Chapter 10. Covering Spaces

Covering Spaces

10.1 Definition: A **covering** consists of two topological spaces X and $\widetilde{X}$ and a continuous map $p:\widetilde{X}\to X$ such that every point in X is contained in an open set U in X which has the property that $p^{-1}(U)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is homeomorphic to U under the map p. The space X is called the **base space**, the space $\widetilde{X}$ is called the **covering space** and the map $p:\widetilde{X}\to X$ is called the **covering map**. An open set U in X with the above-stated property is called an **elementary** open set in X.

10.2 Example: For $n\in\mathbb{Z}^+$, the map $p:\mathbb{S}^1\to\mathbb{S}^1$ given by $p(z)=z^n$ is a covering map. The map $q:\mathbb{R}^1\to\mathbb{S}^1$ given by $q(t)=e^{i\,t}$ is a covering map.

10.3 Example: For $n\in\mathbb{Z}^+$, the map $p:\mathbb{C}^*\to\mathbb{C}^*$ given by $p(z)=z^n$. The map $q:\mathbb{C}\to\mathbb{C}^*$ given by $q(z)=e^z$ is a covering map. A closely related covering map is the polar coordinates map $g:\mathbb{R}^+\times\mathbb{R}\to\mathbb{C}^*$ given by $g(r,\theta)=re^{i\theta}$.

10.4 Example: When $p:\widetilde{X}\to X$ and $q:\widetilde{Y}\to Y$ are covering maps, the map $g:\widetilde{X}\times\widetilde{Y}\to X\times Y$ given by $g(x,y)=\big(p(x),q(y)\big)$ is a covering map.

10.5 Example: For $n,m\in\mathbb{Z}^+$, the map $p:\mathbb{T}^2\to\mathbb{T}^2$ given by $p(z,w)=(z^n,w^m)$ is a covering map. The map $q:\mathbb{R}^1\times\mathbb{S}^1\to\mathbb{T}^2$ given by $q(s,w)=\big(e^{i\,s},w^m\big)$ is a covering map. The map $g:\mathbb{R}^2\to\mathbb{T}^2$ given by $g(s,t)=\big(e^{is},e^{it}\big)$ is a covering map.

10.6 Example: In $\mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1$, let $A=\mathbb{S}^1\times\{1\}$ and $B=\{1\}\times\mathbb{S}^1$. Note that $A\cup B$ is homeomorphic to the wedge product of two circles. The three covering maps p, q and g in the previous example, restricted to the inverse image of $A\cup B$, give three covering maps of the wedge product of two circles.

Path Lifting and Homotopy Lifting

10.7 Definition: Let $p:\widetilde{X}\to X$ be a covering and let $f:Y\to X$ be a continuous map, where Y is a topological space. A **lift** of f is a continuous map $\widetilde{f}:Y\to\widetilde{X}$ such that $p\circ\widetilde{f}=f$.

10.8 Example: Let $p:\mathbb{R}^+\times\mathbb{R}\to\mathbb{C}^*$ be the polar coordinates map $p(r,\theta)=re^{i\theta}$. Given a path $\alpha:[0,1]\to\mathbb{C}^*$, a lift of α is a path $\tilde{\alpha}:[0,1]\to\mathbb{R}^+\times\mathbb{R}$, say given by $\tilde{\alpha}(t)=\big(r(t),\theta(t)\big)$, such that $\alpha(t)=r(t)e^{i\,\theta(t)}$ (giving a representation of α in polar coordinates).

10.9 Theorem: *Let $p : \widetilde{X} \to X$ be a covering.*
(1) (Path Lifting) Given a path α in X with $\alpha(0) = a$, and given $\tilde{a} \in p^{-1}(a)$, there exists a unique lift $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = \tilde{a}$.
(2) (Homotopy Lifting) Given a continuous map $F : [0,1] \times Y \to X$, and given a lift $\widetilde{f} : Y \to \widetilde{X}$ of the function $f : Y \to X$ given by $f(y) = F(0,y)$, there exists a unique lift $\widetilde{F}$ of F with $\widetilde{F}(0,y) = \widetilde{f}(y)$ for all y.

Proof: To prove Part 1, let $\alpha : [0,1] \to X$ with $\alpha(0) = a$, and let $\tilde{a} \in p^{-1}(a)$. Since X is covered by elementary sets, the sets $\alpha^{-1}(U)$ with U elementary form an open cover of $[0,1]$, which is compact. Choose a Lebesgue number $\lambda > 0$ for this cover, and choose $n \in \mathbb{Z}^+$ with $\frac{1}{n} < \lambda$. Then for each interval $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$, the image $\alpha(I_j)$ lies in an elementary set, say U_j, in X. Let $j \geq 1$ and suppose, inductively, that we have constructed a lift $\tilde{\alpha} : \left[0, \frac{j-1}{n}\right] \to \widetilde{X}$ of the restriction $\alpha : \left[0, \frac{j-1}{n}\right] \to X$, with $\tilde{\alpha}(0) = \tilde{a}$, and suppose this lift is unique. We wish to extend the constructed lift to the interval $\left[0, \frac{j}{n}\right]$: we have constructed $\tilde{\alpha}\left(\frac{j-1}{n}\right)$, and we need to extend $\tilde{\alpha}$ to the rest of the interval I_j. We know that $\alpha(I_j) \subseteq U_j$, so in order to obtain $p \circ \tilde{\alpha} = \alpha$, we must have $\tilde{\alpha}(I_j) \subseteq p^{-1}(U_j)$. Since U_j is elementary, $p^{-1}(U_j)$ is a disjoint union of open sets, each of which is homeomorphic to U_j under the (restriction of the) map p. Let $\widetilde{U}_j$ be the open set in $p^{-1}(U_j)$ which contains the point $\tilde{\alpha}\left(\frac{j-1}{n}\right)$, and let $p_j : \widetilde{U}_j \to U_j$ be the homeomorphism given by restricting p. In order for $\tilde{\alpha}$ to be continuous, the image $\tilde{\alpha}(I_j)$ must be connected, so since it contains the point $\tilde{\alpha}\left(\frac{j-1}{n}\right)$, it must lie entirely in the set $\widetilde{U}_j$ (otherwise the disjoint open sets $\widetilde{U}_j$ and $p^{-1}(U_j) \setminus \widetilde{U}_j$ would be nonempty, and they would separate $\tilde{\alpha}(I_j)$). Thus, in order to have $p \circ \tilde{\alpha} = \alpha$, the lift $\tilde{\alpha}$ must be given by $\tilde{\alpha}(t) = {p_j}^{-1}(\alpha(t))$ for all $t \in I_j$. Also note that by defining $\tilde{\alpha}$ according to this formula for all $t \in I_j$, we have uniquely extended the lift to the interval $\left[0, \frac{j}{n}\right]$ (the extension is continuous by the glueing lemma).

