Chapter 1. Topological Spaces and Continuous Maps

Topological Spaces

1.1 Note: Recall that, given an inner product on a real vector space V, we can define a norm on V by defining $\|u\| = \sqrt{\langle u, u\rangle}$ and that, given a norm on a vector space V, we can define a metric on V by defining $d(u,v) = \|v-u\|$. Recall also that, given a metric on a set X, we define the open balls in X as follows: given $a \in X$ and $r > 0$ we define the open ball centred at a of radius r to be the set $B(a,r) = \big\{x \in X \,\big|\, d(a,x) < r\big\}$, and then we define the open subsets of X by stipulating that $A \subseteq X$ is open when it has the property that for every $a \in A$ there exists $r > 0$ such that $B(a,r) \subseteq A$. Finally, recall that when X is a metric space, the open sets in X satisfy the following basic properties:

(1) $\emptyset$ and X are both open,
(2) the union of any set of open sets in X is also open, and
(3) the intersection of any finite set of open sets in X is open.

We take the above three basic properties as the defining properties for what we call a topological space.

1.2 Notation: When $\mathcal{S}$ is a set of sets, the **union** of $\mathcal{S}$ is the set

$$\bigcup \mathcal{S} = \bigcup_{A\in\mathcal{S}} A = \big\{x \,\big|\, x\in A \text{ for some } A\in\mathcal{S}\big\}$$

and, in the case that $\mathcal{S}$ is nonempty, the **intersection** of $\mathcal{S}$ is the set

$$\bigcap \mathcal{S} = \bigcap A \in \mathcal{S} A = \big\{x \,\big|\, x\in A \text{ for all } A\in\mathcal{S}\big\}.$$

When $\mathcal{S} = \{A_k \,|\, k \in K\}$, where K is a set which we call the **index set**, we also write $\bigcup \mathcal{S} = \bigcup_{k\in K} A_k$ and, in the case that K is nonempty, we write $\bigcap \mathcal{S} = \bigcap_{k\in K} A_k$.

We remark that the empty union is empty, that is $\bigcup \emptyset = \emptyset$ and $\bigcup_{k\in\emptyset} A_k = \emptyset$ but that, in general, there is no standard way of defining the empty intersection. In the case that all of the elements in $\mathcal{S}$, or all of the sets A_k, are known to live in a specific set X, sometimes called the **universal set**, it is natural to define the empty intersection to be equal to X, that is $\bigcap \emptyset = X$ and $\bigcap_{k\in\emptyset} A_k = X$.

1.3 Definition: Let X be a set. A **topology** on X is a set $\mathcal{T}$ of subsets of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
(2) If K is a nonempty set and $A_k \in \mathcal{T}$ for all $k \in K$, then $\bigcup_{k\in K} A_k \in \mathcal{T}$.
(3) If K is a nonempty finite set and $A_k \in \mathcal{T}$ for all $k \in K$, then $\bigcap_{k\in K} A_k \in \mathcal{T}$.

Note that, by induction, in order to verify that Property 3 holds, it suffices to show that if $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$. We say that $\mathcal{T}$ is **closed under arbitrary union** to mean that $\emptyset \in \mathcal{T}$ and $\mathcal{T}$ satisfies Property 2. We say that $\mathcal{T}$ is **closed under finite intersection** to mean that $X \in \mathcal{T}$ and $\mathcal{T}$ satisfies Property 3.

A **topological space** is a set X together with a topology $\mathcal{T}$. The sets in $\mathcal{T}$ are called the **open** subsets of X, and their complements are called the **closed** subsets of X. When $a \in X$, an **open neighbourhood** of a in X is an open set $U \in \mathcal{T}$ with $a \in U$.

When $\mathcal{S}$ and $\mathcal{T}$ are two topologies on the same set X with $\mathcal{S} \subseteq \mathcal{T}$ we say that $\mathcal{S}$ is **coarser** than $\mathcal{T}$ and that $\mathcal{T}$ is **finer** than $\mathcal{S}$. When $\mathcal{S} \subsetneq \mathcal{T}$ we say that $\mathcal{S}$ is **strictly coarser** than $\mathcal{T}$ and that $\mathcal{T}$ is **strictly finer** than $\mathcal{S}$.

