# Chapter 3. Connected, Path-Connected, and Compact Spaces

## Connectedness and Connected Components

**3.1 Definition:** Let $X$ be a topological space. We say that two subsets $A, B \subseteq X$ **separate** $X$ when $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. We say that $X$ is **connected** when there do not exist two open sets in $X$ which separate $X$. Equivalently (as you can verify) $X$ is connected when $\emptyset$ and $X$ are the only two subsets of $X$ which are both open and closed in $X$. We say that $X$ is **disconnected** when it is not connected.

**3.2 Exercise:** Prove that the connected subspaces of $\mathbb{R}$ are the intervals (including $\emptyset$, 1-point sets, and $\mathbb{R}$), and the nonempty connected subspaces of $\mathbb{Q}$ are the 1-point sets.

**3.3 Theorem:** *The image of a connected space under a continuous map is connected. In particular, if two spaces are homeomorphic and one is connected, then so is the other.*

Proof: Let $f : X \to Y$ be a continuous map between topological spaces. Note that, by Theorem 1.35, $f : X \to f(X)$ is also continuous. Suppose that $f(X)$ is disconnected. Choose disjoint nonempty open sets $A$ and $B$ in $f(X)$ with $A \cup B = f(X)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty open sets in $X$ with $f^{-1}(A) \cup f^{-1}(B) = X$, so $X$ is disconnected.

**3.4 Lemma:** *Let $X$ be a subspace of $Y$. Suppose that $Y$ is disconnected and that $A$ and $B$ are open sets in $Y$ which separate $Y$. If $X$ is connected, then either $X \subseteq A$ or $X \subseteq B$.*

Proof: Suppose that $X$ is connected. Note that $A \cap X$ and $B \cap X$ are disjoint open sets in $X$. If both of the sets $A \cap X$ and $B \cap X$ were nonempty, then they would be open sets in $X$ which separate $X$. Since $X$ is connected, this is not possible, so either $A \cap X = \emptyset$ or $B \cap X = \emptyset$. If $A \cap X = \emptyset$ then we have $X = X \cap Y = X \cap (A \cup B) = (X \cap A) \cup (X \cap B) = X \cap B$ so that $X \subseteq B$. Similarly, if $B \cap X = \emptyset$ then $X \subseteq A$.

**3.5 Theorem:** *Let $X = \bigcup_{k \in K} A_k$. If each $A_k$ is a connected subspace of $X$ and $\bigcap_{k \in K} A_k \neq \emptyset$ then $X$ is connected.*

Proof: Suppose that each $A_k$ is connected with $p \in \bigcap_{k \in K} A_k$. Suppose, for a contradiction, that $X = \bigcup_{k \in K} A_k$ is disconnected. Choose open sets $U$ and $V$ in $X$ which separate $X$. Note that $p$ lies either in $U$ or in $V$ (but not both), say $p \in U$ (and $p \notin V$). Let $k \in K$ be arbitrary. Since $A_k$ is connected, by the above lemma, either $A_k \subseteq U$, or $A_k \subseteq V$. Since $p \in A_k$ and $p \notin V$, we do not have $A_k \subseteq V$ so we must have $A_k \subseteq U$. Since $k \in K$ was arbitrary, we have $A_k \subseteq U$ for every $k \in K$, and hence $X = \bigcup_{k \in K} A_k \subseteq U$. But this contradicts the fact that $U$ and $V$ separate $X$, giving the desired contradiction.

**3.6 Lemma:** *Let $X$ be a subspace of $Y$ and let $A$ and $B$ be subsets of $X$ which separate $X$. Then $A$ and $B$ are open in $X$ (so that $X$ is disconnected) if and only if $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$ (where $\overline{A}$ and $\overline{B}$ are the closures in $Y$).*

Proof: Suppose that $A$ and $B$ are open in $X$ Note that $A = B^c = X \setminus B$ is closed in $X$ so we have $A = \mathrm{Cl}_X(A) = \overline{A} \cap X = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cup (\overline{A} \cap B)$ and hence $\overline{A} \cap B = \emptyset$. Similarly $\overline{B} \cap A = \emptyset$.

Suppose, conversely, that there exist disjoint nonempty sets $A, B \subseteq X$ with $A \cup B = X$ such that $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. Since $\overline{A} \cap B = \emptyset$, we have $\overline{A} \cap X = A$ so that $\mathrm{Cl}_X(A) = \overline{A} \cap X = A$ and hence $A$ is closed. Similarly $B$ is closed. Since $X$ is the disjoint union of $A$ and $B$, it follows that $A$ and $B$ are both open in $X$ hence $X$ is not connected.

