies to Polar Coordinates)*

(1) (Path Lifting) Let $\alpha : [0,1] \subseteq \mathbb{R} \to \mathbb{C}^*$ be continuous with $\alpha(0) = p$. Let $r_0 = |p|$ and choose $\theta_0 \in \mathbb{R}$ such that $p = r_0 e^{i\,\theta_0}$. Then there exist unique continuous maps $r, \theta : [0,1] \subseteq \mathbb{R} \to \mathbb{R}$ with $r(t) > 0$ for all $t \in [0,1]$ and with $\theta(0) = \theta_0$ such that $\alpha(t) = r(t)\, e^{i\,\theta(t)}$ for all $t \in [0,1]$.

(2) (Homotopy Lifting) Let $F : [0,1] \times [0,1] \to \mathbb{C}^*$ be continuous with $F(0,0) = p$. Let $r_0 = |p|$ and choose $\theta_0 \in \mathbb{R}$ such that $p = r_0 e^{i\,\theta_0}$. Then there exist unique continuous maps $R, \Theta : [0,1] \subseteq \mathbb{R} \to \mathbb{R}$ with $R(s,t) > 0$ for all s, t and with $\Theta(0,0) = \theta_0$ such that $F(s,t) = R(s,t)\, e^{i\,\Theta(s,t)}$ for all $s, t \in [0,1]$.

Proof: We prove Part 1 and leave the proof of Part 2 as an exercise (Part 2 is proven more generally, for covering spaces, in Chapter 10). Write $\alpha(t) = \big(x(t), y(t)\big) = x(t) + i\, y(t)$ where $x, y : [0,1] \to \mathbb{R}$, and note that x and y are continuous. It is clear that, in order to have $\alpha(t) = r(t) e^{i\,\theta(t)}$, the map r must be given by $r(t) = \big|\alpha(t)\big| = \sqrt{x(t)^2 + y(t)^2}$ which is continuous. Let us explain how to construct the map θ. Let

$$\begin{aligned} U_1 &= \big\{x+iy \,\big|\, x>0\big\} \\ U_2 &= \big\{x+iy \,\big|\, y>0\big\} \\ U_3 &= \big\{x+iy \,\big|\, x<0\big\} \\ U_4 &= \big\{x+iy \,\big|\, y<0\big\} \end{aligned}$$

and, for $k = 1, 2, 3, 4$, define $\theta_k : U_k \to \mathbb{R}$ by

$$\begin{aligned} \theta_1(x,y) &= \sin^{-1} \tfrac{y}{\sqrt{x^2+y^2}} \\ \theta_2(x,y) &= \cos^{-1} \tfrac{x}{\sqrt{x^2+y^2}} \\ \theta_3(x,y) &= \pi + \sin^{-1} \tfrac{y}{\sqrt{x^2+y^2}} \\ \theta_4(x,y) &= 2\pi - \cos^{-1} \tfrac{x}{\sqrt{x^2+y^2}}. \end{aligned}$$

Note that when $\alpha(t) \in U_k$, we must have $\theta(t) = \theta\big(\alpha(t)\big) + 2\pi n_k$ for some $n_k = n_k(t) \in \mathbb{Z}$. In order for θ to be continuous, the map $n_k(t)$ must be continuous. Since $n_k(t)$ takes values in the discrete set $\mathbb{Z}$, it must be locally constant, meaning that it is constant in any interval $I \subseteq \alpha^{-1}(U_k) \subseteq [0,1]$.

Note that the sets $\alpha^{-1}(U_k)$ are open in $[0,1]$ and they cover $[0,1]$. Choose a Lebesgue number $\lambda > 0$ for this cover. Choose $m \in \mathbb{Z}^+$ large enough so that $\frac{1}{m} < \lambda$, and note that each subinterval $I_j = \left[\frac{j-1}{m}, \frac{j}{m}\right]$ is contained in the open ball $B_j = B_{[0,1]}\left(\frac{j}{m}, \lambda\right)$ in $[0,1]$, and B_j is contained in one of the four open sets $\alpha^{-1}(U_k)$, say $B_j \subseteq \alpha^{-1}(U_{k_j})$. It follows that, in order for the required continuous map $\theta(t)$ to exist, there must exist constants $n_j \in \mathbb{Z}$ such that for all $t \in B_j$ we have $\theta(t) = \theta_{k_j}\big(\alpha(t)\big) + 2\pi n_j$.