1.4 Definition: When X is a metric space, the set $\mathcal{T}$ of all open sets in X (defined using open balls as in Note 1.1) is a topology, which we call the **metric topology** on X.

1.5 Definition: When X is any set, the set $\{\emptyset, X\}$ is a topology on X, which we call the **trivial topology**, and the power set $\mathcal{P}(X) = \{A \,|\, A \subseteq X\}$ is a topology on X which we call the **discrete topology**.

1.6 Example: Let $X = \{1,2,3\}$, $\mathcal{R} = \big\{\emptyset, \{1\}, \{1,2,3\}\big\}$, $\mathcal{S} = \big\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\big\}$, $\mathcal{T} = \big\{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\big\}$, and $\mathcal{B} = \big\{\emptyset, \{1\}, \{2\}, \{1,2,3\}\big\}$. Than, as you can verify, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are all topologies on X, but $\mathcal{B}$ is not.

1.7 Theorem: *Let $\mathcal{F}$ be the set of all closed sets in a topological space. Then*
(1) $\emptyset \in \mathcal{F}$ and $X \in \mathcal{F}$.
(2) If K is a nonempty set and $A_k \in \mathcal{F}$ for every $k \in K$ then $\bigcap_{k\in K} A_k \in \mathcal{F}$.
(3) If K is a nonempty finite set and $A_k \in \mathcal{F}$ for every $k \in K$ then $\bigcup_{k\in K} A_k \in \mathcal{F}$.

Proof: This follows from Definition 1.3 by taking complements in X, noting that $\emptyset^c = X$, $X^c = \emptyset$, $\big(\bigcup_{k\in K} A_k\big)^c = \bigcap_{k\in K} A^c$ and $\big(\bigcap_{k\in K} A_k\big)^c = \bigcup_{k\in K} A^c$.

1.8 Definition: Let X be a topological space and let $A \subseteq X$ be a subset. The **interior** of A in X, denoted by A^o or by $\text{Int}(A)$ or by $\text{Int}_X(A)$, is the union of the set of all open sets in X which are contained in A. The **closure** of A in X, denoted by $\overline{A}$ or by $\text{Cl}(A)$ or by $\text{Cl}_X(A)$, is the intersection of the set of all closed sets in X which contain X. The **boundary** of A in X, denoted by ∂A or by $\text{Bound}(A)$ or by $\text{Bound}_X(A)$ is the set $\partial A = \overline{A} \setminus A^o$. Note that $\overline{A}$ is the disjoint union $\overline{A} = A^o \cup \partial A$.

1.9 Theorem: *Let X be a topological space and let $A \subseteq X$. Then*
(1) A^o is the largest open set in X which is contained in A, and $\overline{A}$ is the smallest closed set in X which contains A.
(2) A is open if and only if $A = A^o$, and A is closed if and only if $A = \overline{A}$.
(3) $(A^o)^o = A^o$ and $\overline{\overline{A}} = \overline{A}$.

Proof: We prove the properties involving A^o. Note that A^o is open because it is a union of open sets, and A^o is contained in A because it is a union of sets each of which is contained in A, and it is the largest open set contained in A because, by its definition, it contains every open set which contains A. This proves Part 1. If A is open then clearly A is the largest open set contained in itself, so $A^o = A$. Conversely, if $A = A^o$ then it is open (because A^o is open). This proves Part 2. Finally, note that since A^o is open, we have $(A^o)^o = A^o$ by Part 2. This proves Part 3.

1.10 Definition: Let X be a topological space and let $A \subseteq X$. A **limit point** of A in X is a point $a \in X$ with the property that for every open set U in X with $a \in U$ we have $\big(U \setminus \{a\}\big) \cap A \neq \emptyset$. The set of all limit points of A in X is denoted by A' or by $\text{Lim}(A)$ or by $\text{Lim}_X(a)$.

