

Chapter 4. Countability and Separation Axioms

The Countability Axioms

4.1 Definition: Let X be a topological space. For a subset $A \subseteq X$, we say that A is **dense** in X when $\overline{A} = X$. Equivalently, A is dense in X when every nonempty open set in X contains a point in A .

4.2 Definition: Let X be a topological space. We say that X is **first-countable** when for every $a \in X$ there exists a (finite or) countable set \mathcal{B}_a of open sets such that for every set U in X with $a \in U$ there is a set $B \in \mathcal{B}_a$ such that $a \in B \subseteq U$. We say that X is **second-countable** when X has a (finite or) countable basis for its topology. We say that X is **Lindelöf** when every open cover of X contains a (finite or) countable sub-cover. We say that X is **separable** when X has a (finite or) countable dense subset.

4.3 Theorem: Let X be a metric space. Then X is first-countable and the following are equivalent:

- (1) X is second-countable.
- (2) X is Lindelöf.
- (3) X is separable

Proof: The proof is left as an exercise (this is often proven in a real analysis course).

4.4 Theorem: Every second-countable topological space is Lindelöf and separable.

Proof: Let X be a second-countable topological space. Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a (finite or) countable basis for X . We claim that X is Lindelöf. Let \mathcal{S} be any open cover of X . Let $K = \{k \in \mathbb{Z}^+ \mid \exists U \in \mathcal{S} \ B_k \subseteq U\}$. For each $k \in K$, choose $U_k \in \mathcal{S}$ such that $B_k \subseteq U_k$. Then the (finite or) countable set $\{U_k \mid k \in K\}$ is a subcover of \mathcal{S} because for every $a \in X$ we can choose $U \in \mathcal{S}$ with $a \in U$, then we can choose $B_k \in \mathcal{B}$ with $a \in B_k \subseteq U$ and then we have $k \in K$ and $a \in B_k \subseteq U_k$.

We claim that X is separable. Let $K = \{k \in \mathbb{Z}^+ \mid B_k \neq \emptyset\}$. For each $k \in K$, choose $a_k \in B_k$. Then the (finite or) countable set $A = \{a_k \mid k \in K\}$ is dense in X : indeed given any nonempty open set U in X , we can choose $x \in U$, then we can choose $B_k \in \mathcal{B}$ with $x \in B_k$, and then we have $a_k \in B_k \cap A \subseteq U \cap A$.

4.5 Theorem: Every subspace of a first-countable set is first-countable. Every subspace of a second-countable space is second countable.

Solution: We prove the first statement and leave the proof of the second statement as an exercise. Let Y be a first-countable space and let $X \subseteq Y$ be a subspace. Let $a \in X$. Since Y is first-countable and $a \in Y$, we can choose a (finite or) countable set \mathcal{C} of open sets in Y such that for every open set V in Y with $a \in V$, there exists $C \in \mathcal{C}$ such that $a \in C \subseteq V$. Let $\mathcal{B} = \{C \cap X \mid C \in \mathcal{C}\}$. Then \mathcal{B} is (finite or) countable and, given any open set U in X with $a \in U$ we can choose an open set V in Y such that $U = V \cap X$, then we can choose $C \in \mathcal{C}$ such that $a \in C \subseteq V$, and then for $B = C \cap X$ we have $B \in \mathcal{B}$ with $a \in B = C \cap X \subseteq V \cap X = U$.

4.6 Theorem: *The product of two first-countable spaces is first-countable. The product of two second-countable spaces is second-countable. The product of two separable spaces is separable*

Proof: We prove the first statement. Let X and Y be first-countable spaces. Let $a \in X$ and $b \in Y$ so that $(a, b) \in X \times Y$. Choose a (finite or) countable set \mathcal{B} of open sets in X so that for every open set U in X with $a \in U$ there is a set $B \in \mathcal{B}$ such that $a \in B \subseteq U$, and choose a (finite or) countable set \mathcal{C} of open sets in Y so that for every open set V in Y with $b \in V$ there is a set $C \in \mathcal{C}$ such that $b \in C \subseteq V$. Let $\mathcal{P} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$, and note that \mathcal{P} is (finite or) countable. Let W be an open set in $X \times Y$ with $(a, b) \in W$. Since the sets of the form $U \times V$ with U open in X and V open in Y form a basis for the topology on $X \times Y$, we can choose U open in X and V open in Y such that $(a, b) \in U \times V \subseteq W$. Since $a \in U$ we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq U$, and since $b \in V$ we can choose $C \in \mathcal{C}$ such that $b \in C \subseteq V$. Then we have $B \times C \in \mathcal{P}$ and $(a, b) \in B \times C \subseteq U \times V \subseteq W$.