To prove Part 2, let $F : [0,1] \times Y \to X$ be continuous, define $f : Y \to X$ by $f(y) = F(0,y)$, and let $\widetilde{f} : Y \to \widetilde{X}$ be a lift of f. Let $u \in Y$. For each $s \in [0,1]$, the point $F(s,u)$ is contained in an open elementary set, say U_s, in X, and so (s,u) is contained in the open set $F^{-1}(U_s)$. Choose an open neighbourhood J_s of s in $[0,1]$ and an open neighbourhood V_s of u in Y so that $J_s \times V_s \subseteq F^{-1}(U_s)$. The sets J_s with $s \in [0,1]$ form an open cover of $[0,1]$, which is compact. Choose a Lebesgue number $\lambda > 0$ for this open cover, and choose $n \in \mathbb{Z}^+$ with $\frac{1}{n} < \lambda$. Then for each interval $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$, we have $I_j \subseteq J_s$ for some $s \in [0,1]$, say $I_j \subseteq J_{s_j}$, and hence $I_j \times V_{s_j} \subseteq F^{-1}(U_{s_j})$. Let $V = \bigcap_{j=1}^{n} V_{s_j}$ and simplify notation by writing U_{s_j} simply as U_j. Then for all indices j we have $I_j \times V \subseteq F^{-1}(U_j)$, hence we have $F(I_j \times V) \subseteq U_j$ with U_j elementary.

Let $W_0 = V$ and define $\widetilde{F}(0,y) = \widetilde{f}(y)$ for all $y \in W_0$. Let $j \geq 1$ and suppose, inductively, that we have constructed an open set W_{j-1} in Y with $u \in W_{j-1} \subseteq V$, and a lift $\widetilde{F} : \left[0, \frac{j-1}{n}\right] \times W_{j-1} \to \widetilde{X}$ of the restriction $F : \left[0, \frac{j-1}{n}\right] \times W_{j-1} \to X$, with $\widetilde{F}(0,y) = \widetilde{f}(y)$ for all $y \in W_{j-1}$. Since $W_{j-1} \subseteq Y$ we have $F(I_j \times W_{j-1}) \subseteq U_j$. To get $p \circ \widetilde{F} = F$ we need $\widetilde{F}(I_j \times W_{j-1}) \subseteq p^{-1}(U_j)$. Since U_j is elementary, $p^{-1}(U_j)$ is a disjoint union of open sets, each of which is homeomorphic to U_j under the restriction of p. Let $\widetilde{U}_j$ be the one which contains the point $\widetilde{F}\left(\frac{j-1}{n}, u\right)$, and let $p_j : \widetilde{U}_j \to U_j$ be the homeomorphism obtained by restricting p. We remark that we do not know that W_{j-1} is connected so we cannot deduce that the image $\widetilde{F}\left(\{\frac{j-1}{n} \times W_{j-1}\right)$ must be contained entirely in the set $\widetilde{U}_j$, and so we shall restrict the open neighbourhood W_{j-1} of u.

Note that the restriction $\widetilde{F} : \left\{\frac{j-1}{n}\right\} \times W_{j-1} \to \widetilde{X}$ is continuous, so the inverse image $\widetilde{F}^{-1}(\widetilde{U}_j)$ is open in $\left\{\frac{j-1}{n}\right\} \times W_{j-1}$. Choose an open set W_j with $u \in W_j \subseteq W_{j-1}$ such that $\widetilde{F}\left(\left\{\frac{j-1}{n}\right\} \times W_j\right) \subseteq \widetilde{U}_j$. Restrict the domain of the constructed lift $\widetilde{F}$ from $\left[0, \frac{j-1}{n}\right] \times W_{j-1}$ to $\left[0, \frac{j-1}{n}\right] \times W_j$, then we extend this to obtain a lift $\widetilde{F} : \left[0, \frac{j}{n}\right] \times W_j \to \widetilde{X}$ by defining $\widetilde{F}(t, y) = p_j{}^{-1}(F(t, y)) \in \widetilde{U}_j$ for all $t \in I_j$ and all $y \in W_j$. This extension is continuous by the glueing lemma. Thus, by induction, writing $W = W_n$, we can construct a lift $\widetilde{F} : [0,1] \times W \to \widetilde{X}$ of the restriction $F : [0,1] \times W \to X$ with $\widetilde{F}(0, y) = \widetilde{f}(y)$ for all $y \in W$.