1.11 Theorem: *Let X be a topological space, let $A \subseteq X$, and let $a \in X$.*
(1) $a \in A^o$ if and only if there exists an open set U in X with $a \in U \subseteq A$.
(2) $a \in \overline{A}$ if and only if for every open set U in X with $a \in U$ we have $U \cap A \neq \emptyset$.
(3) $\overline{A} = A \cup A'$.
(4) A is closed in X if and only if $A' \subseteq A$.

Proof: We leave the proof as an exercise.

Bases for Topologies

1.12 Theorem: *The intersection of any set of topologies on X is a topology on X.*

Proof: We remark that since every topology on X is a subset of the power set of X, we consider the empty intersection to be the power set of X, which is the discrete topology. Let $\mathcal{R}$ be any nonempty sey of topologies on X and let $\mathcal{S} = \bigcap \mathcal{R}$. We need to show that $\mathcal{S}$ is a topology on X. Since $\emptyset \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$, we also have $\emptyset \in \mathcal{S}$. Likewise, since $X \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$, we also have $X \in \mathcal{S}$. This shows that $\mathcal{S}$ satisfies Property 1 in the definition of a topological space.

To show that $\mathcal{S}$ satisfies Property 2, let K be a nonempty set and let $U_k \in \mathcal{S}$ for each $k \in K$. We need to show that $\bigcup_{k\in K} U_k \in \mathcal{S}$. For each index $k \in K$, since $U_k \in \mathcal{S} = \bigcap \mathcal{R}$ we have $U_k \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$. It follows that for every $\mathcal{T} \in \mathcal{R}$, we have $U_k \in \mathcal{T}$ for every $k \in K$, so that $\bigcup_{k\in K} U_k \in \mathcal{T}$. Since $\bigcup_{k\in K} U_k \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$, it follows that $\bigcup_{k\in K} U_k \in \bigcap \mathcal{R} = \mathcal{S}$, as required.

To show that $\mathcal{S}$ satisfies Property 3, let $U, V \in \mathcal{S}$. We need to show that $U \cap V \in \mathcal{S}$. Since $U \in \mathcal{S} = \bigcap \mathcal{R}$, we have $U \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$. Likewise, since $V \in \mathcal{S} = \bigcap \mathcal{R}$, we have $V \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$. Thus for every $\mathcal{T} \in \mathcal{R}$ we have $U \in \mathcal{T}$ and $V \in \mathcal{T}$, and hence $U \cap V \in \mathcal{T}$. Since $U \cap V \in \mathcal{T}$ for every $\mathcal{T} \in \mathcal{R}$, it follows that $U \cap V \in \bigcap \mathcal{R} = \mathcal{S}$, as required.

1.13 Corollary: *Given any set $\mathcal{S}$ of subsets of X, there exists a unique smallest (or coarsest) topology $\mathcal{T}$ on X which contains $\mathcal{S}$.*

Proof: The unique smallest topology $\mathcal{T}$ which contains $\mathcal{S}$ is the intersection of the set of all topologies on X which contain $\mathcal{S}$ (noting that the set of all topologies on X which contain X is not empty since the discrete topology contains $\mathcal{S}$).

1.14 Definition: Given a set $\mathcal{S}$ of subsets of X, the unique smallest topology $\mathcal{T}$ which contains $\mathcal{S}$ is called the topology on X **generated by** $\mathcal{S}$.

1.15 Definition: Let X be a set. A **basis** of sets in X is a set $\mathcal{B}$ of subsets of X such that

(1) for all $a \in X$ there exists $B \in \mathcal{B}$ such that $a \in B$, and
(2) for all $a \in X$ and all $C, D \in \mathcal{B}$ with $a \in C \cap D$, there exists $B \in \mathcal{B}$ with $a \in B \subseteq C \cap D$.

When $\mathcal{B}$ is a basis of sets in X and $\mathcal{T}$ is the topology on X generated by $\mathcal{B}$, we say that $\mathcal{B}$ is a **basis** for the topology $\mathcal{T}$, and the elements in $\mathcal{B}$ are called **basic open sets**.