4.7 Exercise: Generalize the above theorem to include countable products, using the product topology.

4.8 Exercise: Let \mathbb{R}_ℓ be the set \mathbb{R} using the lower limit topology. Show that \mathbb{R}_ℓ is first-countable and Lindelöf and separable, but not second countable.

4.9 Exercise: Let $I = [0, 1]$ and let I_o^2 be the set I^2 using the order topology for the dictionary order on I^2 . Show that I_o^2 is first countable and compact, hence Lindelöf, but not separable. Also, show that the subspace $A = I \times (0, 1)$ is not Lindelöf.

4.10 Exercise: The **Moore plane** Γ is the closed upper half plane $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ using the topology with basis consisting of the balls $B((a, b), r)$ with $0 < r < b$ and the sets $B((a, r), r) \cup \{(a, 0)\}$. Show that Γ is first countable and separable, but not Lindelöf. Also, show that the subspace $\mathbb{R} \times \{0\}$ is not separable.

4.11 Exercise: Let \mathbb{R}_{cf} be the set \mathbb{R} using the co-finite topology (in which the closed proper subsets of \mathbb{R} are the finite sets), let \mathbb{R}_{cc} be the set \mathbb{R} using the co-countable topology (in which the closed proper subsets of \mathbb{R} are the finite or countable sets), and let \mathbb{R}_d be the set \mathbb{R} using the discrete topology (in which all subsets are both open and closed). For each of the spaces \mathbb{R}_{cf} , \mathbb{R}_{cc} and \mathbb{R}_d , determine whether the space is first countable, whether it is Lindelöf, and whether it is separable.

4.12 Exercise: Let \mathbb{R}_ℓ be the set \mathbb{R} using the lower limit topology. Recall, from Exercise 4.7, that \mathbb{R}_ℓ is Lindelöf. Show that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf.

The Separation Axioms

4.13 Definition: Let X be a topological space. We say that X is $T1$ when for all $a, b \in X$ with $a \neq b$, there exists an open set V in X with $a \notin V$ and $b \in V$. We say that X is $T2$, or that X is **Hausdorff**, when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$. We say that X is $T3$, or that X is **regular**, when X is $T1$ and for every point a in X and every closed set B in X with $a \notin B$, there exist disjoint open sets U and V in X with $a \in U$ and $B \subseteq V$. We say that X is $T4$, or that X is **normal**, when X is $T1$ and for every pair of disjoint closed sets A and B in X there exist disjoint open sets U and V in X with $A \subseteq U$ and $B \subseteq V$.

4.14 Theorem: Let X be a topological space. Then X is $T1$ if and only if the 1-point subsets of X are closed in X .

Proof: If X is $T1$ and we let $a \in X$, then for each $b \in X$ we can choose an open set V_b in X such that $a \notin V_b$ and $b \in V_b$, and then we have $\{a\}^c = \bigcup_{b \in X} V_b$, which is open. Conversely, if the 1-point subsets of X are closed in X , then given $a, b \in X$ with $a \neq b$ we can let V be the open set $V = \{a\}^c$, and then $a \notin V$ and $b \in V$.

4.15 Theorem: Let X be a $T1$ topological space. Then X is regular if and only if for every $a \in X$ and for every open set W in X with $a \in W$, there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$.

Proof: Suppose that X is regular. Let W be open in X with $a \in W$. Then W^c is closed with $a \notin W^c$. Since X is regular, we can choose disjoint open sets U and V in X with $a \in U$ and $W^c \subseteq V$. Since $W^c \subseteq V$ we have $V^c \subseteq W$. Since $U \cap V = \emptyset$ we have $U \subseteq V^c$, which is closed, and hence $a \in U \subseteq \overline{U} \subseteq V^c \subseteq W$.

