Chapter 6. Tensor Algebras and Differential Forms

Multilinear Maps

6.1 Definition: Recall that for a vector space U over a field F, we define the **dual space** of U to be the vector space

$$U^* = \big\{\text{linear maps } L : U \to F\big\}.$$

Recall also, that when U is finite dimensional and $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ is a basis for U, we can define linear maps $f_i : U \to F$ for $i = 1, 2, \cdots, n$ by requiring that $f_i(u_j) = \delta_{ij}$, and then $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ is a basis for U^* which is called the **dual basis** to $\mathcal{B}$. We shall sometimes identify the double dual $U^{**} = (U^*)^*$ with U by identifying the element $u \in U$ with the linear map $u : U^* \to F$ given by $u(f) = f(u)$.

6.2 Definition: Let $U_1, U_2, \cdots, U_k$ and V be vector spaces over a field F. A map

$$L : U_1 \times U_2 \times \cdots \times U_k \to V$$

is called k-linear when

$$\begin{aligned} L(u_1, \cdots, t\, u_i, \cdots, u_k) &= t\, L(u_1, \cdots, u_i, \cdots, u_k) \text{ , and} \\ L(u_1, \cdots, v + w, \cdots, u_k) &= L(u_1, \cdots, v, \cdots, u_k) + L(u_1, \cdots, w, \cdots, u_k) \end{aligned}$$

for all indices i, all vectors $u_1, \cdots, u_k, v, w$ in the appropriate vector spaces, and all $t \in F$. When $U_1, U_2, \cdots, U_k$ are finite dimensional, the **tensor product** of $U_1, U_2, \cdots, U_k$ is the vector space

$$U_1 \otimes U_2 \otimes \cdots \otimes U_k = \Big\{\ k\text{-linear maps } L : U_1^* \times U_2^* \times \cdots \times U_k^* \to F \Big\}.$$

For $u_1, u_2, \cdots, u_k$ with each $u_i \in U_i$, we define $(u_1 \otimes u_2 \otimes \cdots \otimes u_k) \in U_1 \otimes U_2 \otimes \cdots \otimes U_k$ by

$$(u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_1, g_2, \cdots, g_k) = g_1(u_1) g_2(u_2) \cdots g_k(u_k),$$

where each $g_i \in U_i^*$.

6.3 Example: The dot product $\bullet : (F^n)^2 \to F$ given by $u \bullet v = v^T u$ is a 2-linear map.

6.4 Example: An inner product $\langle\ ,\ \rangle : U^2 \to \mathbb{R}$ on a vector space U over $\mathbb{R}$ is 2-linear.

6.5 Example: The determinant $D : (F^n)^n \to F$ given by $D(u_1, u_2, \cdots, u_n) = \det(A)$, where $A = (u_1, u_2, \cdots, u_n) \in M_{n \times n}(F)$, is an n-linear map (indeed, it is the unique n-linear map $D : (F^n)^n \to F$ with $D(I) = 1$).

6.6 Example: The generalized cross product $X : (F^n)^{n-1} \to F$ (defined in Appendix 2) is $(n-1)$-linear.

6.7 Theorem: *Let $U_1, U_2, \cdots, U_k$ be finite dimensional vector spaces. For each index i, let $\mathcal{B}_i$ be a basis for U_i. Then the set*

$$\left\{u_1 \otimes u_2 \otimes \cdots \otimes u_k \,\middle|\, \text{each } u_i \in \mathcal{B}_i\right\}$$

is a basis for $U_1 \otimes U_2 \otimes \cdots \otimes U_k$. In particular $\dim(U_1 \otimes U_2 \otimes \cdots \otimes U_k) = \prod_{i=1}^{k} \dim(U_i)$.