Note that the constants n_j can be determined, in a unique way, so that the resulting map $\theta(t)$ is continuous with $\theta(0) = \theta_0$: indeed the value of n_1 is uniquely determined so that $\theta_{k_1}(\alpha(0)) + 2\pi n_1 = \theta(0) = \theta_0$, then the value of n_2 is uniquely determined so that $\theta_{k_2}\big(\alpha\big(\frac{1}{m}\big)\big) + 2\pi n_2 = \theta\big(\frac{1}{m}\big) = \theta_{k_1}\big(\alpha\big(\frac{1}{m}\big)\big) + 2\pi n_1$, then the value of n_3 is uniquely determined so that $\theta_{k_3}\big(\alpha\big(\frac{2}{m}\big)\big) + 2\pi n_3 = \theta\big(\frac{2}{m}\big) = \theta_{k_2}\big(\alpha\big(\frac{2}{m}\big)\big) + 2\pi n_2$, and so on, so that all values n_j are uniquely determined. Finally note that, with these chosen values of n_j, the resulting map $\theta : [0,1] \to \mathbb{R}$ given by $\theta(t) = \theta_{k_j}(\alpha(t)) + 2\pi n_j$ when $t \in I_j$ (or when $t \in B_j$) is continuous, by the Glueing Lemma.

6.10 Definition: For α, r_0 and θ_0, as in Part 1 of the above theorem, the uniquely determined path $\widetilde{\alpha} : [0,1] \subseteq \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ given by $\widetilde{\alpha}(t) = \big(r(t), \theta(t)\big)$ is called the **lift** of α at (r_0, θ_0) to polar coordinates. In the same vein, for F, r_0 and θ_0, as in Part 2 of the above theorem, the unique map $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}^+ \times \mathbb{R}$ given by $F(s,t) = \big(R(s,t), \Theta(s,t)\big)$ is called the **lift** of F at (r_0, θ_0) to polar coordinates.

6.11 Note: If $\alpha : [0,1] \subseteq \mathbb{R} \to \mathbb{C}^*$ is differentiable (or $\mathcal{C}^1$, or piecewise $\mathcal{C}^1$, or $\mathcal{C}^\infty$) then so is its lift $\widetilde{\alpha}$ to polar coordinates at (r_0, θ_0). Indeed, when $\alpha(t) = x(t) + i\,y(t)$ is differentiable (or $\mathcal{C}^1$ etc), so are the maps x and y. The proof shows that for all t in B_j, which is open in $[0,1]$, we have $r(t) = \sqrt{x^2 + y^2}$ and $\theta(t) = \theta_{k_j}\big(x(t), y(t)\big)$, and these are elementary functions composed with the functions x and y, so they are also differentiable (or $\mathcal{C}^1$, etc).

The Winding Number of a Path in the Plane

6.12 Definition: Let $\alpha : [0,1] \to \mathbb{C} \setminus \{u\}$ be continuous with $\alpha(0) = a$. Let $p = a - u$, let $r_0 = |p|$ and choose $\theta_0 \in \mathbb{R}$ so that $p = r_0 e^{i\theta_0}$. Define $\beta : [0,1] \to \mathbb{C}^*$ by $\beta(t) = \alpha(t) - u$, and let $(r(t), \theta(t))$ be the (unique) lift of β at (r_0, θ_0) so that

$$\alpha(t) = u + r(t)e^{i\,\theta(t)}.$$

We define the **winding number** of α about u to be

$$\text{wind}(\alpha, u) = \tfrac{\theta(1)-\theta(0)}{2\pi}.$$

Note that this does not depend on the choice of θ_0 because, for any $k \in \mathbb{Z}$ we have $u + re^{i\theta(t)} = u + r(t)e^{i(\theta(t)+2\pi k)}$ for all t, so the unique lift of β at $(r_0, \theta_0 + 2\pi k)$ is equal to $(r(t), \phi(t))$ with $\phi(t) = \theta(t) + 2\pi k$ for all t, so that $\phi(1) - \phi(0) = \theta(1) - \theta(0)$. Also note that in the case that α is a loop at a in $\mathbb{C} \setminus \{u\}$, we have $\text{wind}(\alpha, u) \in \mathbb{Z}$.