**3.7 Theorem:** *Let $X$ be a topological space and let $A \subseteq X$. Suppose that $A \subseteq B \subseteq \overline{A}$. If $A$ is connected, as a subspace of $X$, then so is $B$.*

Proof: Suppose that $A$ is connected, and suppose, for a contradiction, that $B$ is disconnected. Let $C$ and $D$ be open sets in $B$ which separate $B$. By Lemma 3.5, we have $\overline{C} \cap D = \emptyset$ and $\overline{D} \cap C = \emptyset$. Since $A$ is connected, by Lemma 3.3, either $A \subseteq C$ or $A \subseteq D$. Say $A \subseteq C$. Then we have $B \subseteq \overline{A} \subseteq \overline{C}$. Since $D \subseteq B \subseteq \overline{C}$ and $\overline{C} \cap D = \emptyset$, we have $D = \emptyset$, which contradicts the fact that $C$ and $D$ separate $B$.

**3.8 Theorem:** *The cartesian product of two connected spaces is connected.*

Proof: Let $X$ and $Y$ be connected topological spaces. If $X = \emptyset$ or $Y = \emptyset$ then $X \times Y = \emptyset$, which is connected. Suppose that $X \neq \emptyset$ and $Y \neq \emptyset$. Choose $a \in X$ and $b \in Y$. Note that $X \times \{b\}$ is connected, since it is homeomorphic to $X$. Likewise, for each $x \in X$, the subspace $\{x\} \times Y$ is connected since it is homeomorhic to $Y$. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected with $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$ it follows, from Theorem 3.5, that $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected. Since $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected for every $x \in X$ and $(a, b) \in \bigcap_{x \in X} (X \times \{b\}) \cup (\{x\} \times Y)$ it follows, again from Theorem 3.5, that $X \times Y = \bigcup_{x \in X} (X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

**3.9 Theorem:** *The cartesian product of an arbitrary set of connected spaces is connected using the product topology.*

Proof: Let $X_k$ be a connected topological space for each $k \in K$. If $X_k = \emptyset$ for some $k \in K$ then $\prod_{k \in K} X_k = \emptyset$. Suppose that $X_k \neq \emptyset$ for all $k \in K$. For each $k \in K$ choose $a_k \in X_k$ and let $a$ be the element in $\prod_{k \in K} X_k$ given by $a(k) = a_k$ for all $k \in K$. Let $\mathcal{F}$ be the set of all finite subsets of $K$. For each $J \in \mathcal{F}$, let $Y_J = \left\{ y \in \prod_{k \in K} X_k \,\middle|\, y_k = a_k \text{ for all } k \notin J \right\}$, using the subspace topology. We claim that $Y_J \cong \prod_{j \in J} X_j$. Define $f : Y_J \to \prod_{j \in J} X_j$ by $f(y)(j) = y_j$. This map is continuous because given $U_j$ open in $X_j$ for each $j \in J$, so that $\prod_{j \in J} U_j$ is a basic open set in $\prod_{j \in J} X_j$, and letting $U_k = X_k$ for all $k \in K \setminus J$, we have $f^{-1}\big(\prod_{j \in J} U_j\big) = \big\{ y \in Y_J \big| y_j \in U_j \text{ for all } j \in J \big\} = \big\{ y \in Y_J \big| \, y_k \in U_k \text{ for all } k \in K \big\} = \big(\prod_{k \in K} U_k\big) \cap Y_J$, which is a basic open set in $Y_J$ (using the subspace topology). The inverse of $f$ is the map $g = f^{-1} : \prod_{j \in J} X_j \to Y_J$ by $g(x)(k) = \left\{ \begin{matrix} x_k \text{ if } k \in J \\ a_k \text{ if } k \notin J \end{matrix} \right\}$. This map is continuous because given $I \in \mathcal{F}$ and given open sets $U_k$ in $X_k$ with $U_k = X_k$ for all $k \notin I$, so that the set $\big(\prod_{k \in K} U_k\big) \cap Y_J$ is a basic open set in $Y_J$, we have $g^{-1}\Big(\big(\prod_{k \in K} U_k\big) \cap Y_J\Big) = \big\{ x \in \prod_{j \in J} X_j \big| x_k \in U_k \text{ for all } k \in J \text{ and } a_k \in U_k \text{ for all } k \notin J \big\} = \big\{ x \in \prod_{j \in J} X_j \big| x_k \in U_k \text{ for all } k \in J \cap I \big\} = \prod_{j \in J} V_j$ where $V_j = U_j$ for $j \in J \cap I$ and $V_j = X_j$ for $j \in J \setminus I$, and this is a basic open set in $\prod_{j \in J} X_j$. This we have $Y_J \cong \prod_{j \in J} X_j$, as claimed.