The above construction was carried out after selecting an arbitrary element $u \in Y$, so we have proven that for every $u \in Y$ there exists an open neighbourhood W_u of u in Y and a lift $\widetilde{F}_u : [0,1] \times W_u \to \widetilde{X}$ of (the restriction of) F with $\widetilde{F}_u(0, y) = \widetilde{f}(y)$ for all $y \in W_u$. To finish the proof, we note that these lifts are unique, and they glue together by the glueing lemma to form a well-defined and continuous lift $\widetilde{F} : [0,1] \times Y \to \widetilde{X}$ because for each $u \in Y$ and each $y \in W_u$, the restriction of $\widetilde{F}_u$ to the set $[0,1] \times \{y\}$ is uniquely determined by the uniqueness of path liftings: indeed the path $\tilde{\alpha}(t) = \widetilde{F}_u(t, y)$ must be equal to the unique lift of the path $\alpha(t) = F(t, y)$ with $\tilde{\alpha}(0) = \widetilde{F}_u(0, y) = \widetilde{f}(y)$.

10.10 Corollary: *Let $p : \widetilde{X} \to X$ be a covering. Let α and β be two paths from a to b in X. Let $\tilde{a} \in p^{-1}(a)$ and let $\tilde{\alpha}$ and $\tilde{\beta}$ be the lifts of α and β starting at the point $\tilde{a}$ (that is with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{a}$). If $\alpha \sim \beta$ in X then $\tilde{\alpha}$ and $\tilde{\beta}$ end at the same point.*

Proof: Let F be a homotopy from α to β in X. Note that the constant function $\widetilde{f}(s) = \tilde{a}$ is a lift of the constant function $f(s) = F(s, 0) = a$. Let $\widetilde{F}$ be the lift of F with $\widetilde{F}(0, t) = \widetilde{f}(t) = \tilde{a}$. Then the unique lift of α starting at $\tilde{a}$ is $\tilde{\alpha}(t) = \tilde{F}(0, t)$, the unique lift of β starting at $\tilde{a}$ is $\beta(t) = \widetilde{F}(1, t)$, the unique lift of the constant function a starting at $\tilde{a}$ is the constant function $\widetilde{F}(s, 0)$, and the unique lift of the constant function b starting at $\tilde{b} = \tilde{\alpha}(1) = \tilde{F}(0, 1)$ is the constant function $\widetilde{F}(s, 1)$. Thus $\tilde{\alpha}(1) = \widetilde{F}(0, 1) = \tilde{b} = \widetilde{F}(1, 1) = \tilde{\beta}(1)$, as required.

10.11 Corollary: *Let $p : \widetilde{X} \to X$ be a covering. Let α be a path from a to b in X. Let $\tilde{a} \in p^{-1}(a)$, let $\tilde{\alpha}$ be the lift of α starting at $\tilde{a}$, and let $\tilde{b} = \tilde{\alpha}(1)$.*

(1) Let $\widetilde{\alpha^{-1}}$ be the lift of α^{-1} starting at $\tilde{b}$. Then $\widetilde{\alpha^{-1}} = \tilde{\alpha}^{-1}$.

(2) Let β be a path from b to c in X. Let $\widetilde{\alpha\beta}$ be the lift of $\alpha\beta$ starting at $\tilde{a}$, and let $\tilde{\beta}$ be the lift of β starting at the point $\tilde{b}$. Then $\widetilde{\alpha\beta} = \tilde{\alpha}\,\tilde{\beta}$.

Proof: Since $p\big(\tilde{\alpha}^{-1}(t)\big) = p\big(\tilde{\alpha}(1-t)\big) = \alpha(1-t) = \alpha^{-1}(t)$ with $\tilde{\alpha}^{-1}(1) = \tilde{\alpha}(1) = \tilde{b}$, it follows that $\tilde{\alpha}^{-1}$ is a lift of α^{-1} starting at $\tilde{b}$. By the uniqueness of lifts, it must be equal to $\widetilde{\alpha^{-1}}$. Similarly, $\tilde{\alpha}\,\tilde{\beta}$ is a lift of $\alpha\beta$ starting at $\tilde{a}$, so it must be equal to $\widetilde{\alpha\beta}$.

10.12 Corollary: *Let $p : \widetilde{X} \to X$ be a covering. Let $a \in X$ and let $\tilde{a} \in p^{-1}(a)$. Then the induced map $p_* : \pi_1(\widetilde{X}, \tilde{a}) \to \pi_1(X, a)$ is injective. The image of p_* consists of the elements of the form $[\alpha]$ where α is a loop at a in X whose lift $\tilde{\alpha}$, starting at $\tilde{a}$, is a loop.*

Proof: Let $\tilde{\alpha}$ be a loop at $\tilde{a}$ in $\widetilde{X}$. Let $\alpha = p \circ \tilde{\alpha}$, so that α is a loop at a in X and we have $p_*(\tilde{\alpha}] = [\alpha]$. Suppose that $[\tilde{\alpha}] \in \operatorname{Ker} \phi$. Then we have $[\alpha] = 0 \in \pi_1(X, a)$, which means that $\alpha \sim \kappa$ in X where κ is the constant loop at a in X. The lift of κ starting at $\tilde{a}$ is the constant loop $\tilde{\kappa}$ at $\tilde{a}$ in $\widetilde{X}$. A homotopy from α to κ in X lifts to a homotopy from $\tilde{\alpha}$ to $\tilde{\kappa}$ in $\widetilde{X}$, and so we have $[\tilde{\alpha}] = 0$ in $\pi_1(\widetilde{X}, \tilde{a})$. This shows that $\operatorname{Ker} p_* = 0$ so that p_* is injective. We leave it as an exercise to prove the statement about the image of p_*.