6.13 Theorem: *Let $a, b, c \in \mathbb{C} \setminus \{u\}$, let α be a path from a to b in $\mathbb{C} \setminus \{u\}$, and let β be a path from b to c in $\mathbb{C} \setminus \{u\}$. Then*
(1) $\text{wind}(\kappa_a, u) = 0$,
(2) $\text{wind}(\alpha^{-1}, u) = -\text{wind}(\alpha, u)$,
(3) $\text{wind}(\alpha\beta, a) = \text{wind}(\alpha, u) + \text{wind}(\beta, u)$.

Proof: The proof is left as an exercise.

6.14 Definition: When a piecewise continuous map $g : [0,1] \subseteq \mathbb{R} \to \mathbb{C}$ is given by $g(t) = \big(x(t), y(t)\big) = x(t) + i\,y(t)$ with $x, y : [0,1] \to \mathbb{R}$, we define the **integral** $\int_0^1 g$ to be

$$\int_0^1 g = \int_0^1 g(t)\,dt = \int_0^1 x(t)\,dt + i\int_0^1 y(t)\,dt.$$

When $\alpha : [0,1] \subseteq \mathbb{R} \to U \subseteq \mathbb{C}$ is piecewise $\mathcal{C}^1$ and $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is continuous, we define the **path integral** $\int_\alpha f$ to be

$$\int_\alpha f = \int_\alpha f(z)\,dz = \int_0^1 f(\alpha(t))\alpha'(t)\,dt.$$

6.15 Theorem: *Let $\alpha : [0,1] \subseteq \mathbb{R} \to \mathbb{C} \setminus \{u\}$ be piecewise $\mathcal{C}^1$ with $\alpha(0) = a$ and $\alpha(1) = b$. Let $p = a - u$, let $r_0 = |p|$ and choose $\theta_0 \in \mathbb{R}$ so that $p = r_0 e^{i\theta_0}$. Let $(r(t), \theta(t))$ be the lift of $\beta(t) = \alpha(t) - u$ at (r_0, θ_0), so that $\alpha(t) = u + r(t)e^{i\theta(t)}$ for all t. Then*

$$\int_\alpha \frac{dz}{z-u} = \ln\tfrac{r(1)}{r(0)} + i\,(\theta(1) - \theta(0)) = \ln\tfrac{|b-u|}{|a-u|} + 2\pi\,i\;\text{wind}(\alpha, u)\,.$$

In particular, in the case that $a = b$ we have $r(1) = r(0)$ and $\text{wind}(\alpha, u) \in \mathbb{Z}$ with

$$\text{wind}(\alpha, u) = \frac{1}{2\pi\,i}\int_\alpha \frac{dz}{z-u}.$$

Proof: The proof is a straightforward calculation: we have

$$\begin{aligned}\int_\alpha \frac{dz}{z-u} &= \int_{t=0}^1 \frac{\alpha'(t)\,dt}{\alpha(t)-u} = \int_{t=0}^1 \frac{r'(t)e^{i\theta(t)} + r(t)e^{i\theta(t)}\cdot i\theta'(t)}{r(t)e^{i\theta(t)}}\,dt \\ &= \int_{t=0}^1 \frac{r'(t)}{r(t)} + i\,\theta'(t)\;dt = \Big[\ln r(t) + i\,\theta(t)\Big]_0^1 = \ln\tfrac{r(1)}{r(0)} + i\,(\theta(1) - \theta(0)).\end{aligned}$$

The Fundamental Group of an Annulus

6.16 Theorem: *Let I be an interval in $\mathbb{R}$ with $a \in I$, and let $A = \left\{z \in \mathbb{C} \,\middle|\, \|z\| \in I\right\}$. Then*
(1) For loops α and β at a in A, $\alpha \sim \beta \iff \text{wind}(\alpha, 0) = \text{wind}(\beta, 0)$.
(2) $\pi_1(A, a) = \big\langle [\sigma] \big\rangle \cong \mathbb{Z}$ where $\sigma : [0,1] \to A$ is the loop at a given by $\sigma(t) = a\, e^{i\,2\pi t}$.