Proof: Let $\mathcal{B}_i = \{u_{i1}, u_{i2}, \cdots, u_{i,n_i}\}$ be a basis for U_i and let $\mathcal{F}_i = \{f_{i1}, f_{i2}, \cdots, f_{i,n_i}\}$ be the dual basis for U_i^*. Then for appropriate indices $i_1, i_2, \cdots, i_k$ and $j_1, j_2, \cdots, j_k$ (that is for $1 \leq i_1 \leq n_1$, $1 \leq i_2 \leq n_2$, $\cdots$, $1 \leq i_k \leq n_k$ and similarly for $j_1, j_2, \cdots, j_k$) we have

$$\begin{aligned}(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})(f_{1,j_1}, f_{2,j_2}, \cdots, f_{k,j_k}) &= f_{1,j_1}(u_{1,i_1}) f_{2,j_2}(u_{2,i_2}) \cdots f_{k,i_k}(u_k, i_k) \\ &= \delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_k,j_k} = \begin{cases} 1 \text{ if } i_1 = j_1, i_2 = j_2, \cdots, i_k = j_k \\ 0 \text{ otherwise.} \end{cases}\end{aligned}$$

It follows that the set of elements of the form $(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})$ is linearly independent because if $0 = \alpha = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k}(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots, \otimes u_{k,i_k})$ then for all appropriate choices of indices $j_1, j_2, \cdots, j_k$ we have

$$0 = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k}(u_{1,i_1} \otimes \cdots \otimes u_{k,i_k})(f_{1,j_1}, \cdots, f_{k,j_k}) = a_{j_1 j_2 \cdots j_k}$$

More generally, for $g_i \in U_i^*$ with say $g_i = \sum c_{ij} f_{ij}$ we have

$$\begin{aligned}&(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})(g_1, g_2, \cdots, g_k) \\ &\quad= (u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})\Big(\sum_{j_1} c_{1,j_1} f_{1,j_1}, \sum_{j_2} c_{2,j_2} f_{2,j_2}, \cdots, \sum_{j_k} c_{2,j_2} f_{2,j_2}\Big) \\ &\quad= \sum_{j_1,j_2,\cdots,j_k} c_{1,j_1} c_{2,j_2} \cdots c_{k,j_k} (u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,j_k})(f_{1,j_1}, f_{2,j_2}, \cdots, f_{k,j_k}) \\ &\quad= \sum_{j_1,j_2,\cdots,j_k} c_{1,j_1} c_{2,j_2} \cdots c_{k,j_k} \delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_k,j_k} = c_{1,i_1} c_{2,i_2} \cdots c_{k,i_k}.\end{aligned}$$

It follows that the set of elements of the form $(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})$ spans $U_1 \otimes U_2 \otimes \cdots \otimes U_k$ because given $L \in U_1 \otimes U_2 \otimes \cdots \otimes U_k$, for $g_1, g_2, \cdots, g_k$ with each $g_i \in U_i^*$, with say $g_i = \sum c_{ij} f_{ij}$, we have

$$\begin{aligned}L(g_1, g_2, \cdots, g_k) &= L\Big(\sum_{i_1} c_{1,i_1} f_{1,i_1}, \sum_{i_2} c_{2,i_2} f_{2,i_2}, \cdots, \sum_{i_k} c_{k,i_k} f_{k,i_k}\Big) \\ &= \sum_{i_1,i_2,\cdots,i_k} c_{1,i_1} c_{2,i_2} \cdots c_{k,i_k} L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k}) \\ &= \sum_{i_1,i_2,\cdots,i_k} L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k})(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})(g_1, g_2, \cdots, g_k)\end{aligned}$$

so $L = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k}(u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes u_{k,i_k})$ with $a_{i_1 i_2 \cdots i_k} = L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k})$.

6.8 Example: For finite dimensional vector spaces U and V, there is a natural isomorphism $U^* \otimes V \cong \mathrm{Lin}(U,V)$ obtained by identifying the element $f \otimes v \in U^* \otimes V$ with the linear map $f \otimes v : U \to V$ given by $(f \otimes v)(u) = f(u)v$.

6.9 Remark: For finite dimensional vector spaces $U_1, U_2, \cdots, U_k$ and V, there is a natural isomorphism between the space of k-linear maps $L : U_1 \times U_2 \times \cdots \times U_k \to V$ and the space of linear maps $M : U_1 \otimes U_2 \otimes \cdots \otimes U_k \to V$. This isomorphism sends the k-linear map $L : U_1 \times U_2 \times \ldots \times U_k \to V$ to the liner map $M : U_1 \otimes U_2 \otimes \cdots \otimes U_k \to V$ given by $M(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = L(u_1, u_2, \cdots, u_k)$ for all $u_i \in U_i$.