Since $J$ is finite, and each $X_j$ is connected, the space $\prod_{j \in J} X_j$ is connected by the previous theorem (and by induction), and hence $Y_J = g\big(\prod_{j \in J} X_j\big)$ is connected by Theorem. 3.3. Since $Y_J$ is connected for every $J \in \mathcal{F}$, and since $a \in Y_J$ for all $J \in \mathcal{F}$, it follows from Theorem 3.5 that $\bigcup_{J \in \mathcal{F}} Y_J$ is connected. Finally, we note that $\overline{\bigcup_{J \in \mathcal{F}} Y_J} = \prod_{k \in K} X_k$: indeed, given $I \in \mathcal{F}$ and given open sets $U_k \subseteq X_k$ with $U_k = X_k$ for all $k \notin I$ , so that $\prod_{k \in K} U_k$ is a basic open set in $\prod_{k \in K} X_k$, we have $\emptyset \neq \prod_{k \in K} U_k \cap Y_I \subseteq \prod_{k \in K} X_k \cap \bigcup_{J \in \mathcal{F}} Y_J$. Since $\bigcup_{J \in \mathcal{F}} Y_J$ is connected, and $\prod_{k \in K} X_k = \overline{\bigcup_{J \in \mathcal{F}} Y_J}$, it follows that $\prod_{k \in K} X_k$ is connected by Theorem 3.7.

**3.10 Example:** The result of the above theorem does not necessarily hold when $\prod_{k\in K} X_k$ uses the box topology. For example you can verify that in the space $\mathbb{R}^\omega = \prod_{k=1}^\infty \mathbb{R}$ using the box topology, the sets $U = \left\{x \in \mathbb{R}^\omega \middle| \, \|x\|_\infty < \infty\right\}$ and $V = \left\{x \in \mathbb{R}^\omega \middle| \, \|x\|_\infty = \infty\right\}$ are open sets which separate $\mathbb{R}^\omega$.

**3.11 Definition:** Let $X$ be a topological space. Define a relation $\sim$ on $X$ by setting $x \sim y$ if and only if there exists a connected subspace of $X$ which contains both $x$ and $y$. We note that $\sim$ is an equivalence relation on $X$: indeed, given $x, y, z \in X$, we have $x \sim x$ because $\{x\}$ is connected, and if $x \sim y$ then clearly $y \sim x$, and if $x \sim y$ and $y \sim z$ then we can choose connected spaces $A, B \subseteq X$ with $x, y \in A$ and $y, z \in B$ and then, by Theorem 3.5, since $y \in A \cap B$ it follows that $A \cup B$ is connected, and we have $x, z \in A \cup B$. The equivalence classes under this equivalence relation are called the **connected components** of $X$. Note that if $X$ is connected then the only connected component of $X$ is $X$ itself.

**3.12 Theorem:** *The connected components of a topological space $X$ are connected, and every non-empty connected subspace of $X$ is contained in exactly one of the connected components.*

Proof: We claim that every nonempty connected subspace of $X$ is contained in exactly one connected component. Let $A$ be a nonempty connected subspace of $X$. Let $a \in A$. Let $C$ be the equivalence class of $a$, that is $C = [a] = \{x \in X \,|\, x \sim a\}$ and note that $A \cap C \neq \emptyset$ since $a \in A \cap C$. Let $D$ be any equivalence class with $A \cap D \neq \emptyset$. Choose $b \in A \cap D$. Since $A$ is connected with $a \in A$ and $b \in A$ we have $a \sim b$ so that $C = [a] = [b] = D$. Thus $A$ intersects with exactly one connected component, namely $C$. Since the connected components (being equivalence classes) cover $X$, it follows that $A \subseteq C$.

We claim that each connected component of $X$ is connected. Let $C$ be a connected component and let $a \in C$ so that $C = [a] = \{x \in X \,|\, x \sim a\}$. For each $x \in C$, since $x \sim a$ we can choose a connected set $A_x$ in $X$ with $a, x \in A$. By the previous claim, since $A_x$ is connected with $a \in A_x \cap C$, it follows that $A_x \subseteq C$. Since $x \in A_x \subseteq C$ for all $x \in C$, we have $C = \bigcup_{x\in C} A_x$. This is connected by Theorem 3.5, since each $A_x$ is connected and $a \in \bigcap_{x\in C} A_x$.