Proof: To prove Part 1, let α and β be loops at a in A. Suppose first that $\text{wind}(\alpha, 0) = \text{wind}(\beta, 0) = n \in \mathbb{Z}$. Let $\widetilde{\alpha}(t) = (r(t), \theta(t))$ and $\widetilde{\beta}(t) = (\rho(t), \phi(t))$ be the lifts of α and β at $(r_0, \theta_0) = (a, 0)$. Since $\text{wind}(\alpha, 0) = \text{wind}(\beta, 0) = n$, we have $\theta(1) = \phi(1) = 2\pi n$. Note that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are paths from $(a, 0)$ to $(a, 2\pi n)$ in the convex set $I \times \mathbb{R}$, so they are homotopic in $I \times \mathbb{R}$, with a homotopy given by $F(s,t) = \big(r(t) + s(\rho(t) - r(t)\,,\; \theta(t) + s(\phi(t) - \theta(t))\big)$. It follows that α and β are homotopic in A with a homotopy $G : [0,1] \times [0,1] \to A$ given by

$$G(s,t) = \big(r(t) + s(\rho(t) - r(t)\big) e^{\,i\,(\theta(t) + s(\phi(t) - \theta(t))}.$$

Now suppose $\alpha \sim \beta$ in A, and let $G : [0,1] \times [0,1] \to A$ be a homotopy from α to β in A. Let $\widetilde{\alpha}(t) = (r(t), \theta(t))$, $\widetilde{\beta}(t) = (\rho(t), \phi(t))$ and $\widetilde{G}(s,t) = \big(R(s,t), \Theta(s,t)\big)$ be the lifts of α, β and G at $(r_0, \theta_0) = (a, 0)$, and note that the lift $\widetilde{\kappa}_0$ of the constant loop κ_a at $(a, \theta(0))$ is given by $\widetilde{\kappa}_0 = (a, 0)$ and the lift $\widetilde{\kappa}_1$ of κ_a at $(a, \theta(1))$ is given by $\widetilde{\kappa}_1(t) = (a, \theta(1))$. By the uniqueness of lifts of paths (with the same initial point), since $G(0,t) = \alpha(t)$ for all t with $\widetilde{G}(0,0) = (a,0) = \widetilde{\alpha}(0)$, lifting at $(a, 0)$ gives

$$\big(R(0,t), \Theta(0,t)\big) = \widetilde{G}(0,t) = \widetilde{\alpha}(t) = (r(t), \theta(t)) \;\text{ for all } t \in [0,1]$$

so that $R(0,t) = r(t)$ and $\Theta(0,t) = \theta(t)$ for all t. In particular, we have $R(0,1) = a$ and $\Theta(0,1) = \theta(1)$. Since $G(s,1) = a = \kappa_a(s)$ for all s with $\widetilde{G}(0,1) = (a, \theta(1))$, lifting at κ_a at $(a, \theta(1))$ gives

$$\big(R(s,1), \Theta(s,1)\big) = \widetilde{G}(s,1) = \widetilde{\kappa}_1(s) = (a, \theta(1)) \;\text{ for all } s \in [0,1]$$

so that $R(s,1) = a$ and $\Theta(s,1) = \theta(1)$ for all s, hence, in particular, $\Theta(1,1) = \theta(1)$. Similarly, since $G(s,0) = \kappa_a(t)$ with $\widetilde{G}(0,0) = (a,0)$, lifting κ_a at $(a,0)$ gives $R(1,0) = a$ and $\Theta(1,0) = 0$, then since $G(1,t) = \beta(t)$ with $\widetilde{G}(1,0) = (a,0) = \widetilde{\beta}(0)$, lifting β at $(a,0)$ gives $\Theta(1,1) = \phi(1)$. Thus $\theta(1) = \Theta(1,1) = \phi(1)$ so that

$$\text{wind}(\alpha, 0) = \tfrac{\theta(1)}{2\pi} = \tfrac{\phi(1)}{2\pi} = \text{wind}(\beta(, 0).$$