Suppose, conversely, that for every $a \in X$ and every open set W in X with $a \in W$ there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$. Let $a \in X$ and let B be a closed set in X with $a \notin B$. Then $a \in B^c$, which is open, so we can choose an open set U in X with $a \in U \subseteq \overline{U} \subseteq B^c$. Let $V = \overline{U}^c$, which is open. Since $U \subseteq \overline{U} = V^c$ we have $U \cap V = \emptyset$. Since $V^c = \overline{U} \subseteq B^c$ we have $B \subseteq V$.

4.16 Definition: Let X be a topological space. We say that X is **metrizable** when there exists a metric on X for which the topology on X is the metric topology.

4.17 Theorem: Every metrizable space is normal, every normal space is regular, every regular space is Hausdorff, and every Hausdorff space is $T1$.

Proof: The proof is left as an exercise.

4.18 Theorem: Every subspace of a $T1$ space is $T1$, every subspace of a Hausdorff space is Hausdorff, and every subspace of a regular space is regular.

Proof: Let X be a subspace of Y . If the 1-point subsets of Y are closed in Y , then given $a \in X$, since $\{a\}$ is closed in Y and $\{a\} = \{a\} \cap X$, it follows that $\{a\}$ is closed in X .

Suppose Y is Hausdorff. Let $a, b \in X$. Choose disjoint open sets U and V in Y with $a \in U$ and $b \in V$. Then $U \cap X$ and $V \cap X$ are disjoint open sets in X with $a \in U \cap X$ and $b \in V \cap X$.

Now suppose that Y is regular. As shown above, X is $T1$. Let $a \in X$ and let B be a closed set in X with $a \notin B$. Then we have $B = \text{Cl}_X(B) = \overline{B} \cap X$ where $\overline{B} = \text{Cl}_Y(B)$. Since $a \in X$ and $a \notin B = \overline{B} \cap X$, it follows that $a \notin \overline{B}$. Since Y is regular, we can find open sets U and V in Y with $a \in U$ and $\overline{B} \subseteq V$. Then the sets $U \cap X$ and $V \cap X$ are open in X with $a \in U \cap X$ and $B = \overline{B} \cap X \subseteq V \cap X$.

4.19 Theorem: When using the product or the box topology, the product of an indexed set of T_1 spaces is T_1 , the product of an indexed set of Hausdorff spaces is Hausdorff, and the product of an indexed set of regular spaces is regular.

Proof: Let X_k be a topological space for each $k \in K$. Suppose first that each X_k is T_1 . Let $a \in \prod_{k \in K} X_k$. For each $k \in K$, since X_k is T_1 , the set $\{a_k\}$ is closed in X_k so that $\overline{\{a_k\}} = \{a_k\}$. Note that $\{a\} = \prod_{k \in K} \{a_k\}$. By Theorem 2.16 we have $\overline{\{a\}} = \overline{\prod_{k \in K} \{a_k\}} = \prod_{k \in K} \overline{\{a_k\}} = \prod_{k \in K} \{a_k\} = \{a\}$ and hence $\{a\}$ is closed in $\prod_{k \in K} X_k$.

Now suppose that each X_k is Hausdorff. Let $a, b \in \prod_{k \in K} X_k$ with $a \neq b$. Since $a \neq b$ we can choose $\ell \in K$ so that $a_\ell \neq b_\ell$. Since X_ℓ is Hausdorff, we can choose disjoint open sets U_ℓ and V_ℓ in X_ℓ with $a_\ell \in U_\ell$ and $b_\ell \in V_\ell$. For each $k \in K$ with $k \neq \ell$, let $U_k = V_k = X_k$. Then the sets $U = \prod_{k \in K} U_k$ and $V = \prod_{k \in K} V_k$ are disjoint basic open sets in $\prod_{k \in K} X_k$ with $a \in U$ and $b \in V$.

Finally, suppose that each X_k is regular. Note that since each X_k is T_1 it follows, from the first paragraph, that $\prod_{k \in K} X_k$ is T_1 . Using Theorem 4.15, in order to show that $\prod_{k \in K} X_k$ is regular, it suffices to show that for every open set W in $\prod_{k \in K} X_k$ and for every $a \in W$, there exists an open set U in $\prod_{k \in K} X_k$ with $a \in U \subseteq \overline{U} \subseteq W$. Let W be an open set in $\prod_{k \in K} X_k$ and let $a \in W$. Choose a basic open set V in $\prod_{k \in K} X_k$ with $a \in V \subseteq W$, say $V = \prod_{k \in K} V_k$ where each V_k is open in X_k with $a_k \in V_k$. For each $k \in K$, since X_k is regular, by Theorem 4.15, we can choose an open set U_k in X_k with $a \in U_k \subseteq \overline{U_k} \subseteq V_k$. Let $U = \prod_{k \in K} U_k$ and note that $a \in U$. By Theorem 2.16, we have $\overline{U} = \overline{\prod_{k \in K} U_k} = \prod_{k \in K} \overline{U_k} \subseteq \prod_{k \in K} V_k = V \subseteq W$, so that $a \in U \subseteq \overline{U} \subseteq W$, as required.