**3.13 Note:** The connected components of a topological space are closed: indeed if $C$ is a connected component of $X$ then, by the above theorem, $C$ is a maximal connected set in $X$, and by Theorem 3.7, $\overline{C}$ is a connected connected set with $C \subseteq \overline{C}$, and hence $\overline{C} = C$.

**3.14 Example:** Since $\mathbb{R}$ is connected, it has only one connected component, namely $\mathbb{R}$. The one-point sets are the connected components of $\mathbb{Q}$.

## Path-Connectedness and Path-Components

**3.15 Definition:** Let $X$ be a topological space and let $a, b \in X$. A (continuous) **path** from $a$ to $b$ in $X$ is a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. A **loop** at $a$ in $X$ is a path from $a$ to $a$ in $X$. We say that $X$ is **path-connected** when for every $a, b \in X$ there exists a path from $a$ to $b$ in $X$.

**3.16 Theorem:** *The image of a path-connected space under a continuous map is path-connected. In particular, if $X \cong Y$ and $X$ is path-connected, then so is $Y$.*

Proof: Let $f : X \to Y$ be continuous and suppose $X$ is path connected. Let $c, d \in f(X)$. Choose $a, b \in X$ such that $f(a) = c$ and $f(b) = d$. Let $\alpha$ be a path in $X$ from $a$ to $b$. Then $\beta = f \circ \alpha$ is a path in $Y$ from $c$ to $d$.

**3.17 Theorem:** *Every path-connected topological space is connected.*

Proof: Let $X$ be a path-connected topological space. Suppose, for a contradiction, that $X$ is not connected. Choose nonempty disjoint open sets $U$ and $V$ in $X$ which separate $X$ (meaning that $X = U \cup V$). Choose $a \in U$ and $b \in V$. Let $\alpha : [0,1] \to X$ be a path from $a$ to $b$ in $X$. Then $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are nonempty disjoint open sets in $[0,1]$ which separate $[0,1]$. This is not possible since $[0,1]$ is connected.

**3.18 Example:** Every **convex set** in a normed linear space is path-connected, hence connected. Indeed if $X$ is a convex set then, given $a, b \in X$, the map $\alpha : [0,1] \to X$ given by $\alpha(t) = a + t(b-a)$ is a path from $a$ to $b$ in $X$ ($\alpha$ takes values in $X$ because $X$ is convex).

**3.19 Theorem:** *Let $X$ be a topological space. The relation $\sim$ on $X$, given by $a \sim b$ when there exists a path from $a$ to $b$ in $X$, is an equivalence relation on $X$, which we call* **path-equivalence**.

Proof: We have $a \sim a$ because the constant path $\kappa = \kappa_a : [0,1] \to X$, given by $\kappa(t) = a$ for all $t$, is a path from $a$ to $a$ in $X$. Note that if $a \sim b$ then $b \sim a$: indeed if $\alpha$ is a path from $a$ to $b$ in $X$ then the map $\beta = \alpha^{-1} : [0,1] \to X$ given by $\beta(t) = \alpha(1-t)$ is a path from $b$ to $a$ in $X$. Finally, note that if $a \sim b$ and $b \sim c$ then $a \sim c$: indeed if $\alpha$ is a path from $a$ to $b$ in $X$ and $\beta$ is a path from $b$ to $c$ in $X$ then the map $\gamma = \alpha\beta : [0,1] \to X$ given by $\gamma(t) = \alpha(2t)$ when $0 \le t \le \frac{1}{2}$, and by $\gamma(t) = \beta(2t-1)$ when $\frac{1}{2} \le t \le 1$, is a path from $a$ to $c$ in $X$ ($\gamma$ is continuous by Theorem 1.36).

**3.20 Definition:** Let $X$ be a topological space. The equivalence classes under the path-equivalence relation on $X$ are called the **path-components** of $X$.

**3.21 Theorem:** *The path components of a topological space $X$ are the maximal path-connected subspaces of $X$: indeed each path-component of $X$ is path-connected, and every nonempty path-connected subset of $X$ is contained in exactly one of the path-components.*

Solution: Let $P$ be a path-component of $X$, say $a \in P$ so that $P = [a] = \{x \in X \,|\, x \sim a\}$. Then $P$ is path-connected because if $b, c \in P$ then we have $b \sim a$ and $c \sim a$, and hence $b \sim c$. Now let $S$ be any path-connected subset of $X$. Suppose that $S$ intersects two path-components, say $P$ and $Q$, of $X$. Choose $p \in P \cap S$ and $q \in Q \cap S$. Since $p, q \in S$ and $S$ is path-connected, we have $p \sim q$. Since $p \sim q$ we have $P = [p] = [q] = Q$.