10.13 Corollary: *Let $p:\widetilde{X}\to X$ be a covering with X path-connected. Then the sets $p^{-1}(a)$, $a\in X$, all have the same cardinality.*

Proof: Suppose that X is path-connected. Let $a,b\in X$. Let α be a path in X from a to b, and let $\beta=\alpha^{-1}$. Define $\phi:p^{-1}(a)\to p^{-1}(b)$ by $\phi(\tilde a)=\tilde\alpha(1)$ where $\tilde\alpha$ is the lift of α starting at $\tilde a$, and define $\psi:p^{-1}(b)\to p^{-1}(a)$ by $\psi(\tilde b)=\tilde\beta(1)$ where $\tilde\beta$ is the lift of β starting at $\tilde b$. We claim that ϕ and ψ are inverses of one another.

Let $\tilde a\in p^{-1}(a)$, let $\tilde b=\phi(\tilde a)=\tilde\alpha(1)$, let $\tilde\alpha$ be the lift of α starting at $\tilde a$, and let $\tilde\beta$ be the lift of β starting at $\tilde b$. We have $\alpha\beta=\alpha\alpha^{-1}\sim\kappa$ in X, where κ is the constant loop at a in X. The lift of κ at $\tilde a$ is the constant loop $\tilde\kappa$ at $\tilde a$ in $\widetilde{X}$. Since $\alpha\beta\sim\kappa$ in X, the lifts $\widetilde{\alpha\beta}$ and $\tilde\kappa$ have the same endpoints, and so

$$\psi(\phi(\tilde a))=\psi(\tilde b)=\tilde\beta(1)=(\tilde\alpha\tilde\beta)(1)=\widetilde{\alpha\beta}(1)=\tilde\kappa(1)=\tilde a.$$

This shows that $\psi(\phi(\tilde a))=\tilde a$ for every $\tilde a\in p^{-1}(a)$, and a similar argument shows that $\phi(\psi(\tilde b))=\tilde b$ for all $\tilde b\in p^{-1}(b)$.

10.14 Corollary: *Let $p:\widetilde{X}\to X$ be a covering with $\widetilde{X}$ path-connected, and let $a\in X$.*
(1) Given $\widetilde a\in p^{-1}(a)$, the index of $p_*\big(\pi_1(\widetilde{X},\tilde a)\big)$ in $\pi_1(X,a)$ is the cardinality of $p^{-1}(a)$.
(2) The set $\big\{p_*\big(\pi_1(\widetilde{X},\tilde a)\big)\,\big|\,\widetilde a\in p^{-1}(a)\big\}$ is a conjugacy class of subgroups of $\pi_1(X,a)$.

Proof: Let us write $H=p_*(\pi_1(\widetilde{X},\tilde a))$. Define a map ϕ from the set of right cosets of H in $\pi_1(X,a)$ to $p^{-1}(a)$ by $\phi\big(H[\alpha]\big)=\tilde\alpha(a)$, where α is a loop at a in X and $\tilde\alpha$ is the lift of α starting at $\tilde a$. We claim that this map is well-defined. First, note that $\tilde\alpha(1)$ does lie in $p^{-1}(a)$ because $p(\tilde\alpha(1))=\alpha(1)=a$. Next note that if α and β are two loops at a in X with $[\alpha]=[\beta]\in\pi_1(X,a)$, that is with $\alpha\sim\beta$ in X, then the lifts $\tilde\alpha$ and $\tilde\beta$, starting at $\tilde a$, have the same endpoint, that is $\tilde\alpha(1)=\tilde\beta(1)$. Finally, suppose that α and β are two loops at a in X such that $H[\alpha]=H[\beta]$. Then we have $[\alpha][\beta]^{-1}\in H$, hence $[\alpha\beta^{-1}]\in H=p_*(\widetilde{X},\tilde a)$. By Corollary 2.12), the lift $\widetilde{\alpha\beta^{-1}}$ of $\alpha\beta^{-1}$, starting at $\tilde a$, is a loop. By Corollary 2.11), we have $\widetilde{\alpha\beta^{-1}}=\tilde\alpha\widetilde{\beta^{-1}}=\tilde\alpha\tilde\beta^{-1}$, so the path $\tilde\beta^{-1}$ starts at $\tilde\alpha(1)$ and ends at $\tilde\alpha(0)=\tilde a$, and hence $\tilde\beta(1)=\tilde\alpha(1)$. This completes the proof that ϕ is well-defined.

We claim ϕ is injective. Let α and β be loops at a in X such that $\phi(H[\alpha])=\phi(H[\beta])$, that is such that $\tilde\alpha(1)=\tilde\beta(1)$ where $\tilde\alpha$ and $\tilde\beta$ are the lifts of α and β starting at $\tilde a$. Let $\tilde b=\tilde\alpha(1)=\tilde\beta(1)$. Note that $\alpha\beta^{-1}$ is a loop at a in X, and let $\widetilde{\alpha\beta^{-1}}$ be the lift at $\tilde a$. By Corollary 2.11, since $\tilde\alpha$ and $\tilde\beta$ are both paths from $\tilde a$ to $\tilde b$, we have $\widetilde{\alpha\beta^{-1}}=\tilde\alpha\tilde\beta^{-1}$, which is a loop at $\tilde a$ in $\widetilde{X}$. Since $\widetilde{\alpha\beta^{-1}}$ is a loop, it follows from Corollary 2.12 that $\alpha\beta^{-1}\in p_*(\pi_1(\tilde X,\tilde a))=H$. Thus we have $[\alpha][\beta^{-1}]=[\alpha\beta^{-1}]\in H$. This completes the proof that ϕ is injective.