4.20 Exercise: Show that \mathbb{R}_ℓ is normal but not metrizable.

4.21 Exercise: Show that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is regular but not normal.

4.22 Exercise: Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. The K -topology on \mathbb{R} is the topology generated by the sets of the form (a, b) with $a < b$ together with the sets of the form $(a, b) \setminus K$ with $a < b$. We write \mathbb{R}_K for the set \mathbb{R} equipped with the K -topology. Show that \mathbb{R}_K is Hausdorff but not regular.

4.23 Exercise: Let X be an infinite set using the co-finite topology. Show that X is T_1 but not Hausdorff.

4.24 Remark: A subspace of a normal space is not necessarily normal. As an example, without proof, when K is uncountable, the space $[0, 1]^K$ is compact, using the product topology (by Tichanoff's Theorem), and Hausdorff (by Theorem 4.19), so it is normal (by Theorem 4.24, below), but it can be shown (see Exercise 9 of Chapter 32 on page 206 of Munkres' book) that the subspace $(0, 1)^K$ is not normal. As another example, without proof, the space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal, by Exercise 4.21, but it can be shown (see Theorem 33.2 on page 211 and Theorem 34.3 on page 218 of Munkres' book) that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is homeomorphic to a subspace of $[0, 1]^K$ for some (necessarily uncountable) set K .

4.25 Theorem: *Every compact Hausdorff space is normal.*

Proof: Let X be a compact Hausdorff space. First, we claim that X is regular. Let $a \in A$ and let B be a closed set in X with $a \notin B$. For each $b \in B$, since X is Hausdorff we can choose disjoint open sets U_b and V_b in X with $a \in U_b$ and $b \in V_b$. Note that for $\mathcal{S} = \{V_b \mid b \in B\}$ we have $B \subseteq \bigcup \mathcal{S}$. Since B is a closed subspace of the Hausdorff space X , it is compact, so we can choose $b_1, b_2, \dots, b_m \in B$ such that $B \subseteq \bigcup_{k=1}^m V_{b_k}$. Let $U = \bigcap_{k=1}^m U_{b_k}$ and $V = \bigcup_{k=1}^m V_{b_k}$. Then U and V are disjoint open sets in X with $a \in U$ and $B \subseteq V$. This shows that X is regular, as claimed.

Now let us show that X is normal. Let A and B be disjoint closed sets in X . For each $a \in A$, since X is regular we can choose disjoint open sets U_a and V_a in X with $a \in U_a$ and $B \subseteq V_a$. Note that for $\mathcal{R} = \{U_a \mid a \in A\}$, we have $A \subseteq \bigcup \mathcal{R}$. Since A is a closed subspace of the Hausdorff space X , it is compact, so we can choose $a_1, a_2, \dots, a_n \in A$ such that $A \subseteq \bigcup_{k=1}^n U_{a_k}$. Let $U = \bigcup_{k=1}^n U_{a_k}$ and let $V = \bigcap_{k=1}^n V_{a_k}$. Then U and V are disjoint open sets in X with $A \subseteq U$ and $B \subseteq V$. Thus X is normal, as required.

4.26 Theorem: *Every regular space with a (finite or) countable basis is normal.*

Proof: Let X be a regular space with a countable basis \mathcal{B} . First note that given $a \in X$ and given an open set W in X with $a \in W$, by Theorem 4.15 we can choose an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$, then we can choose a basic open set $C \in \mathcal{B}$ with $a \in C \subseteq U$, and then we have $\overline{C} \subseteq \overline{U}$ so that $a \in C \subseteq \overline{C} \subseteq W$.