**3.22 Note:** In a topological space $X$, since each path-component is path-connected, hence connected, it is contained in one of the connected components of $X$. It follows that each connected component of $X$ is the (disjoint) union of the path-components which it contains.

**3.23 Exercise:** Let $A = \{(x, \sin\frac{1}{x}) \,|\, x > 0\}$. The closure $\overline{A}$ of $A$ in $\mathbb{R}^2$ is called the **topologist's sine curve**. Note that $\overline{A} = A \cup B$ where $B = \{(0, y) \,|\, y \in [-1, 1]\}$. Show that $\overline{A}$ is connected but not path-connected, and the sets $A$ and $B$ are the path-components.

## Compactness

**3.24 Definition:** Let $X$ be a topological space. For a set $\mathcal{S}$ of subsets of $X$, we say that $\mathcal{S}$ **covers** $X$ when $\bigcup \mathcal{S} = X$. An **open cover** of $X$ is a set $\mathcal{S}$ of open sets which covers $X$. When $\mathcal{S}$ is an open cover of $X$, a **subcover** of $\mathcal{S}$ is a subset $\mathcal{R} \subseteq \mathcal{S}$ with $\bigcup \mathcal{R} = X$. We say that $X$ is **compact** when every open cover of $X$ has a finite subcover. Equivalently, $X$ is compact when it has the property that if $K$ is a set, and $U_k$ is an open set in $X$ for each $k \in K$, and $\bigcup_{k \in K} U_k = X$, then there is a finite subset $L \subseteq K$ such that $\bigcup_{k \in L} U_k = X$.

**3.25 Theorem:** *(The Heine-Borel Theorem) A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof: We omit the proof. This theorem is proven in a real analysis course.

**3.26 Theorem:** *The image of a compact space under a continuous map is compact. In particular, if $X \cong Y$ and $X$ is compact, then so is $Y$.*

Proof: Let $f : X \to Y$ be continuous and suppose $X$ is compact. By restricting the codomain, the map $f : X \to f(X)$ is also continuous. Let $\mathcal{T}$ be an open cover of $f(X)$. Then the set $\mathcal{S} = \left\{ f^{-1}(V) \,\middle|\, V \in \mathcal{T} \right\}$ is an open cover of $X$. Since $X$ is compact, $\mathcal{S}$ has a finite subcover, so we can choose $V_1, V_2, \cdots, V_n \in \mathcal{T}$ such that $X = \bigcup_{k=1}^n f^{-1}(V_k)$. Then $f(X) = \bigcup_{k=1}^n V_k$ so that $\{V_1, V_2, \cdots, V_n\}$ is a finite subcover of $\mathcal{T}$.

**3.27 Theorem:** *Let $X$ be a subspace of $Y$. Then $X$ is compact if and only if for every set $\mathcal{T}$ of open sets in $Y$ with $X \subseteq \bigcup \mathcal{T}$ there exists a finite set $\mathcal{Q} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup \mathcal{Q}$.*

Proof: Suppose that $X$ is compact. Let $\mathcal{T}$ be a set of open sets in $Y$ such that $X \subseteq \bigcup \mathcal{T}$. Let $\mathcal{S} = \left\{ V \cap X \,\middle|\, V \in \mathcal{T} \right\}$. Then $\mathcal{S}$ is an open cover of $X$. Since $X$ is compact, we can choose $V_1, \cdots, V_n \in \mathcal{T}$ such that $X = \bigcup_{k=1}^n (V_k \cap X)$. Since $X = \bigcup_{k=1}^n (V_k \cap X) = \left( \bigcup_{k=1}^n V_k \right) \cap X$ we have $X \subseteq \bigcup_{k=1}^n V_k$.