6.10 Remark: When some of the vector spaces $U_1, U_2, \cdots, U_k$ are infinite dimensional, for vectors $u_1, u_2, \cdots, u_k$ with each $u_i \in U_i$, we can still define the k-linear map

$$u_1 \otimes u_2 \otimes \cdots \otimes u_k : U_1^* \times U_2^* \times \cdots \times U_k^* \to F$$

by

$$(u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_1, g_2, \cdots, g_k) = g_1(u_1)g_2(u_2)\cdots g_k(u_k).$$

When $\mathcal{B}_i$ is a basis for U_i for each i, the set of k-linear maps

$$\mathcal{S} = \left\{ (u_1 \otimes u_2 \otimes \cdots \otimes u_k) \Big| \text{ each } u_i \in \mathcal{U}_i \right\}$$

is linearly independent (but does not span the vector space of all k-linear maps). In this case we define the tensor product $U_1 \otimes U_2 \otimes \cdots \otimes U_k$ to be the span of $\mathcal{S}$.

6.11 Example: We have natural isomorphisms $F[x] \otimes F[x] \cong F[x] \otimes F[y] \cong F[x,y]$. The element $f(x) \otimes g(x) \in F[x] \otimes F[x]$ corresponds to the element $f(x) \otimes g(y) \in F[x] \otimes F[y]$ which corresponds to the element $f(x)g(y) \in F[x,y]$.

6.12 Definition: For $k \in \mathbb{Z}^+$ we let S_k denote the set of all permutations of $\{1, 2, \cdots, k\}$, that is the set of all bijective maps $\pi : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$. Recall that for a permutation $\pi \in S_k$ we denote the **sign** of π by $(-1)^\pi$, in other words $(-1)^\pi = \det(P_\pi)$ where P_π is the $k \times k$ permutation matrix $P_\pi = (e_{\pi(1)}, e_{\pi(2)}, \cdots e_{\pi(k)})$

6.13 Definition: Let U and V be vector spaces over a field F. Let $L : U^k \to V$ be k-linear. We say that L is **symmetric** when

$$L(u_1, \cdots, u_i, \cdots, u_j, \cdots, u_k) = L(u_1, \cdots, u_j, \cdots, u_i, \cdots, u_k)$$

for all indices i, j and all vectors $u_1, u_2, \cdots, u_k \in U$. Equivalently L is symmetric when

$$L(u_1, u_2, \cdots, u_k) = L(u_{\pi(1)}, u_{\pi(2)}, \cdots, u_{\pi(k)})$$

for all vectors $u_1, u_2, \cdots, u_k \in U$ and for every permutation $\pi \in S_k$. We say that L is **alternating** (or **skew-symmetric**) when

$$L(u_1, \cdots, u_i, \cdots, u_j, \cdots, u_k) = -L(u_1, \cdots, u_j, \cdots, u_i, \cdots, u_k)$$

for all indices i, j and all vectors $u_1, u_2, \cdots, u_k \in U$. Equivalently, L is skew-symmetric when

$$L(u_1, u_2, \cdots, u_k) = (-1)^\pi L(u_{\pi(1)}, u_{\pi(2)}, \cdots, u_{\pi(k)})$$

for all vectors $u_1, u_2, \cdots, u_k \in U$ and all permutations $\pi \in S_k$.

k-Forms and Tensor Algebras

6.14 Definition: Let U be a finite dimensional vector space over a field F. For $k \in \mathbb{Z}^+$, a **k-form** (or a **k-tensor**) on U is a k-linear map $L : U^k \to F$. We define the space of **k-forms** on U, the space of **symmetric k-forms** on U, and the space of **alternating k-forms** on U to be

$$\begin{aligned} T^k U &= \bigotimes_{i=1}^{k} U = U \otimes U \otimes \cdots \otimes U = \big\{ k\text{-linear maps } L : (U^*)^k \to F \big\}, \\ S^k U &= \big\{ S \in T^k U \big| S \text{ is symmetric} \big\}, \\ \Lambda^k U &= \big\{ A \in T^k U \big| A \text{ is alternating} \big\}. \end{aligned}$$

We also let $T^0 U = S^0 U = \Lambda^0 U = F$.

6.15 Example: We have $T^1 U = S^1 U = \Lambda^1 U = \big\{ \text{linear maps } L : U^* \to F \big\} = U^{**}$, which we identify with U.