Finally, we note that ϕ is surjective because $\widetilde{X}$ is path-connected: indeed, given $\tilde b\in p^{-1}(a)$, we can choose a path $\tilde\alpha$ from $\tilde a$ to $\tilde b$ in $\widetilde{X}$ and let α be the loop at a in X given by $\alpha=p\circ\tilde\alpha$, and then we have $\phi(H[\alpha])=\tilde\alpha(1)=\tilde b$. This completes the proof of Part 1.

We leave the proof of Part 2 as an exercise.

Local Connectedness and Path-Connectedness

10.15 Definition: Let X be a topological space. We say that X is **locally connected at the point** $a \in X$ when every open neighbourhood of a in X contains a connected open neighbourhood of a in X. We say that X is **locally connected** when it is locally connected at every point. Similarly, we say that X is **locally path-connected at the point** $a \in X$ when every open neighbourhood of a in X contains a path-connected open neighbourhood of a in X, and we say that X is **locally path-connected** when it is locally path-connected at every point.

10.16 Example: The topologist's sine curve is locally path-connected (hence locally connected) at every point $a \in A = \left\{\left(x, \sin \frac{1}{x}\right) \,\middle|\, x > 0\right\}$, but is not locally path-connected (or locally connected) at any point $b \in B = \{0\} \times [-1, 1]$.

10.17 Theorem: *Let X be a topological space.*

(1) X is locally connected if and only if X has the property that for every open set U in X, the connected components of U are open in X.
(2) X is locally path-connected if and only if X has the property that for every open set U in X, the path-connected components of U are open in X.

Proof: We prove Part 2 (the proof of Part 1 is similar). Suppose that X is path-connected. Let U be an open set in X. Let P be a path-component of U. For each $a \in P$, since X is locally path-connected and the open neighbourhood U of a contains a path-connected open neighbourhood, say C_a, of a in X. Since P is the path-component of a in X and C_a is path-connected, we must have $C_a \subseteq P$. Thus $P = \bigcup_{a \in P} C_a$, which is open.

Suppose, conversely, that every path-component, of every open set in X, is open in X. Let $a \in X$ and let U be an open neighbourhood of a in X. Let P be the path-component of U containing a. Then, by our supposition, P is open in X, and so P is a path-connected open neighbourhood of a which is contained in U.

10.18 Theorem: *Let X be a topological space. If X is locally path-connected then the path-components of X are equal to the connected components of X.*

Proof: Suppose that X is locally path-connected. Let C be a connected component of X. We know that C is a union of path-components. Suppose, for a contradiction, that C contains at least two distinct path-components, and let P be one of the path-components of X which is contained in C. Since X is locally path-connected, each of its path-components is open. In particular P is open and $C \setminus P$ (which is a union of path-components) is open. But then P and $C \setminus P$ are nonempty disjoint open sets which cover C, which is not possible, since C is connected.

10.19 Example: Every topological n-manifold is locally path-connected (since every neighbourhood of a point contains an open neighbourhood of the point which is homeomorphic to an open ball in $\mathbb{R}^n$, and every open ball in $\mathbb{R}^n$ is convex, hence path-connected). Consequently, in an n-manifold, the connected components are equal to the path-components.

10.20 Definition: When $(\widetilde{X}, \tilde{a})$ and (X, a) are based topological spaces and $p : \widetilde{X} \to X$ is a covering with $p(\tilde{a}) = a$, we write $p : (\widetilde{X}, \tilde{a}) \to (X, a)$ and say that p is a **based covering**. For a continuous map $f : (Y, b) \to (X, a)$, a **based lift** of f to $(\widetilde{X}, \tilde{a})$ is a continuous map $\widetilde{f} : (Y, b) \to (\widetilde{X}, \tilde{a})$ such that $p \circ \widetilde{f} = f$.

10.21 Theorem: *let Y be path-connected and locally path-connected with $b \in Y$. Let $p : (\widetilde{X}, \tilde{a}) \to (X, a)$ be a based covering, and let $f : (Y, b) \to (X, a)$ be a continuous map of based spaces. Then there exists a based lift $\widetilde{f} : (Y, b) \to (\widetilde{X}, \tilde{a})$ if and only if $f_*(\pi_1((Y, b)) \subseteq p_*(\pi_1(\widetilde{X}, \tilde{a}))$. In this case, the based lift is unique*

Proof: I may include a proof later.

10.22 Definition: Let X be a topological space. We say that X is **locally path-connected** when for every $a \in X$, every open set V with $a \in V$ contains an open set U with $a \in U \subseteq V$ such that $\pi_1(U, a) = 0$. We say that X is **semi-locally path-connected** when every point $a \in X$ is contained in an open neighbourhood U with the property that every loop at a in U is homotopic in X to the constant loop at a.

10.23 Example: The shrinking wedge of circles discussed in Example 6.8 is not locally (or semi-locally) simply connected.

10.24 Definition: Let $p_1 : \widetilde{X} \to X$ and $p_2 : \widetilde{X}_1 \to X$ be coverings. A covering space **homomorphism** from $\widetilde{X}_1$ to $\widetilde{X}_2$ is a continuous map $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ f = p_1$. In the case that f is a homeomorphism, it is called a covering space **isomorphism**.