Suppose, conversely, that for every set $\mathcal{T}$ of open sets in $Y$ with $X \subseteq \bigcup \mathcal{T}$ there is a finite subset $\mathcal{Q} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup \mathcal{Q}$. Let $\mathcal{S}$ be a set of open sets in $X$ with $X = \bigcup \mathcal{S}$. For each $U \in \mathcal{S}$, since $U$ is open in $X$ we can choose an open set $V_U$ in $Y$ such that $U = V_U \cap X$. Let $\mathcal{T} = \left\{ V_U \,\middle|\, U \in \mathcal{S} \right\}$. Choose a finite subset $\left\{ V_{U_1}, V_{U_2}, \cdots, V_{U_n} \right\}$ of $\mathcal{T}$ such that $X \subseteq \bigcup_{k=1}^n V_{U_k}$. Then we have $\bigcup_{k=1}^n U_k = \bigcup_{k=1}^n (V_{U_k} \cap X) = \left( \bigcup_{k=1}^n V_{U_k} \right) \cap X = X$ so that $\{U_1, U_2, \cdots, U_n\}$ is a finite subcover of $\mathcal{S}$.

**3.28 Theorem:** *Every closed subspace of a compact space is compact.*

Proof: Let $X$ be a closed subspace of the compact space $Y$. Let $\mathcal{T}$ be a set of open sets in $Y$ such that $X \subseteq \bigcup \mathcal{T}$. Then $\mathcal{T} \cup \{X^c\}$ is an open cover of $Y$, where $X^c = Y \setminus X$. Since $Y$ is compact, we can choose $V_1, V_2, \cdots, V_n \in \mathcal{T}$ such that $V_1 \cup V_2 \cup \cdots \cup V_n \cup X^c = Y$. It follows that $X \subseteq V_1 \cup V_2 \cup \cdots \cup V_n$. Thus $X$ is compact by Theorem 3.27.

**3.29 Theorem:** *Every compact subspace of a Hausdorff space is closed.*

Proof: Let $X$ be a compact subspace of the Hausdorff space $Y$. To show that $X$ is closed in $Y$, we show that for every $b \in X^c = Y \setminus X$ there exists an open set $V$ in $Y$ with $b \in V \subseteq X^c$. Let $b \in X^c$. For each $a \in X$, since $Y$ is Hausdorff we can choose disjoint open sets $U_a$ and $V_a$ in $Y$ with $a \in U_a$ and $b \in V_a$. Since $X \subseteq \bigcup_{a \in X} U_a$ and $X$ is compact, by Theorem 3.27 we can choose $a_1, a_2, \cdots, a_n \in X$ such that $X \subseteq \bigcup_{k=1}^n U_{a_k}$. Let $U = \bigcup_{k=1}^n U_{a_k}$ and $V = \bigcap_{k=1}^n V_{a_k}$. Note that $U$ and $V$ are open in $Y$ with $X \subseteq U$ and $b \in V$. Also note that $U$ and $V$ are disjoint because, for $y \in Y$, if $y \in U = \bigcup_{k=1}^n U_{a_k}$ then we can choose an index $\ell$ such that $y \in U_{a_\ell}$, and then $y \notin V_{a_\ell}$ and hence $y \notin \bigcap_{k=1}^n V_{a_k} = V$. Since $X \subseteq U$ and $b \in V$ and $U \cap V = \emptyset$, we have found an open set $V$ in $Y$ with $b \in V \subseteq X^c$, as required.

**3.30 Theorem:** *If $X$ is compact and $Y$ is Hausdorff and $f : X \to Y$ is continuous and bijective, then the inverse of $f$ is also continuous, so that $f$ is a homeomorphism.*

Proof: Let $X$ be compact, let $Y$ be Hausdorff, let $f : X \to Y$ be continuous and bijective, and let $g = f^{-1} : Y \to X$. Let $U$ be an open set in $X$. Then $U^c$ is closed in $X$, where $U^c = X \setminus U$. By Theorem 3.26, since $U^c$ is closed in $X$ and $X$ is compact, it follows that $U^c$ is compact. By Theorem 3.25, since $U^c$ is compact and $f$ is continuous, it follows that $f(U^c)$ is compact. Since $f$ is bijective, we have $f(U^c) = f(U)^c = Y \setminus f(U)$. By Theorem 3.27, since $f(U)^c$ is compact and $Y$ is Hausdorff, it follows that $f(U)^c$ is closed in $Y$, and hence $g^{-1}(U) = f(U)$ is open in $Y$. Thus $g$ is continuous, as required.

**3.31 Example:** Show that no two of the spaces $(0,1)$, $(0,1]$ and $[0,1]$ are homeomorphic.

Solution: Since $[0,1]$ is compact while $(0,1)$ and $(0,1]$ are not, we see that $[0,1]$ cannot be homeomorphic either to $(0,1)$ or to $(0,1]$. Also note that $(0,1] \setminus \{1\}$ is connected while $(0,1) \setminus \{p\}$ is not connected for any $p \in (0,1)$, and so it follows that $(0,1]$ cannot be homeomorphic to $(0,1)$. Indeed if $f : (0,1] \to (0,1)$ was a homeomorphism with $p = f(0)$ then the map $f : (0,1] \setminus \{0\} \to (0,1) \setminus \{p\}$ would also be a homeomorphism.