Let A and B be disjoint closed sets in X . For each $a \in A$, choose a basic open set $C_a \in \mathcal{B}$ with $a \in C_a \subseteq \overline{C_a} \subseteq B^c$ and note that $\overline{C_a} \cap B = \emptyset$. Note that $\mathcal{S} = \{C_a \mid a \in A\}$ is an open cover of A . Since $\mathcal{S} \subseteq \mathcal{B}$ and \mathcal{B} is (finite or) countable, it follows that \mathcal{S} is (finite or) countable, so we can choose a_1, a_2, a_3, \dots in A so that $\mathcal{S} = \{C_{a_1}, C_{a_2}, \dots\}$, and note that for each $a \in A$ we have $C_a = C_{a_k}$ for some k . Similarly, for each $b \in B$, we choose $D_b \in \mathcal{B}$ with $b \in D_b$ such that $\overline{D_b} \cap A = \emptyset$, and say $\mathcal{T} = \{D_b \mid b \in B\} = \{D_{b_1}, D_{b_2}, \dots\}$. Let $U = \bigcup_{n \geq 1} U_n$ where $U_n = C_{a_n} \setminus \bigcup_{k=1}^n \overline{D_{b_k}}$, and let $V = \bigcup_{n \geq 1} V_n$ where $V_n = D_{b_n} \setminus \bigcup_{k=1}^n \overline{C_{a_k}}$. Note that U and V are open sets in X . Note that $A \subseteq U$ because for each $a \in A$, a lies in one of the sets C_{a_ℓ} and a lies in none of the sets $\overline{D_{b_k}}$. Similarly, we have $B \subseteq V$. Finally, note that $U \cap V = \emptyset$ because if we had $x \in U \cap V$ then we would have $x \in U_\ell$ for some ℓ and $x \in V_m$ for some m , but if say $\ell \leq m$ then $x \in V_m \implies x \notin \bigcup_{k=1}^m \overline{C_{a_k}} \implies x \notin \overline{C_{a_\ell}} \implies x \notin U_{a_\ell}$.

4.27 Exercise: Show that every ordered set with a minimum element (or with a maximum element) is normal.

Urysohn's Lemma

4.28 Theorem: (Urysohn's Lemma) Let X be a normal topological space. For any disjoint closed sets $A, B \subseteq X$ there exists a continuous map $f : X \rightarrow [0, 1]$ with $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Proof: Let $A, B \subseteq X$ be closed. Say $[0, 1] \cap \mathbb{Q} = \{a_0, a_1, a_2, a_3, \dots\}$ where the terms a_k are distinct with $a_0 = 0$ and $a_1 = 1$. Choose disjoint open sets $U_0, V_0 \subseteq X$ with $A \subseteq U_0$ and $B \subseteq V_0$. Note that

$$U_0 \cap V_0 = \emptyset \implies U_0 \subseteq V_0^c \implies \overline{U_0} \subseteq V_0^c \implies \overline{U_0} \subseteq B^c.$$

Let $U_1 = B^c$ so that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1 = B^c$. Let $n \geq 2$ and suppose, inductively, that we have defined open sets $U_{a_0}, U_{a_1}, \dots, U_{a_{n-1}}$ such that when $a_k < a_\ell$ we have $\overline{U_{a_k}} \subseteq U_{a_\ell}$. Define U_{a_n} as follows. Rearrange the terms in the set $\{a_0, a_1, \dots, a_n\}$ in increasing order and say $a_k < a_n < a_\ell$ are consecutive. Since $\overline{U_{a_k}} \subseteq U_{a_\ell}$, we have $\overline{U_{a_k}} \cap U_{a_\ell}^c = \emptyset$, so we can choose disjoint open sets $U_{a_n}, V_{a_n} \subseteq X$ with $\overline{U_{a_k}} \subseteq U_{a_n}$ and $U_{a_\ell}^c \subseteq V_{a_n}$, and then

$$U_{a_n} \cap V_{a_n} = \emptyset \implies U_{a_n} \subseteq V_{a_n}^c \implies \overline{U_{a_n}} \subseteq V_{a_n}^c \subseteq U_{a_\ell}.$$

Recursively, we have defined U_{a_n} for all $n \geq 0$, so we have defined U_r for all $r \in [0, 1] \cap \mathbb{Q}$. For $r \in \mathbb{Q}$ with $r < 0$ we define $U_r = \emptyset$, and for $r \in \mathbb{Q}$ with $r > 1$ we define $U_r = X$, and then we have defined U_r for all $r \in \mathbb{Q}$ so that whenever $r < s$ we have $\overline{U_r} \subseteq U_s$.