1.16 Example: Any topology on a set is a basis for itself.

1.17 Example: On any set X, the set $\{X\}$ is a basis for the trivial topology.

1.18 Example: On any set X, the set of all one-point subsets of X is a basis for the discrete topology on X (this follows easily from the Theorem 1.20 below).

1.19 Example: The set of open balls in a metric space X is a basis for the metric topology (this follows from Theorem 1.21 below).

1.20 Theorem: *Let $\mathcal{B}$ be a basis for the topology $\mathcal{T}$ on X.*

(1) $\mathcal{T}$ is the set of all $U \subseteq X$ such that for every $a \in U$ there exists $B \in \mathcal{B}$ with $a \in B \subseteq U$.
(2) $\mathcal{T}$ is the set of all unions of (sets of) elements in $\mathcal{B}$.

Proof: Let $\mathcal{R}$ be the set of all unions of (sets of) elements in $\mathcal{B}$, that is the set of all sets of the form $U = \bigcup_{k \in K} B_k$ where K is an index set and each $B_k \in \mathcal{B}$ (with the empty union giving the empty set), and let $\mathcal{S}$ be the set of all sets $U \subseteq X$ with the property that for every $a \in U$ there exists $B \in \mathcal{B}$ with $a \in B \subseteq U$. We need to show that $\mathcal{R} = \mathcal{S} = \mathcal{T}$. We do this by showing that $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{R} \subseteq \mathcal{T}$.

Since $\mathcal{T}$ is the intersection of the set of all topologies which contain $\mathcal{B}$, in order to show that $\mathcal{S} \subseteq \mathcal{T}$, it suffices to show that $\mathcal{S}$ is a topology which contains $\mathcal{B}$. It is easy to see that $\mathcal{S}$ contains $\mathcal{B}$: indeed given $U \in \mathcal{B}$ and given $a \in U$, we can choose $B = U$ to get $B \in \mathcal{B}$ with $a \in B \subseteq U$, showing that $U \in \mathcal{S}$. It remains to show that $\mathcal{S}$ is a topology. To see that $\mathcal{S}$ satisfies Property 1 in the definition of a topology, note that $\emptyset \in \mathcal{S}$ vacuously, and note that given $a \in X$, since $\mathcal{B}$ is a basis of sets we can choose $B \in \mathcal{B}$ such that $a \in B$ and then we have $a \in B \subseteq X$ showing that $X \in \mathcal{S}$. To show that Property 2 holds, let K be an index set and let $U_k \in \mathcal{S}$ for each $k \in K$. Let $a \in \bigcup_{k \in K} U_k$. Choose an index $\ell \in K$ such that $a \in U_\ell$. Since $U_\ell \in \mathcal{S}$ we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq U_\ell$. Since $U_\ell \subseteq \bigcup_{k \in K} U_k$, we also have $a \in B \subseteq \bigcup_{k \in K} U_k$, and hence $\bigcup_{k \in K} U_k \in \mathcal{S}$, showing that Property 2 holds. To show that Property 3 holds, let $U, V \in \mathcal{S}$. Let $a \in U \cap V$. Since $U \in \mathcal{S}$ and $V \in \mathcal{S}$ we can choose sets $C, D \in \mathcal{B}$ such that $a \in C \subseteq U$ and $a \in D \subseteq V$, and then we have $a \in C \cap D \subseteq U \cap V$. Since $\mathcal{B}$ is a basis of sets, we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq C \cap D \subseteq U \cap V$, showing that $U \cap V \in \mathcal{S}$ so that Property 3 holds. This completes the proof that $\mathcal{S}$ is a topolgy on X which contains $\mathcal{B}$, and hence that $\mathcal{T} \subseteq \mathcal{S}$.

To show that $\mathcal{S} \subseteq \mathcal{R}$, let $U \in \mathcal{S}$. For each $a \in U$, choose $B_a \in \mathcal{B}$ such that $a \in B_a \subseteq U$. Then we have $U = \bigcup_{a \in U} B_a \in \mathcal{R}$. This shows that $\mathcal{S} \subseteq \mathcal{R}$.