When $p_1 : (\widetilde{X}_1, \tilde{a}_1) \to (X, a)$ and $p_2 : (\widetilde{X}_2, \tilde{a}_2) \to (X, a)$ are based coverings, a **based homomorphism** (or **isomorphism**) from $(\widetilde{X}_1, \tilde{a}_1)$ to $(\widetilde{X}_2, \tilde{a}_2)$ is a homomorphism (or isomorphism) $f : \widetilde{X}_1 \to \widetilde{X}_2$ with $f(\tilde{a}_1) = \tilde{a}_2$.

10.25 Theorem: *(The Classification of Covering Spaces) Let X be path-connected, locally path-connected, and semi-locally simply connected, and let $a \in X$. There is a bijective correspondence between the set of path-connected based coverings $p : (\widetilde{X}, \tilde{a}) \to (X, a)$ and the set of subgroups $H \subseteq \pi_1(X, a)$. The correspondence associates the based covering $p : (\widetilde{X}, \tilde{a}) \to (X, a)$ with the group $H = p_* \pi_1(\widetilde{X}, \tilde{a})$. In particular, there exists a cover $p : (\widetilde{X}, \tilde{a}) \to (X, a)$, unique up to isomorphism, with $\widetilde{X}$ simply connected, and this cover is called the* **universal cover** *of (X, a).*

Proof: Step 1. First, let us construct a universal cover. Let $\widetilde{X}$ be the set of homotopy classes of paths in X which start at the point a, that is

$$\widetilde{X} = \big\{ [\alpha] \, \big| \, \alpha \text{ is a path in } X \text{ with } \alpha(0) = a \big\},$$

let $\tilde{a} = [\kappa]$ where κ is the constant path at a, and define $p : (\widetilde{X}, \tilde{a}) \to (X, a)$ by $p([\alpha]) = \alpha(1)$. Note that p is well-defined because homotopic paths have the same endpoints.

We need to define a topology on $\widetilde{X}$ under which p is a covering map. Let $\mathcal{B}$ be the set of path-connected open sets U in X with the property that every loop in U is homotopic, in X, to a constant loop. Note that $\mathcal{B}$ is a basis for the topology on X because X is locally path-connected and semi-locally simply connected. Given $U \in \mathcal{B}$ and given a path α in X with $\alpha(0) = a$ and $\alpha(1) \in U$, define

$$\widetilde{U}_{[\alpha]} = \big\{ [\alpha\lambda] \, \big| \, \lambda \text{ is a path in } U \text{ with } \lambda(0) = \alpha(1) \big\}.$$

We claim that the sets $\widetilde{U}_{[\alpha]}$, where $U \in \mathcal{B}$ and α is a path in X with $\alpha(0) = a$ and $\alpha(1) \in U$, form a basis for a topology on $\widetilde{X}$. Note that when α is a path with $\alpha(0) = a$ and $\alpha(1) \in U \in \mathcal{B}$, we have $[\alpha] \in \widetilde{U}_{[\alpha]}$. It follows that the sets $\widetilde{U}_{[\alpha]}$ cover $\widetilde{X}$. Now suppose that $[\gamma] \in \widetilde{U}_{[\alpha]} \cap \widetilde{V}_{[\beta]}$ where $U, V \in \mathcal{B}$, say $[\gamma] = [\alpha\lambda]$ and $[\gamma] = [\beta\mu]$ where $\alpha(1) = \lambda(0) \in U$ and $\beta(1) = \mu(0) \in V$. Since $\alpha\lambda \sim \gamma \sim \beta\mu$, the paths λ, γ and μ have the same endpoint, say $b = \lambda(1) = \gamma(1) = \mu(1)$. Since $U, V \in \mathcal{B}$ with $b \in U \cap V$, we can choose $W \in \mathcal{B}$ with $b \in W \subseteq U \cap V$. Note that $\widetilde{W}_{[\gamma]} \subseteq \widetilde{U}_{[\alpha]}$: indeed a point in $\widetilde{W}_{[\gamma]}$ is of the form $[\gamma\nu]$ where ν is a path in W with $\nu(0) = \gamma(1)$, and we have $\gamma\nu \sim \alpha\lambda\nu$ so that $[\gamma\nu] = [\alpha\lambda\nu] \in \widetilde{U}_{[\alpha]}$. Similarly, $\widetilde{W}_{[\gamma]} \subseteq \widetilde{V}_{[\beta]}$ and so we have $[\gamma] \in \widetilde{W}_{[\gamma]} \subseteq \widetilde{U}_{[\alpha]}\widetilde{V}_{[\beta]}$, and this completes the proof that the sets $\widetilde{U}_{[\alpha]}$ form a basis for a topology on $\widetilde{X}$.

We now have a topology on the space $\widetilde{X}$. Using this topology, note that the map $p : (\widetilde{X}, \tilde{a}) \to (X, a)$ is continuous because, given a basic open set $U \in \mathcal{B}$ we have

$$p^{-1}(U) = \big\{[\alpha] \,\big|\, \alpha(1) \in U\big\} = \textstyle\bigcup_{\alpha(1)\in U} \widetilde{U}_{[\alpha]}$$

which is a union of basic open sets in $\widetilde{X}$.