6.16 Definition: Let U be a finite dimensional vector space. For $u_1, u_2, \cdots, u_k \in U$, we defined the **tensor product** $(u_1 \otimes u_2 \otimes \cdots \otimes u_k) \in T^k U$ by

$$(u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_1, g_2, \cdots, g_k) = g_1(u_1) g_2(u_2) \cdots g_k(u_k)$$

where each $g_i \in U^*$. We also define the **symmetric product** $u_1 \odot u_2 \odot \cdots \odot u_k \in S^k U$ by

$$\begin{aligned} (u_1 \odot u_2 \odot \cdots \odot u_k)(g_1, g_2, \cdots, g_k) &= \sum_{\pi \in S_k} (u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_1, g_2, \cdots, g_k) \\ &= \sum_{\pi \in S_k} g_{\pi(1)}(u_1) g_{\pi(2)}(u_2) \cdots g_{\pi(k)}(u_k). \end{aligned}$$

and we define the **exterior product** (or **wedge product**) $u_1 \wedge u_2 \wedge \cdots \wedge u_k \in \Lambda^k U$ by

$$\begin{aligned} (u_1 \wedge u_2 \wedge \cdots \wedge u_k)(g_1, g_2, \cdots, g_k) &= \sum_{\pi \in S_k} (-1)^\pi (u_1 \otimes u_2 \otimes \cdots \otimes u_k)(g_{\pi(1)} g_{\pi(2)} \cdots, g_{\pi(k)}) \\ &= \sum_{\pi \in S_k} (-1)^\pi g_{\pi(1)}(u_1) g_{\pi(2)}(u_2) \cdots g_{\pi(k)}(u_k) \\ &= \det \begin{pmatrix} g_1(u_1) & g_1(u_2) & \cdots & g_1(u_k) \\ g_2(u_1) & g_2(u_2) & \cdots & g_2(u_k) \\ \vdots & \vdots & & \vdots \\ g_k(u_1) & g_k(u_2) & \cdots & g_k(u_k) \end{pmatrix} \end{aligned}$$

6.17 Theorem: *Let U be a finite dimensional vector space. Let $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ be a basis for U. Then*

(1) $\big\{ (u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_k}) \big| 1 \le i_1, i_2, \cdots, i_k \le n \big\}$ *is a basis for* $T^k U$,

(2) $\big\{ (u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_k}) \big| 1 \le i_1 \le i_2 \le \cdots \le i_k \le n \big\}$ *is a basis for* $S^k U$, *and*

(3) $\big\{ (u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}) \big| 1 \le i_1 < i_2 < \cdots < i_k \le n \big\}$ *is a basis for* $\Lambda^k U$.

In particular we have $\dim\big(T^k U\big) = n^k$, $\dim\big(S^k U\big) = \binom{n+k-1}{k}$ *and* $\dim\big(\Lambda^k U\big) = \binom{n}{k}$.

Proof: Part (1) follows immediately from Theorem 6.7. We shall prove Part (3) and leave the proof of Part (2) as an exercise. Let $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ be the dual basis for U^*. Note that

$$\begin{aligned}
\big(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}\big)\big(f_{j_1}, f_{j_2}, \cdots, f_{j_k}\big) &= \det \begin{pmatrix} f_{j_1}(u_{i_1}) & f_{j_1}(u_{i_2}) & \cdots & f_{j_1}(u_{i_k}) \\ \vdots & \vdots & & \vdots \\ f_{j_k}(u_{i_1}) & f_{j_k}(u_{i_2}) & \cdots & f_{j_k}(u_{i_k}) \end{pmatrix} \\
&= \det \begin{pmatrix} \delta_{i_1,j_1} & \delta_{i_1,j_2} & \cdots & \delta_{i_1,j_k} \\ \vdots & \vdots & & \vdots \\ \delta_{i_1,j_k} & \delta_{i_2,j_k} & \cdots & \delta_{i_k,j_k} \end{pmatrix} \\
&= \begin{cases} 0 & \text{if for some } l \text{ we have } i_l \neq j_m \text{ for all } m \\ 0 & \text{if } i_l = i_m \text{ for some } l \neq m \\ (-1)^\pi & \text{if } i_l = j_{\pi(l)} \text{ for all } l \text{ and some } \pi \in S_k. \end{cases}
\end{aligned}$$

In particular, when $I = (i_1, i_2, \cdots, i_k)$ and $J = (j_1, j_2, \cdots, j_k)$ are increasing (that is when $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$) we have

$$\big(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}\big)\big(f_{j_1}, f_{j_2}, \cdots, f_{j_k}\big) = \begin{cases} 0 \text{ if } I = J \\ 1 \text{ if } I \neq J. \end{cases}$$