Finally, to show that $\mathcal{R} \subseteq \mathcal{T}$, let $U \in \mathcal{R}$, say $U = \bigcup_{k \in K} B_k$. Note that every topology on X which contains $\mathcal{B}$ must contain the set U by Property 2 in the definition of a topology, and hence U lies in the intersection of the set of all such topologies, that is $U \in \mathcal{T}$. This shows that $\mathcal{R} \subseteq \mathcal{T}$.

1.21 Theorem: *Let $\mathcal{T}$ be a topology on X and let $\mathcal{B} \subseteq \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if for every $a \in X$ and every $U \in \mathcal{T}$ with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.*

Proof: Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Let $a \in X$ and let $U \in \mathcal{T}$ with $a \in U$. By Property 1 of the above theorem, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq U$.

Suppose, conversely, that for every $U \in \mathcal{T}$ and every $a \in U$ there exists $B \in \mathcal{B}$ with $a \in B \subseteq U$. We need to show that $\mathcal{B}$ is a basis of sets in X and that the topology $\mathcal{T}$ is generated by $\mathcal{B}$. Property 1 in the definition of a basis of sets holds because given $a \in X$ and taking $U = X$ so that $U \in \mathcal{T}$ with $a \in U$, we can choose $B \in \mathcal{B}$ with $a \in B \subseteq U$. Property 2 holds because given $a \in X$ and given $C, D \in \mathcal{B}$ with $a \in C \cap D$, and taking $U = C \cap D$, we have $U \in \mathcal{T}$ with $a \in U$ so we can choose $B \in \mathcal{B}$ with $a \in B \subseteq U = C \cap D$. This shows that $\mathcal{B}$ is a basis of sets in X. To see that $\mathcal{T}$ is generated by $\mathcal{B}$, we use Part 2 of the previous theorem: Since $\mathcal{B} \subseteq \mathcal{T}$, every union of elements in $\mathcal{B}$ is also a union of elements in $\mathcal{T}$, and hence is an element of $\mathcal{T}$ (since $\mathcal{T}$ is a topology so it is closed under unions). On the other hand, given $U \in \mathcal{T}$, for each $a \in U$ we can choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$ and then we have $U = \bigcup_{a \in U} B_a$ (which is a union of elements of $\mathcal{B}$).

Hausdorff Spaces

1.22 Definition: A topological space X is said to be **Hausdorff** when for all $a, b \in X$ with $a \neq b$, there exist open sets $U, V \subseteq X$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

1.23 Example: Every metric space X is a Hausdorff space because, given $a, b \in X$, we can let $U = B(a, r)$ and $V = B(b, r)$ with $r = \frac{1}{2} d(a, b)$ to get $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

The Subspace Topology

1.24 Theorem: *Let $\mathcal{S}$ be a topology on Y, let $X \subseteq Y$, and let $\mathcal{T} = \big\{V \cap X \,\big|\, V \in \mathcal{S}\big\}$. Then $\mathcal{T}$ is a topology on X.*

Proof: Note that $\mathcal{T}$ satisfies Property 1 (in the definition of a topology) because $\emptyset \in \mathcal{S}$ so that $\emptyset = \emptyset \cap X \in \mathcal{T}$ and because $Y \in \mathcal{S}$ so that $X = Y \cap X \in \mathcal{T}$.

To show that Property 2 holds, let $U_k \in \mathcal{T}$ for each $k \in K$. For each $k \in K$, choose $V_k \in \mathcal{S}$ such that $U_k = V_k \cap X$. Since $\mathcal{S}$ is a topology, we have $\bigcup_{k \in K} V_k \in \mathcal{S}$, and hence $\bigcup_{k \in K} U_k = \bigcup_{k \in K} (V_k \cap X) = \big(\bigcup_{k \in K} V_k\big) \cap X \in \mathcal{T}$.