Define $f : X \rightarrow [0, 1]$ by

$$f(x) = \inf \{r \in \mathbb{Q} \mid x \in U_r\}$$

Note that f does take values in $[0, 1]$: indeed for all $x \in X$, we have $f(x) \geq 0$ because $r < 0 \implies U_r = \emptyset \implies x \notin U_r$, and we have $f(x) \leq 1$ because $r > 1 \implies U_r = X \implies x \in U_r$. Also note that when $x \in A$ we have $x \in U_0$ so that $f(x) = 0$ and when $x \in B$ and $r \leq 1$ we have $U_r \subseteq U_1 = B^c$ so that $x \notin U_r$, and so $f(x) = 1$.

It remains to show that f is continuous. We shall show that the inverse image of every open interval is open. Let $c, d \in \mathbb{R}$ with $c < d$. Let $a \in f^{-1}(c, d)$ so we have $c < f(a) < d$. Choose $r, s \in \mathbb{Q}$ with $c < r < f(a) < s < d$. We claim that $a \in U_s \setminus \overline{U_r} \subseteq f^{-1}(c, d)$. First we make two observations: for $x \in X$ and $p \in \mathbb{Q}$,

- (1) if $x \in \overline{U_p}$ then $x \in U_r$ for all $r > p$ and so $f(x) \leq p$, and
- (2) if $x \notin U_p$ then $x \notin U_r$ for any $r \leq p$ and so $f(x) \geq p$.

Since $r < f(a)$ it follows from the first observation that $a \notin \overline{U_r}$, and since $f(a) < s$ it follows from the second observation that $a \in U_s$, and this shows that $a \in U_s \setminus \overline{U_r}$. On the other hand, when $x \in U_s \setminus \overline{U_r}$, since $x \in U_s$ it follows from the first observation that $f(x) \leq s$, and since $x \notin \overline{U_r}$ it follows from the second observation that $f(x) \geq r$, and so we have $f(x) \in [r, s] \subseteq (c, d)$. Thus we have $a \in U_s \setminus \overline{U_r} \subseteq f^{-1}(c, d)$, as claimed. Since $U_s \setminus \overline{U_r}$ is open, we can choose a basic open set V with $a \in V \subseteq U_s \setminus \overline{U_r} \subseteq f^{-1}(c, d)$. Since for every $a \in f^{-1}(c, d)$ there is a basic open set V with $a \in V \subseteq f^{-1}(c, d)$, it follows that $f^{-1}(c, d)$ is open, so that f is continuous, as required.

The Tietze Extension Theorem

4.29 Theorem: (*The Tietze Extension Theorem*) Let X be a normal topological space, let $A \subseteq X$ be closed, and let $a, b \in \mathbb{R}$ with $a < b$.

- (1) Every continuous map $f: A \rightarrow [a, b]$ can be extended to a continuous map $g: X \rightarrow [a, b]$.
- (2) Every continuous map $f: A \rightarrow (a, b)$ can be extended to a continuous map $g: X \rightarrow (a, b)$.

Proof: Note that since $[a, b]$ is homeomorphic to the interval $[-1, 1]$, we may replace $[a, b]$ by $[-1, 1]$. Suppose that $f: A \rightarrow [-1, 1]$ is continuous.

We begin with an observation. If $h: A \rightarrow [-r, r]$ is continuous, then $h^{-1}([-r, -\frac{r}{3}])$ and $h^{-1}([\frac{r}{3}, r])$ are disjoint closed sets in X , so by scaling and translating the map given by Urysohn's Lemma, we can construct a map $g: X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$ with $g(x) = -\frac{r}{3}$ for all $x \in h^{-1}([-r, -\frac{r}{3}])$ and $g(x) = \frac{r}{3}$ for all $x \in h^{-1}([\frac{r}{3}, r])$. We then have $|g(x)| \leq \frac{r}{3}$ for all $x \in X$, and we have $|h(x) - g(x)| \leq \frac{2r}{3}$ for all $x \in A$.