For Part 2, let $\sigma^0 = \kappa_a$, let $\sigma^1 = \sigma$ and for $n \in \mathbb{Z}^+$ let $\sigma^{n+1} = \sigma^n \sigma$ and $\sigma^{-n} = (\sigma^n)^{-1}$. For all $n \in \mathbb{Z}$, we have $[\sigma^n] = [\sigma]^n$ and, by Theorem 6.13, we have $\text{wind}(\sigma^n, u) = n$. By Part 1, when α is any loop at a in A with $\text{wind}(\alpha, 0) = n$, we have $\alpha \sim \sigma^n$ so that $[\alpha] = [\sigma]^n$ in $\pi_1(A, a)$. This proves that $\pi_1(A, a) = \big\langle [\sigma] \big\rangle$ (the cyclic group generated by $[\sigma]$). Finally, we note that $\big\langle [\sigma] \big\rangle \cong \mathbb{Z}$ because when $n \in \mathbb{Z}^+$ we have $\text{wind}(\sigma^n, 0) = n \neq 0 = \text{wind}(\kappa_a, 0)$ so that by Part 1, $\sigma^n \not\sim \kappa_a$, and hence $[\sigma]^n = [\sigma^n] \neq [\kappa_a]$

6.17 Remark: For students who have seen Cauchy's Theorem for Paths, from complex analysis, here is an alternate proof that when $\alpha \sim \beta$ we have $\text{wind}(\alpha, 0) = \text{wind}(\beta, 0)$. Suppose that $\alpha \sim \beta$ and let $\text{wind}(\alpha, 0) = n$ and $\text{wind}(\beta, 0) = m$. For $k \in \mathbb{Z}$, let ω_k be the loop at a in A given by $\omega_k(t) = a e^{i\,2\pi k t}$ and note that ω_k is $\mathcal{C}^1$ with $\text{wind}(\omega_k, 0) = k$. By the first part of the proof of Part 1, since $\text{wind}(\alpha, 0) = n = \text{wind}(\omega_n, 0)$, we have $\alpha \sim \omega_n$ and since $\text{wind}(\beta, 0) = m = \text{wind}(\omega_m, 0)$ we have $\beta \sim \omega_m$. Thus we have $\omega_n \sim \alpha \sim \beta \sim \omega_m$. Since the function $f : \mathbb{C}^* \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ is holomorphic in $\mathbb{C}^*$, and since $\omega_n \sim \omega_m$ in $A \subseteq \mathbb{C}^*$, it follows from Cauchy's Theorem for paths (which, we remark, holds for piecewise $\mathcal{C}^1$ paths, not for continuous paths), we have $n = \frac{1}{2\pi\, i} \int_{\omega_n} f = \frac{1}{2\pi\, i} \int_{\omega_m} f = m$.

Basic Properties of the Fundamental Group

6.18 Note: When X is path-connected and P is the path-component of X which contains the point $a \in X$, we have $\pi_1(X,a) = \pi_1(P,a)$. Indeed, every loop α at a in X also lies in P, and when α and β are homotopic loops at a in X, every homotopy F from α to β in X takes values in P so that it is also a homotopy from α to β in P.

6.19 Note: When γ is a path from a to b in X, the map $\phi_\gamma : \pi_1(X,a) \to \pi_1(X,b)$ given by $\phi_\gamma([\alpha]) = [\gamma^{-1}\alpha\gamma]$ is a well-defined group isomorphism: It is well-defined because for loops α, β at a in X, if $\alpha \sim \beta$ then $\gamma^{-1}\alpha\,\gamma \sim \gamma^{-1}\beta\gamma$ in X. It is a group homomorphism because for loops α, β at a in X, we have $\gamma^{-1}\alpha\beta\,\gamma \sim \gamma^{-1}\alpha\gamma\,\gamma^{-1}\beta\gamma$ in X. It is bijective because it has an inverse ${\phi_\gamma}^{-1} = \phi_{\gamma^{-1}} : \pi_1(X,b) \to \pi_1(X,a)$ which is given by $\phi_{\gamma^{-1}}([\beta]) = \gamma\beta\gamma^{-1}$.