To show that Property 3 holds, let $U_1, U_2 \in \mathcal{T}$. Choose $V_1, V_2 \in \mathcal{S}$ with $U_1 = V_1 \cap X$ and $U_2 = V_2 \cap X$. Since $\mathcal{S}$ is a topology, we have $V_1 \cap V_2 \in \mathcal{S}$, and hence we have $U_1 \cap U_2 = (V_1 \cap X) \cap (V_2 \cap X) = (V_1 \cap V_2) \cap X \in \mathcal{S}$.

1.25 Definition: When $\mathcal{S}$ is a topology on Y and $X \subseteq Y$, the set $\mathcal{T} = \big\{V \cap X \,\big|\, V \in \mathcal{S}\big\}$ is called the **subspace topology** on X, and when X uses this topology we say that X is a **subspace** of Y. When Y is a topological space and $X \subseteq Y$, unless otherwise indicated we shall assume that X uses the subspace topology.

1.26 Theorem: *Let X be a subspace of Y and let $A \subseteq X$. Then A is closed in X if and only if $A = B \cap X$ for some closed set B in Y.*

Proof: The proof is left as an exercise.

1.27 Theorem: *Let $\mathcal{C}$ be a basis for a topology on Y, and let $X \subseteq Y$. Then the set $\mathcal{B} = \big\{C \cap X \,\big|\, C \in \mathcal{C}\big\}$ is a basis for the subspace topology on X.*

Proof: Let $\mathcal{S}$ be the topology on Y generated by $\mathcal{C}$ and let $\mathcal{T}$ be the subspace topology on X. By Theorem 1.21, it suffices to show that for every $a \in X$ and $U \in \mathcal{T}$, there exists $B \in \mathcal{B}$ with $a \in B \subseteq U$. Let $a \in X$ and $U \in \mathcal{T}$. Choose $V \in \mathcal{S}$ such that $U = V \cap X$. Choose $C \in \mathcal{C}$ such that $a \in C \subseteq V$. Let $B = C \cap X$. Then $B \in \mathcal{B}$ with $a \in B = C \cap X \subseteq V \cap X = U$, as required.

1.28 Theorem: *When Y is a metric space and $X \subseteq Y$ and we give X the metric obtained from Y, the metric topology on X is equal to the subspace topology on X.*

Proof: Let $\mathcal{R}$ be the metric topology on Y, let $\mathcal{T}$ be the metric topology on X, and let $\mathcal{S} = \big\{V \cap X \,\big|\, V \in \mathcal{R}\big\}$ be the subspace topology on X. The set $\mathcal{B}$ of all open balls $B_X(a, r)$ with $a \in X$ and $r > 0$ is a basis for $\mathcal{T}$. By Theorem 1.21, in order to show that $\mathcal{B}$ is also a basis for $\mathcal{S}$, it suffices to show that $\forall\, U \in \mathcal{S}\ \forall\, x \in U\ \exists\, B \in \mathcal{B}\ a \in B \subseteq U$. Let $U \in \mathcal{S}$ and let $a \in U$. Choose $V \in \mathcal{R}$ such that $U = V \cap X$. Since $V \in \mathcal{R}$ with $a \in V$, we can choose $r > 0$ such that $B_Y(a, r) \subseteq V$. Then for $B = B_X(a, r) \in \mathcal{B}$ we have $B = B_X(a, r) = B_Y(a, r) \cap X \subseteq V \cap X = U$, as required.

Continuous Maps

1.29 Definition: Let X and Y be topological spaces. For a map $f : X \to Y$ and an element $a \in X$, we say that f is **continuous at** a when for every open set V in Y, with $f(a) \in V$, there exists an open set U in X with $a \in U \subseteq f^{-1}(V)$. A map $f : X \to Y$ is said to be **continuous** (on X) when $f^{-1}(V)$ is open in X for every open set V in Y. A **homeomorphism** from X to Y is an invertible map $f : X \to Y$ such that both f and its inverse f^{-1} are continuous. We say that X and Y are **homeomorphic**, and we write $X \cong Y$, when there exists a homeomorphism $f : X \to Y$.

1.30 Theorem: *Let $f : X \to Y$ be a map between topological spaces. Then f is continuous (on X) if and only if f is continuous at every point $a \in X$.*

Proof: The proof is left as an exercise.