**3.32 Example:** Show that no two of the spaces $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{S}^1$ and $\mathbb{S}^2$ are homeomorphic.

Solution: Since $\mathbb{S}^1$ and $\mathbb{S}^2$ are compact while $\mathbb{R}^1$ and $\mathbb{R}^2$ are not, neither $\mathbb{S}^1$ nor $\mathbb{S}^2$ can be homeomorphic to either $\mathbb{R}^1$ or $\mathbb{R}^2$. Since $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected while $\mathbb{R} \setminus \{x\}$ is not connected for any $x \in \mathbb{R}$, it follows that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^1$. Since $\mathbb{S}^2 \setminus \{(0,0,1)\} \cong \mathbb{R}^2$, and $\mathbb{S}^1 \setminus \{x\} \cong \mathbb{R}^1$ for any $x \in \mathbb{S}^1$ (under the composite of a rotation with the stereographic projection), and since $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^1$, it follows that $\mathbb{S}^2$ is not homeomorphic to $\mathbb{S}^1$.

**3.33 Theorem:** *Let $X$ and $Y$ be topological spaces.*

*(1) If $X$ and $Y$ are connected then so is $X \times Y$.*
*(2) If $X$ and $Y$ are path-connected then so is $X \times Y$.*
*(3) If $X$ and $Y$ are compact then so is $X \times Y$.*

Proof: The proof is left as an exercise.

**3.34 Theorem:** *Let $\sim$ be an equivalence relation on a topological space $X$.*

*(1) If $X$ is connected then so is $X/\sim$.*
*(2) If $X$ is path-connected then so is $X/\sim$.*
*(3) If $X$ is compact then so is $X/\sim$.*

Proof: The proof is left as an exercise.

**3.35 Definition:** Let $X$ be an ordered set. For $A \subseteq X$ and $b \in X$, we say that $b$ is an **upper bound** for $A$ in $X$ when $b \geq x$ for every $x \in A$, and we say that $b$ is the **supremum** (or the **least upper bound**) of $A$ in $X$ when $b$ is an upper bound for $A$ in $X$ and $b \leq c$ for every upper bound $c$ of $A$ in $X$. Note that when $A$ has a supremum in $X$, the supremum is unique, and we denote it by $\sup X$. We say that $X$ has the **supremum property** (or the **least upper bound property) when every nonempty subset of $X$ which has an upper bound in $X$ also has a supremum in $X$.**

**3.36 Theorem:** *Let $X$ be an ordered set with the supremum property. Let $a, b \in X$ with $a < b$. Then the interval $[a,b]$ is compact.*

Proof: We leave the proof as an exercise.

**3.37 Theorem:** *Let $X$ be a topological space. Then $X$ is compact if and only if $X$ has the* **finite intersection property on closed sets***: for every set $T$ of closed sets in $X$, if every finite subset of $T$ has non-empty intersection, then $T$ has non-empty intersection.*

Proof: Suppose that $X$ is compact. Let $T$ be a set of closed sets in $X$. Suppose that $T$ has empty intersection, that is suppose $\bigcap_{A\in T} A = \emptyset$. Then $\bigcup_{A\in T} A^c = X$ so the set $S = \left\{A^c \middle| A \in T\right\}$ is an open cover for $X$. Since $X$ is compact, we can choose a finite subcover, say $\left\{{A_1}^c, \cdots, {A_n}^c\right\}$ of $S$ for $X$. Then we have $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$, showing that some finite subset of $T$ has empty intersection.

Suppose, conversely, that $X$ has the finite intersection property on closed sets. Let $S$ be an open cover of $X$. Let $T = \left\{U^c \,\middle|\, U \in S\right\}$. Since $\bigcup S = X$ we have $\bigcap T = \left(\bigcup S\right)^c = \emptyset$. Since $X$ has the finite intersection on closed sets, there exists a finite subset of $T$ with empty intersection. so we can choose $U_1, U_2, \cdots U_n \in S$ such that ${U_1}^c \cap \cdots \cap {U_n}^c = \emptyset$. It follows that $U_1 \cup \cdots \cup U_n = X$, so $S$ has a finite subcover.