6.20 Notation: When X is path-connected and $a \in X$, it is fairly common to write $\pi_1(X,a)$ simply as $\pi_1(X)$.

6.21 Definition: A **based topological space** (or a **pointed topological space**) is a pair (X,a) where X is a topological space and $a \in X$ (the point a is called the **base point**). A (continuous) **map of based spaces** $f : (X,a) \to (Y,b)$ is a continuous map $f : X \to Y$ with $f(a) = b$. A **homeomorphism** from $(X,a) \to (Y,b)$ is a continuous map $f : (X,a) \to (Y,b)$ with a continuous inverse map $f^{-1} : (Y,b) \to (X,a)$. We say that (X,a) is **homeomorphic** to (Y,b), and write $(X,a) \cong (Y,b)$, when there exists a homeomorphism $f : (X,a) \to (Y,b)$.

6.22 Definition: Given a map $f : (X,a) \to (Y,b)$ of based spaces, we define the **induced group homomorphism** $f_* : \pi_1(X,a) \to \pi_1(Y,b)$ by $f_*([\alpha]) = [f \circ \alpha]$. Note that f_* is well-defined because, for loops α and β at a in X, if F is a homotopy from α to β in X then $G = f \circ F$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$ in Y. Also note that f_* is a group homomorphism because, for loops α and β at a in X, we have $f \circ (\alpha\beta) = (f \circ \alpha)(f \circ \beta)$.

6.23 Note: Note that $\mathrm{id}_* = \mathrm{id}$, meaning that when $\mathrm{id} : (X,a) \to (X,a)$ is the identity map (given by $\mathrm{id}(x) = x$ for all $x \in X$, the induced map $\mathrm{id}_* : \pi_1(X,a) \to \pi_1(X,a)$ is also the identity map. Also note that when $f : (X,a) \to (Y,b)$ and $g : (Y,b) \to (Z,c)$ we have $(g \circ f)_* = g_* \circ f_*$ because $(g \circ f) \circ \alpha = g \circ (f \circ \alpha)$ for all loops α at a in X.

6.24 Remark: The above note can be summarized by saying the we have a (covariant) **functor** F, from the category of based topological spaces to the category of groups, given by $F(X,a) = \pi_1(X,a)$ with $F(f) = f_*$ when $f : (X,a) \to (Y,b)$.

6.25 Theorem: *When $(X,a) \cong (Y,b)$ we have $\pi_1(X,a) \cong \pi_1(Y,b)$.*

Proof: This is immediate from the above note: indeed when $f : (X,a) \to (Y,b)$ is a homeomorphism with inverse $g = f^{-1} : (Y,b) \to (X,a)$, we have $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$, and hence $g_* \circ f_* = (g \circ f)_* = \mathrm{id}_* = \mathrm{id}$ and $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_* = \mathrm{id}$, so that f_* is invertible with inverse $g_* = (f_*)^{-1}$.

6.26 Theorem: $\pi_1\big(X \times Y, (a,b)\big) \cong \pi_1(X,a) \times \pi_1(Y,b)$.

Proof: Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. Since every loop γ at (a,b) in $X \times Y$ is of the form $\gamma(t) = \big(\alpha(t), \beta(t)\big)$ where $\alpha = p \circ \gamma$ (which is a loop at a in X) and $\beta = q \circ \gamma$ (which is a loop at b in Y), the map $\phi : \pi_1\big(X \times Y, (a,b)\big) \to \pi_1(X,a) \times \pi_1(Y,b)$ given by $\phi([\gamma]) = \big([p \circ \gamma], [q \circ \gamma]\big) = \big(p_*([\gamma]), q_*([\gamma])\big)$ is a surjective group homorphism, and ϕ is injective because if F is a homotopy from α to κ_a in X and G is a homotopy from β to κ_b in Y, then (F,G) is a homotopy from (α,β) to $\kappa_{(a,b)}$ in $X \times Y$.

6.27 Example: Using the above theorem, we have $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$.