1.31 Theorem: *Let X and Y be topological spaces. Then a map $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed in X for every closed set B in Y.*

Proof: Note that when $B \subseteq Y$ we have $f^{-1}(B^c) = f^{-1}(B)^c$ (where B^c is the complement of B in Y and $f^{-1}(B)$ is the complement of $f^{-1}(B)$ in X): indeed for $x \in X$ we have

$$x \in f(B)^c \iff x \notin f(B) \iff f(x) \notin B \iff f(x) \in B^c \iff x \in f(B^c).$$

Suppose that f is continuous. Let $B \subseteq Y$ be closed in Y. Then B^c is open in Y so, since f is continuous, $f^{-1}(B^c)$ is open in X, that is $f^{-1}(B)^c$ is open in X, and hence $f^{-1}(B)$ is closed in X.

Suppose, conversely, that $f^{-1}(B)$ is closed in X for every closed set B in Y. Let $V \subseteq Y$ be open in Y. Then V^c is closed in Y, so $f^{-1}(V^c)$ is closed in X, that is $f^{-1}(V)^c$ is closed in X, and hence $f^{-1}(V)$ is open in X. Thus f is continuous.

1.32 Theorem: *Let X and Y be topological spaces, and let $\mathcal{C}$ be a basis for the topology on Y. A map $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is open in X for every $C \in \mathcal{C}$.*

Proof: If f is continuous then of course $f^{-1}(C)$ is open in X for every $C \in \mathcal{C}$ (because the elements in $\mathcal{C}$ are open in Y). Suppose, conversely, that $f^{-1}(C)$ is open in X for every $C \in \mathcal{C}$. Let $V \subseteq Y$ be open in Y. For each $b \in V$, choose $C_b \in \mathcal{C}$ with $b \in C_b \subseteq V$ so that $V = \bigcup_{b\in V} C_b$. Note that $f^{-1}(V) = f^{-1}\big(\bigcup_{b\in V} C_b\big) = \bigcup_{b\in V} f^{-1}(C_b)$: indeed for $x \in X$ we have

$$\begin{aligned} x \in f^{-1}\big(\textstyle\bigcup_{b\in V} C_b\big) &\iff f(x) \in \textstyle\bigcup_{b\in V} C_b \iff \exists\, b\in V\ f(x) \in C_b \\ &\iff \exists\, b\in V\ x \in f^{-1}(C_b) \iff x \in \textstyle\bigcup_{b\in V} f^{-1}(C_b). \end{aligned}$$

Since each $C_b \in \mathcal{C}$, it follows that each of the sets $f^{-1}(C_b)$ is open in X, and hence the union $f^{-1}(V) = \bigcup_{b\in V} f^{-1}(C_b)$ is open in X, as required.

1.33 Theorem: *Let X and Y be topological spaces. If $b \in Y$, then the constant map $f(x) = b$ is continuous. If $X \subseteq Y$ is a subspace, then the inclusion map $f(x) = x$ is continuous.*

Proof: Let $b \in Y$ and let $f : X \to Y$ be the constant map given by $f(x) = b$ for all x. Let $V \subseteq Y$ be open in Y. If $b \in V$ then $f^{-1}(V) = X$ and if $b \notin V$ then $f^{-1}(V) = \emptyset$ and, in either case, $f^{-1}(V)$ is open in X.

Now suppose that $X \subseteq Y$ is a subspace (with X using the subspace topology) and let $f : X \to Y$ be the inclusion map given by $f(x) = x$ for all x. Let $V \subseteq Y$ be open in Y. Then $f^{-1}(V) = V \cap X$, which is open in X (since X is using the subspace topology).