We claim that when $U \in \mathcal{B}$ and α and β are two paths starting at a with endpoints in U, we have

$$\widetilde{U}_{[\alpha]} = \widetilde{U}_{[\beta]} \iff [\beta] \in \widetilde{U}_{[\alpha]}.$$

One direction is immediate: if $\widetilde{U}_{[\alpha]} = \widetilde{U}_{[\beta]}$ then of course since $[\beta] \in \widetilde{U}_{[\beta]}$ we also have $[\beta] \in \widetilde{U}_{[\alpha]}$. Suppose, conversely, that $[\beta] \in \widetilde{U}_{[\alpha]}$, say $[\beta] = [\alpha\lambda]$, that is $\beta \sim \alpha\lambda$, where λ is a path in U with $\lambda(0) = \alpha(1)$. An element in $\widetilde{U}_{[\alpha]}$ is of the form $[\alpha\mu]$ where μ is a path in U with $\mu(0) = \alpha(1)$, and we have $\alpha\mu \sim \beta\lambda^{-1}\mu$ so that $[\alpha\mu] = [\beta\lambda^{-1}\mu] \in \widetilde{U}_{[\beta]}$, and this shows that $\widetilde{U}_{[\alpha]} \subseteq \widetilde{U}_{[\beta]}$. A similar argument shows that $\widetilde{U}_{[\beta]} \subseteq \widetilde{U}_{[\alpha]}$, so that we have $\widetilde{U}_{[\alpha]} = \widetilde{U}_{[\beta]}$, as required.

By the above claim, the distinct sets $\widetilde{U}_{[\alpha]}$ in the inverse image $p^{-1}(U) = \bigcup_{\alpha(1)\in U} \widetilde{U}_{[\alpha]}$ are disjoint. Let us verify that the restriction $p : \widetilde{U}_{[\alpha]} \to U$ is a homeomorphism. Note that p is surjective because the set U is path-connected: given $b \in U$ we can choose a path λ in U from $\alpha(1)$ to b and then $[\alpha\lambda] \in \widetilde{U}_{[\alpha]}$ with $p([\alpha\lambda]) = \lambda(1) = b$. Note that p is injective because U has the property that every loop in U is homotopic, in X, to a constant loop: indeed if $p([\alpha\lambda]) = p([\alpha\mu])$ then we have $\lambda(1) = \mu(1)$ so that $\lambda\mu^{-1}$ is a loop in U, and hence $\lambda\mu^{-1}$ is homotopic, in X, to a constant loop, so we have $[\alpha\lambda] = [\alpha\lambda\mu^{-1}\mu] = [\alpha\mu]$. Finally, note that the inverse $p^{-1} : U \to \widetilde{U}_{[\alpha]}$ is continuous because for a basic open set $\widetilde{V}_{[\beta]} \subseteq \widetilde{U}_{[\alpha]}$ we have $V = p\big(\widetilde{V}_{[\beta]}\big) \subseteq p\big(\widetilde{U}_{[\alpha]}\big) = U$ with V open.

Let us verify that $\widetilde{X}$ is simply connected. To show this, we find a formula for the lift of a path in X. Let $\alpha : [0,1] \to X$ be a path in X with $\alpha(0) = a$. Define $\tilde{\alpha} : [0,1] \to \widetilde{X}$ by $\tilde{\alpha}(s) = [\alpha_s]$ where $\alpha_s : [0,1] \to X$ is given by $\alpha_s(t) = \alpha(st)$. Note that $\tilde{\alpha}(0) = [\alpha_0] = \tilde{a}$ and $\tilde{\alpha}(1) = [\alpha_1] = [\alpha]$. Verify, as an exercise, that $\tilde{\alpha}$ is continuous, and note that $p(\tilde{\alpha}(s)) = p([\alpha_s]) = \alpha_s(1) = \alpha(s)$, and so $\tilde{\alpha}$ is the lift of α starting at $\tilde{\alpha}(0) = [\alpha_0] = \tilde{a}$. To see that $\widetilde{X}$ is path-connected, note that for any path α in X with $\alpha(0) = a$, the lift $\tilde{\alpha}$ is a path in $\widetilde{X}$ from $\tilde{a}$ to $\tilde{\alpha}(1) = [\alpha_1] = [\alpha]$, so $[\alpha]$ is in the same path-component as $\tilde{a}$. To see that $\pi_1(\widetilde{X}, \tilde{a}) = 0$, recall that the group homomorphism $p_* : \pi_1(\widetilde{X}, \tilde{a}) \to (X, a)$ is injective, and the image consists of elements of the form $[\alpha]$ where α is a loop at a in X whose lift $\tilde{\alpha}$ is a loop at $\tilde{a}$ in $\widetilde{X}$. If $[\alpha]$ is in the image of p_* then $\tilde{\alpha}(1) = \tilde{a}$, that is $[\alpha] = \tilde{a} = [\kappa]$ where κ is the constant loop at a, so that $[\alpha] = 0 \in \pi_1(X, a)$. Thus p_* is the zero map and hence (since p_* is injective) we have $\pi_1(\widetilde{X}, \tilde{a}) = 0$.

Step 2. Let $H \subseteq \pi_1(X,a)$. We wish to construct a covering $p_H : (\widetilde{X}_H, \tilde{a}_H) \to (X,a)$ with $p_{H*}\pi_1(\widetilde{X}_H, \tilde{a}_H) = H$. Define a relation on the universal cover $\widetilde{X}$ by stipulating that $[\alpha] \equiv [\beta]$ when $\alpha(1) = \beta(1)$ and $[\alpha\beta^{-1}] \in H$. Note that this is an equivalence relation because H is a subgroup of $\pi_1(X,a)$. Denote the equivalence class of $[\alpha]$ by $[\alpha]_H$, and let $\widetilde{X}_H$ be the quotient space of $\widetilde{X}$ under this equivalence relation, that is

$$\widetilde{X}_H = \widetilde{X}/\!\equiv\; = \big\{[\alpha]_H \,\big|\, \alpha \text{ is a path with } \alpha(0) = a\big\},$$