It follows that the set

$$\mathcal{S} = \big\{u_I = (u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}) \,\big|\, I = (i_1, i_2, \cdots, i_k) \text{ is increasing}\big\}$$

is linearly independent because if $\sum\limits_{I \text{ incr}} a_I u_I = 0$ then for all increasing $J = (j_1, j_2, \cdots, j_k)$ we have

$$0 = \Big(\sum_{I \text{ incr}} a_I u_I\Big)\big(f_{j_1}, f_{j_2}, \cdots, f_{j_k}\big) = a_J.$$

Given $L \in \Lambda^k U$, for each increasing $I = (i_1, i_2, \cdots, i_k)$, let $a_I = L(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k})$. Then for $g_1, g_2, \cdots, g_k \in U^*$ with say $g_j = \sum\limits_i c_{j,i} f_i$, we have

$$\begin{aligned}
L(g_1, g_2, \cdots, g_k) &= L\Big(\sum_{i_1} c_{1,i_1} f_{i_1}, \sum_{i_2} c_{2,i_2} f_{i_2}, \cdots, \sum_{i_k} c_{k,i_k} f_{i_k}\Big) \\
&= \sum_{\text{all } I} \big(c_{1,i_1} c_{2,i_2} \cdots c_{k,i_k}\big) L\big(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k}\big) \\
&= \sum_{I \text{ incr}} \sum_{\pi \in S_k} \big(c_{1,i_{\pi(1)}} c_{2,i_{\pi(2)}} \cdots c_{k,i_{\pi(k)}}\big)(-1)^\pi L\big(f_{1,i_1}, f_{2,i_2}, \cdots, f_{k,i_k}\big) \\
&= \sum_{I \text{ incr}} a_I \sum_{\pi \in S_k} (-1)^\pi c_{1,i_{\pi(1)}} c_{2,i_\pi(2)} \cdots c_{k,i_{\pi(k)}} \\
&= \sum_{I \text{ incr}} a_I \det \begin{pmatrix} c_{1,i_1} & c_{1,i_2} & \cdots & c_{1,i_k} \\ \vdots & \vdots & & \vdots \\ c_{k,i_1} & c_{k,i_2} & \cdots & c_{k,i_k} \end{pmatrix} \\
&= \sum_{I \text{ incr}} a_I \, u_I(g_1, g_2, \cdots, g_k).
\end{aligned}$$

Thus we have $L = \sum\limits_{I \text{ incr}} a_I u_I \in \text{Span}(\mathcal{S})$ and so $\mathcal{S}$ spans $\Lambda^k U$.

6.18 Example: Let $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ and $\mathcal{C} = \{v_1 v_2, \cdots, v_n\}$ be two bases for U. Let $\alpha \in \Lambda^k U$. Say $\alpha = \sum\limits_{I \text{ incr}} a_I u_I = \sum\limits_{J \text{ incr}} b_J v_J$. Determine how a_I and b_J are related.

Solution: Let $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ and $\mathcal{G} = \{g_1, g_2, \cdots, g_n\}$ be the bases for U^* which are dual to $\mathcal{B}$ and $\mathcal{C}$. Let P be the change of basis matrix $P = [I]_{\mathcal{B}}^{\mathcal{C}}$ so that we have $v_j = \sum\limits_i p_{ij} u_i$. Note that

$$f_i(v_j) = f_i\big(\sum_k p_{kj} u_k\big) = \sum p_{kj} f_i(u_k) = \sum_k p_{kj} \delta_{ik} = p_{ij}\,.$$