1.34 Theorem: *The composite of two continuous maps is continuous.*

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be continuous and let $h : X \to Z$ be the composite, given by $h(x) = g(f(x))$ for all x. Let $W \subseteq Z$ be open in Z. Since g is continuous, $g^{-1}(W)$ is open in Y and hence, since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X. To complete the proof, we note that $h^{-1}(W) = f^{-1}(g^{-1}(W))$ because for $x \in X$

$$x\in h^{-1}(W) \iff h(x)\in W \iff g(f(x))\in W \iff f(x)\in g^{-1}(W) \iff x\in f^{-1}(g^{-1}(W)).$$

1.35 Theorem: *Let X, Y and Z be topological spaces and let $f : X \to Y$ be continuous. Then*

(1) If $A \subseteq X$ is a subspace then the restriction $f : A \to Y$ is continuous.
(2) If $Y \subseteq Z$ is a subspace then $f : X \to Z$ is continuous, and if $B \subseteq Y$ is a subspace with $f(X) \subseteq B$ then $f : X \to B$ is continuous.

Proof: The proof is left as an exercise.

1.36 Theorem: *Let X and Y be topological spaces and let $f : X \to Y$. Then*

(1) If $X = \bigcup_{k\in K} A_k$ where K is a set and each A_k is open in X, and if each restriction $f : A_k \to Y$ is continuous, then $f : X \to Y$ is continuous.

(2) If $X = A_1\cup A_2\cup\cdots\cup A_n$ where each A_k closed in X, and if each restriction $f : A_k \to Y$ is continuous, then $f : X \to Y$ is continuous.

Proof: We shall prove Part 1 and leave the proof of Part 2 as an exercise. Suppose that $X = \bigcup_{k\in K} A_k$ where each A_k is open in X. For each k, let $f_k : A_k \to Y$ be the restriction of f to A_k. Suppose that each f_k is continuous (with the understanding that A_k is using the subspace topology in X). Let $V \subseteq Y$ be open in Y. Let $U_k = {f_k}^{-1}(V)$, which is open in A_k since f_k is continuous. Note that since U_k is open in A_k, which is using the subspace topology, we can choose an open set W_k in X such that $U_k = W_k \cap A_k$. Since U_k is the intersection of two open sets in X, it follows that U_k is open in X. Note that

$$f^{-1}(V) = \big\{x\in X \,\big|\, f(x)\in V\big\} = \bigcup_{k\in K} \big\{x\in U_k \,\big|\, f(x)\in V\big\} = \bigcup_{k\in K} {f_k}^{-1}(V) = \bigcup_{k\in K} U_k$$

which is open in X.

1.37 Example: Recall, from calculus, that if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ (where X and Y use the standard metric topology) then every elementary function $f : X \to Y$ is continuous.

1.38 Example: Let $a, b > 0$. Then the circle $x^2 + y^2 = 1$ is homeomorphic to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Indeed, a homeomorphism from the circle to the ellipse is given by $f(x, y) = (ax, ay)$ and its inverse is $g(u, v) = \left(\frac{u}{a}, \frac{v}{b}\right)$.

1.39 Example: The real line $\mathbb{R}$ is homeomorphic to the open interval $(0, 1)$. Indeed a homeomorphism $f : \mathbb{R} \to (0, 1)$ is given by $f(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$ and its inverse is $g(y) = \tan\left(\pi\left(y - \frac{1}{2}\right)\right)$.

1.40 Example: Let $\mathbb{S}^n$ be the unit sphere $\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} \middle| \|x\| = 1\right\}$ and let p be the north pole $p = (0, 0, \cdots, 0, 1) \in \mathbb{S}^n$. Then we have $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$. Indeed a homeomorphism is given by the stereographic projection $f : \mathbb{S}^n \setminus \{p\} \to \mathbb{R}^n$, which is defined by

$$f(x) = \left(\tfrac{x_1}{1-x_{n+1}}, \tfrac{x_2}{1-x_{n+1}}, \cdots, \tfrac{x_n}{1-x_{n+1}}\right).$$

Its inverse is the map $g : \mathbb{R}^n \to \mathbb{S}^n \setminus \{p\}$ given by

$$g(y) = \left(\tfrac{2y_1}{\|y\|^2+1}, \tfrac{2y_2}{\|y\|^2+1}, \cdots, \tfrac{2y_n}{\|y\|^2+1}, \tfrac{\|y\|^2-1}{\|y\|^2+1}\right).$$