Since $f: A \rightarrow [-1, 1]$ is continuous, by the above observation we can construct a continuous map $g_1: X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that $|f(x) - g_1(x)| \leq \frac{2}{3}$ for all $x \in A$. Since $(f - g_1): A \rightarrow [-\frac{2}{3}, \frac{2}{3}]$ is continuous, we can apply the above observation again to construct a continuous map $g_2: X \rightarrow [-\frac{2}{9}, \frac{2}{9}]$ such that $|f(x) - g_1(x) - g_2(x)| \leq \frac{4}{9}$ for all $x \in A$. Repeating this procedure, we construct maps $g_k: X \rightarrow [-\frac{2^{k-1}}{3^k}, \frac{2^{k-1}}{3^k}]$ such that $|f(x) - \sum_{k=1}^n g_k(x)| \leq \frac{2^n}{3^n}$ for all $x \in A$. Since $|g_k(x)| \leq \frac{2^{k-1}}{3^k}$ for all $x \in X$, the series $\sum_{k=1}^{\infty} g_k$ converges uniformly on X by the Weierstrass M-Test. Define $g(x) = \sum_{k=1}^{\infty} g_k(x)$ for all $x \in X$. Note that g is continuous by uniform convergence, note that for all $x \in X$ we have $|g(x)| \leq \sum_{k=1}^{\infty} |g_k(x)| \leq \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 1$ so that $g: X \rightarrow [-1, 1]$, and note that for all $x \in A$, since $|f(x) - \sum_{k=1}^n g_k(x)| \leq \frac{2^n}{3^n}$ we have $f(x) = \sum_{k=1}^{\infty} g_k(x) = g(x)$, and so g extends f . This completes the proof of Part 1.

To prove Part 2, suppose that $f: A \rightarrow (a, b)$ is continuous. Note that f is also continuous as a map $f: A \rightarrow [a, b]$ so, by Part 1, we can extend f to a continuous map $h: X \rightarrow [a, b]$. Let $B = h^{-1}(a) \cup h^{-1}(b)$ and note that B is closed in X and B is disjoint from A . By Urysohn's Lemma, we can construct a continuous map $k: X \rightarrow [0, 1]$ with $k(x) = 0$ for all $x \in B$ and $k(x) = 1$ for all $x \in A$. Then $g = kh: X \rightarrow (a, b)$ is continuous on X with $g(x) = h(x) = f(x)$ for all $x \in A$.

Urysohn's Metrization Theorem

4.30 Exercise: The following theorem shows that \mathbb{R}^{ω} is metrizable, using the product topology. Show that \mathbb{R}^{ω} is not metrizable in the box topology, and show that when K is uncountable, \mathbb{R}^K is not metrizable in the product topology.

4.31 Theorem: Let X_k be metrizable for each $k \in \mathbb{Z}^+$. Then $\prod_{k=1}^{\infty} X_k$ is metrizable, in the product topology

Proof: We outline a proof, and leave the details as an exercise. Verify that when d is a metric on a set X , the map $\bar{d} : X \times X \rightarrow \mathbb{R}$ given by $\bar{d}(x, y) = \min \{d(x, y), 1\}$, is another metric on X which induces the same topology as d . This metric \bar{d} is called the **bounded metric** on X corresponding to d .

For each $k \in \mathbb{Z}^+$, let d_k be a metric which induces the topology on X_k , and let \bar{d}_k be the corresponding bounded metric. For $x, y \in \prod_{k=1}^{\infty} X_k$, let

$$d(x, y) = \sup \left\{ \frac{\bar{d}_k(x_k, y_k)}{k} \mid k \in \mathbb{Z}^+ \right\}.$$

Verify that d is a metric on $\prod_{k=1}^{\infty} X_k$ which induces the product topology.