We have

$$\begin{aligned} a_I = \alpha(f_{i_1}, f_{i_2}, \cdots, f_{i_k}) &= \sum_{J \text{ incr}} b_J v_J(f_{i_1}, f_{i_2}, \cdots, f_{i_k}) \\ &= \sum_{J \text{ incr}} b_J \det \begin{pmatrix} f_{i_1}(v_{j_1}) & f_{i_1}(v_{j_2}) & \cdots & f_{i_1}(v_{j_k}) \\ \vdots & \vdots & & \vdots \\ f_{i_k}(v_{j_1}) & f_{i_k}(v_{j_2}) & \cdots & f_{i_k}(v_{j_k}) \end{pmatrix} \\ &= \sum_{J \text{ incr}} b_J \det \begin{pmatrix} p_{i_1,j_1} & p_{i_1,j_2} & \cdots & p_{i_1,j_k} \\ \vdots & \vdots & & \vdots \\ p_{i_k,j_1} & p_{i_k,j_2} & \cdots & p_{i_k,j_k} \end{pmatrix} \\ &= \sum_{J \text{ incr}} b_J \det P_I^J, \end{aligned}$$

where P_I^J is the matrix obtained from P by selecting rows $i_1, \cdots, i_k$ and columns $j_1, \cdots, j_k$.

6.19 Definition: Given an n-dimensional vector space U, we define vector spaces

$$TU = \bigoplus_{k=0}^{\infty} T^k U \;,\; SU = \bigoplus_{k=0}^{\infty} S^k U \;,\; \Lambda U = \bigoplus_{k=0}^{n} \Lambda^k U.$$

The operations $\otimes$, $\odot$ and $\wedge$, which are defined on basis vectors, determine products on each of the above vector spaces. A vector space with a compatible multiplication operation is called an algebra, so the above three vector spaces, together with their products, are called the **tensor algebra**, the **symmetric algebra**, and the **exterior algebra**.

6.20 Example: If $\alpha \in \Lambda^k U$ and $\beta \in \Lambda^\ell U$ then we have $\alpha \wedge \beta \in \Lambda^{k+\ell} U$. Indeed if $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ is a basis for U and we have $\alpha = \sum\limits_{I \text{ incr}} a_I u_I$ and $\beta = \sum\limits_{J \text{ incr}} b_J u_J$, then

$$\alpha \wedge \beta = \sum_{I \text{ incr}} \sum_{J \text{ incr}} a_I b_J \; u_I \wedge u_J$$

where

$$\begin{aligned} u_I \wedge u_J &= (u_{i_1} \wedge \cdots \wedge u_{i_k}) \wedge (u_{j_1} \wedge \cdots \wedge u_{j_\ell}) \\ &= u_{i_1} \wedge \cdots \wedge u_{i_k} \wedge u_{j_1} \wedge \cdots \wedge u_{j_\ell}. \end{aligned}$$

6.21 Remark: Notice the similarity between the formula in Example 6.18 for the coefficients of an alternating k-form under a change of basis, and the formula in Definition 5.33 for the pullback of a smooth k-form by a smooth map f, which is used in Definition 5.36 to define smooth k-forms on a manifold. We can exploit this similarity to give an alternate algebraic definition for smooth k-forms on manifolds.

An algebraic Definition of Smooth k-forms

6.22 Notation: Let us introduce some notation for tangent vectors, which is commonly used in differential geometry, when we think of the vectors as being differential operators. Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. Let $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$ be a chart on M at p with $\sigma(a) = p$. A tangent vector $X \in T_pM$ determines, and is determined by, the tangent vector $A = (\sigma^{-1})_* X \in T_aU = \mathbb{R}^m$, and we have $X = D\sigma(a)A$. The vector $A \in T_aU = \mathbb{R}^m$ acts (as a differential operator) on a smooth map $g : U \to \mathbb{R}$ by $A(g) = Dg(a)A$. For the vector $A = \sum_{i=1}^m A_i e_i$ (where e_i is the i^{th} standard basis vector) we have

$$A(g) = Dg(a)A = \sum_{i=1}^m A_i \frac{\partial g}{\partial u_i}(a)$$

so that (as a differential operator) we have $A = \sum_{i=1}^m A_i \frac{\partial}{\partial u_i}$. The corresponding vector $X = D\sigma(a)A \in T_pM$ acts (as a differential operator) on a smooth map $f : M \to \mathbb{R}$ by

$$X(f) = A(f\sigma) = \sum_{i=1}^m A_i \frac{\partial(f\sigma)}{\partial u_i}(a).$$