**3.38 Theorem:** *(Tychanoff's Theorem) The product of any indexed set of compact spaces is compact, using the product topology.*

Proof: Let $X_k$ be compact for each $k \in K$. We shall prove that $\prod X_k$ has the finite intersection property on closed sets. Let $T$ be a set of closed sets in $\prod X_k$ such that every finite subset of $T$ has non-empty intersection. We need to show that $\bigcap T \neq \emptyset$. By Zorn's Lemma, we can choose a maximal set $S$ of subsets of $\prod X_k$ with $T \subseteq S$ such that every finite subset of $S$ has non-empty intersection (let $\mathcal{R}$ be the set of all such sets $S$ and note that for every chain $\mathcal{C}$ in $\mathcal{R}$ we have $\bigcup \mathcal{C} \in \mathcal{R}$). Note that the maximality of $S$ implies that $S$ is closed under finite intersection (since if $A_1, \cdots, A_n \in S$ then every intersection of a finite subset of $S \cup \{A_1 \cap \cdots \cap A_n\}$ is also an intersection of a finite subset of $S$).

We shall show that $\bigcap \left\{\overline{A} \,\middle|\, A \in S\right\} \neq \emptyset$, hence $\bigcap T \neq \emptyset$ since if $A \in T$ then $A = \overline{A} \in S$. Let $k \in K$. Note that finite subsets of $\left\{p_k(A) \,\middle|\, A \in S\right\}$ have non-empty intersection (because if $A_1, \cdots, A_n \in S$ then $p_k(A_1) \cap \cdots \cap p_k(A_n) = p_k(A_1 \cap \cdots \cap A_n) \neq \emptyset$), and hence finite subsets of $\left\{\overline{p_k(A)} \,\middle|\, A \in S\right\}$ also have nonempty intersection. Since $X_k$ is compact, so $X_k$ has the finite intersection property on closed sets, it follows that $\bigcap \left\{\overline{p_k(A)} \middle| A \in S\right\} \neq \emptyset$, so we can choose $a_k \in X_k$ such that $a_k \in \overline{p_k(A)}$ for every $A \in S$. We do this for each $k \in K$, that is for each $k \in K$ we choose $a_k \in X_k$ with $a_k \in \overline{p_k(A)}$ for every $A \in S$, then we let $a = (a_k)_{k\in K} \in \prod_{k\in K} X_k$.

We claim that $a \in \overline{A}$ for every $A \in S$. Let $k \in K$. Let $U_k$ be an open set in $X_k$ with $a_k \in U_k$. Then for every $A \in S$, we have $a_k \in \overline{p_k(A)} \cap U_k$ so that $\overline{p_k(A)} \cap U_k \neq \emptyset$ hence $p_k(A) \cap U_k \neq \emptyset$ (if we had $p_k(A) \cap U_k = \emptyset$ then $p_k(A) \subseteq {U_k}^c$ hence $\overline{p_k(A)} \subseteq {U_k}^c$ so that $\overline{p_k(A)} \cap U_k = \emptyset$). For each $A \in S$, since $p_k(A) \cap U_k \neq \emptyset$, we can choose $b \in A$ such that $p_k(b) \in U_k$, that is $b \in {p_k}^{-1}(U_k)$, and hence ${p_k}^{-1}(U_k) \cap A \neq \emptyset$. Since $S$ is closed under finite intersection and ${p_k}^{-1}(U_k) \cap A \neq \emptyset$ for every $A \in S$, the maximality of $S$ implies that ${p_k}^{-1}(U_k) \in S$. Let $V$ be any basic open set in $\prod X_k$ with $a \in V$, say $V = \prod U_k$ where each $U_k \subseteq X_k$ is open with $a_k \in U_k$, and with $U_k = X_k$ for all $k \in F$ where $F$ is a finite subset of $K$. Since ${p_k}^{-1}(U_k) \in S$ for every $k \in K$ and $S$ is closed under finite intersection, we have

$$V = \left\{(x_k)_{k\in K} \,\middle|\, x_k \in U_k \text{ for all } k \in F\right\} = \textstyle\bigcap_{k\in F} {p_k}^{-1}(U_k) \in S.$$

Since $V \in S$ and every finite subset of $S$ has non-empty intersection, we have $A \cap V \neq \emptyset$ for all $A \in S$. Given $A \in S$, since $A \cap V \neq \emptyset$ for every basic open set $V$ in $\prod X_k$ with $a \in V$, it follows that $a \in \overline{A}$. Thus $a \in \overline{A}$ for all $A \in S$, so $\bigcap \left\{\overline{A} \,\middle|\, A \in S\right\} \neq \emptyset$, as required.