and let $q : \widetilde{X} \to \widetilde{X}_H$ be the quotient map. The covering map $p : \widetilde{X} \to X$, which is given by $p([\alpha]) = \alpha(1)$, is clearly constant on equivalence classes (since when $[\alpha] \equiv [\beta]$ we have $\alpha(1) = \beta(1)$) and so it induces a continuous map $p_H : \widetilde{X}_H \to X$ which is given by $p_H([\alpha]_H) = \alpha(1)$. To see that p_H is a covering map, we make the following observation. Let $U \in \mathcal{B}$ and let α and β be two paths in X starting at a and ending at $b \in U$, with $[\alpha] \equiv [\beta]$, that is with $[\alpha\beta^{-1}] \in H$. Then for any path λ in U starting at b, we have $[(\alpha\lambda)(\beta\lambda)^{-1}] = [\alpha\beta^{-1}] \in H$ so that $[\alpha\lambda] \equiv [\beta\lambda]$. Thus every point $[\alpha\lambda] \in \widetilde{U}_{[\alpha]}$ is equivalent to the corresponding point $[\beta\lambda] \in \widetilde{U}_{[\beta]}$. It follows that the equivalence relation $\equiv$ on X induces an equivalence relation on the disjoint sets in the inverse image $p^{-1}(U) = \bigcup_{\alpha(1) \in U} \widetilde{U}_{[\alpha]}$, so that the inverse image $p_H^{-1}(U)$ is the disjoint union $p_H^{-1}(U) = \bigcup_{\alpha(1)\in U} \widetilde{U}_{[\alpha]_H}$, where

$$\widetilde{U}_{[\alpha]_H} = \big\{[\alpha\lambda]_H \,\big|\, \lambda \text{ is a path in } U \text{ starting at } \alpha(1)\big\} = q^{-1}\big(\widetilde{U}_{[\alpha]}\big).$$

The restriction $p : \widetilde{U}_{[\alpha]} \to U$ is a homeomorphism, so the restriction $p_H : \widetilde{U}_{[\alpha]_H} \to U$ is also a homeomorphism: we have $p = p_H \circ q$ and the inverses of the restrictions are related by $p_H^{-1} = q \circ p^{-1}$.

Step 3. It remains to show that for two based coverings $p_1 : (\widetilde{X}_1, \tilde{a}_1) \to (X,a)$ and $p_2 : (\widetilde{X}, \tilde{a}) \to (X,a)$, we have $(\widetilde{X}_1, \tilde{a}_1) \cong (\widetilde{X}_2, \tilde{a}_2) \iff p_{1*}\pi_1(\widetilde{X}_1, \tilde{a}_1) \cong p_{2*}\pi_1(\widetilde{X}_2, \tilde{a}_2)$. Suppose that $(\widetilde{X}_1, \tilde{a}_1) \cong (\widetilde{X}_2, \tilde{a}_2)$ and let $f : (\widetilde{X}_1, \tilde{a}_1) \to (\widetilde{X}_2, \tilde{a}_2)$ be a based isomorphism of coverings, so f is a homeomorphism with $p_1 = p_2 \circ f$. Since f is a homeomorphism, the induced map $f_* : \pi_1(\widetilde{X}_1, \tilde{a}_1) \to \pi_1(\widetilde{X}_2, \tilde{a}_2)$ is an isomorphism. Since $p_1 = p_2 \circ f$ we have $p_{1*} = p_{2*} \circ f_*$ and so $\pi_{1*}\pi_1(\widetilde{X}_1, \tilde{a}_1) = p_{2*}\big(f_*\pi_1(\widetilde{X}_1, \tilde{a}_1)\big) = p_{2*}\pi_1(\widetilde{X}_2, \tilde{a}_2)$, as required.

Suppose, conversely, that $\pi_{1*}\pi_1(\widetilde{X}_1, \tilde{a}_1) = p_{2*}\pi_1(\widetilde{X}_2, \tilde{a}_2)$. By Theorem 10.9, there exists a (unique) lift $\widetilde{p}_1 : (\widetilde{X}_1, \tilde{a}_1) \to (\widetilde{X}_2, \tilde{a}_2)$ of the map $p_1 : (\widetilde{X}_1, \tilde{a}_1) \to (X,a)$, and there exists a (unique) lift $\widetilde{p}_2 : (\widetilde{X}_2, \tilde{a}_2) \to (\widetilde{X}_1, \tilde{a}_1)$ of the map $p_2 : (\widetilde{X}_2, \tilde{a}_2) \to (X,a)$, so we have $p_1 = p_2 \circ \widetilde{p}_2$ and $p_2 = p_1 \circ \widetilde{p}_2$. Consider the map $\widetilde{p}_2 \circ \widetilde{p}_1 : (\widetilde{X}_1, \tilde{a}_1) \to (\widetilde{X}_1, \tilde{a}_1)$. Since $p_1 \circ \widetilde{p}_2 \circ \widetilde{p}_1 = p_2 \circ \widetilde{p}_1 = p_1$ with $\widetilde{p}_2(\widetilde{p}_1(\tilde{a}_1)) = \widetilde{(\tilde{a}_2)} = a$, it follows that $\widetilde{p}_2 \circ \widetilde{p}_1$ is equal to the unique lift of the map $p_1 : (\widetilde{X}_1, \tilde{a}_1) \to (X,a)$, so it must be equal to the identity map $\mathrm{id} : (\widetilde{X}_1, \tilde{a}_1) \to (\widetilde{X}_1, \tilde{a}_1)$ (which is also a lift of p_1). Thus we have $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathrm{id}$, and similarly $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathrm{id}$ so that $\widetilde{p}_1$ and $\widetilde{p}_2$ are inverses of one another. Thus $(\widetilde{X}_1, \tilde{a}_1) \cong (\widetilde{X}_2, \tilde{a}_2)$.