4.32 Theorem: (Urysohn's Metrization Theorem) Every regular space with a countable basis for its topology is metrizable.

Proof: Let X be a regular topological space with a countable basis $\mathcal{B} = \{B_1, B_2, B_3, \dots\}$ for its topology. Note that X is normal, by Theorem 4.25. We shall construct a sequence $(f_n)_{n \geq 1}$ of continuous functions $f_n : X \rightarrow [0, 1]$ with the property that for every $a \in X$ and for every open set U in X with $a \in U$, there exists $n \in \mathbb{Z}^+$ such that $f_n(a) = 1$ and $f_n(x) = 0$ for all $x \in U^c = X \setminus U$. Let $\mathcal{S} = \{(k, \ell) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \bar{B}_k \subseteq B_\ell\}$, and note that \mathcal{S} is countable. For each pair $(k, \ell) \in \mathcal{S}$, since $\bar{B}_k \subseteq B_\ell$ we have $\bar{B}_k \cap B_\ell^c = \emptyset$ so, that by Urysohn's Lemma, we can choose a continuous map $g_{k, \ell} : X \rightarrow [0, 1]$ with $g_{k, \ell}(x) = 1$ for all $x \in B_k$ and $g_{k, \ell}(x) = 0$ for all $x \in B_\ell^c$. Since the set $\{g_{k, \ell} \mid (k, \ell) \in \mathcal{S}\}$ is countable, we can list the elements as $\{g_{k, \ell} \mid (k, \ell) \in \mathcal{S}\} = \{f_1, f_2, f_3, \dots\}$ where $(f_n)_{n \geq 1}$ is a sequence of functions $f_n : X \rightarrow [0, 1]$. Given an open set U in X and given $a \in U$, we can choose a basic open set B_ℓ with $a \in B_\ell \subseteq U$ then, since X is regular, we can choose an open set V with $a \in V \subseteq \bar{V} \subseteq B_\ell$, then we can choose another basic open set B_k with $a \in B_k \subseteq V$, and then we have $a \in B_k \subseteq \bar{B}_k \subseteq \bar{V} \subseteq B_\ell \subseteq U$, and finally we can choose $n \in \mathbb{Z}^+$ so that $f_n = g_{k, \ell}$ to get $f(x) = 1$ for all $x \in B_k$ (so that $f(a) = 1$) and $f(x) = 0$ for all $x \in B_\ell^c$ (so that $f(x) = 0$ for all $x \in U^c$).

Let $f : X \rightarrow [0, 1]^\omega \subseteq \mathbb{R}^\omega$ be the function given by $f(x) = (f_1(x), f_2(x), \dots)$. Note that f is continuous because \mathbb{R}^ω is using the product topology and each f_k is continuous. Note that f is injective because given $a, b \in X$ with $a \neq b$, we can choose disjoint open sets U and V in X with $a \in U$ and $b \in V$, then we can choose $n \in \mathbb{Z}^+$ such that $f_n(a) = 1$ and $f_n(x) = 0$ for all $x \in U^c$ so that $f_n(b) = 0$ and hence, since $f_n(a) \neq f_n(b)$ we have $f(a) \neq f(b)$. By restricting the codomain, we obtain a bijective continuous map $f : X \rightarrow f(X) \subseteq [0, 1]^\omega \subseteq \mathbb{R}^\omega$.

To complete the proof it suffices to show that $g = f^{-1} : f(X) \rightarrow X$ is continuous, when $f(X) \subseteq \mathbb{R}^\omega$ uses the subspace topology. It suffices to show that $g^{-1}(U)$ is open in $f(X)$ for every open set U in X . Let U be open in X and note that $g^{-1}(U) = f(U)$. We must show that $f(U)$ is open in $f(X)$. Let $b \in f(U)$. Let $a \in U$ with $f(a) = b$. Choose $n \in \mathbb{Z}^+$ such that $f_n(a) = 1$ and $f_n(x) = 0$ for all $x \notin U$. Let $p_n : \mathbb{R}^\omega \rightarrow \mathbb{R}$ be the n^{th} projection map. Let $V = p_n^{-1}((0, \infty)) = \{y \in \mathbb{R}^\omega \mid y_n > 0\}$ and note that V is open in \mathbb{R}^ω . Let $W = V \cap f(X)$, which is open in $f(X)$. Note that $b \in W$ because $b_n = p_n(b) = p_n(f(a)) = f_n(a) = 1 > 0$. We claim that $W \subseteq f(U)$. Let $y \in W = V \cap f(X)$. Since $y \in f(X)$ we can choose $x \in X$ such that $y = f(x)$. Since $f(x) = y \in V$ we have $f_n(x) = y_n > 0$. Since $f_n(x) > 0$ we have $x \in U$ (since $f_n(x) = 0$ for all $x \notin U$). Thus $y = f(x) \in f(U)$ so that $W \subseteq f(U)$, as claimed. For each $b \in f(U)$ we have found an open set W in $f(X)$ with $b \in W \subseteq f(U)$, which shows that $f(X)$ is open, as required.