When working with the vector $A \in T_aU = \mathbb{R}^m$, we write $\frac{\partial}{\partial u_i}$ simply as an alternate notation for the i^{th} standard basis vector in $\mathbb{R}^m$, that is $\frac{\partial}{\partial u_i} = e_i$. When working with the vector $X = D\sigma(a)A \in T_pM$, we write $\frac{\partial}{\partial u_i}$ as an alternate notation for the i^{th} column of the Jacobian matrix $D\sigma(a)$, that is $\frac{\partial}{\partial u_i} = D\sigma(a)e_i$. Using this notation, every vector $A \in T_aU = \mathbb{R}^m$ can be written uniquely as $A = \sum_{i=1}^m A_i \frac{\partial}{\partial u_i}$ (where $\frac{\partial}{\partial u_i} = e_i \in T_aU = \mathbb{R}^m$) and the corresponding vector $X = D\sigma(a)A \in T_pM$ is then given by $X = \sum_{i=1}^n A_i \frac{\partial}{\partial u_i}$ (where now $\frac{\partial}{\partial u_i} = D\sigma(a)e_i \in T_pM \subseteq \mathbb{R}^n$).

This notation can be used for any point $a \in U$ and any point $p \in V$, and so it is also used for vector fields. For a smooth vector field $X : M \to \bigcup_{p \in M} T_pM$, the restriction of X to V determines and is determined by the smooth vector field $A : U \to \mathbb{R}^m$ given by $A = (\sigma^{-1})_* X$ so that $X(\sigma(u)) = D\sigma(u)A(u)$ for all $u \in U$. When working with the vector field A on U, we write $\frac{\partial}{\partial u_i}$ to denote the constant vector field $\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i}(u) = e_i$ for all $u \in U$, and when working with the (restriction of the) vector field X on V, we write $\frac{\partial}{\partial u_i}$ to denote the vector field given by $\frac{\partial}{\partial u_i}(\sigma(u)) = D\sigma(u)e_i$ for all $u \in U$. Using this notation, every smooth vector field $A : U \to \mathbb{R}^m$ can be written (uniquely) as

$$A = A(u) = \sum_{i=1}^m A_i(u) \frac{\partial}{\partial u_i} \quad \text{so} \quad A(g)(u) = A(u)(g) = \sum_{i=1}^m A_i(u) \frac{\partial g}{\partial u_i}(u)$$

where $\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i}(u) = e_i$ for all $u \in U$ and each $A_i : U \to \mathbb{R}$ is a smooth map, and the corresponding vector field $X = X(p)$ on V with $X(\sigma(u)) = D\sigma(u)A(u)$ is given by

$$X = X(p) = \sum_{i=1}^m X_i(p) \frac{\partial}{\partial u_i} \quad \text{so} \quad X(f)(p) = X(p)(f) = \sum_{i=1}^m X_i(p) \frac{\partial(f\sigma)}{\partial u_i}(\sigma^{-1}(p))$$

where $\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i}(p) = D\sigma(\sigma^{-1}(p))e_i$ for all $p \in V$ and $X_i(p) = A_i(\sigma^{-1}(p))$.

6.23 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold, let $p \in M$, and let $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$ be a chart on M at p. The dual space $T_p^*M = (T_pM)^*$ of T_pM is called the **cotangent space** of M at p. The basis for T_p^*M which is dual to the basis $\left\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_m}\right\}$ is denoted by $\left\{du_1, \cdots, du_m\right\}$, so du_k is the linear map $du_k : T_pM \to \mathbb{R}$ given by $du_k\left(\frac{\partial}{\partial u_\ell}\right) = \delta_{k,\ell}$. An element $\omega \in \Lambda^k T_p^*M$ is an alternating k-linear map $\omega : (T_pM)^{**} \times \cdots \times (T_pM)^{**} \to \mathbb{R}$. We identify $(T_pM)^{**}$ with T_pM (by identifying the vector $X \in T_pM$ with the linear map $X : T_p^*M \to \mathbb{R}$ given by $X(\alpha) = \alpha(X)$ for all $\alpha \in T_p^*M$) so that an element $\omega \in \Lambda^k T_p^*M$ is an alternating k-linear map $\omega : T_pM \times \cdots \times T_pM \to \mathbb{R}$. The space $\Lambda^k T_p^*M$ has basis $\left\{du_I = du_{i_1} \wedge \cdots \wedge du_{i_k} \,\middle|\, 1 \le i_1 < i_2 < \cdots < i_k \le m\right\}$, so each element $\omega \in \Lambda^k T_p^*M$ can be written uniquely in the form $\omega = \sum\limits_{I \text{ incr}} w_I \, du_I$.

Somewhat confusingly, we use the same notation when working in $T_aU = \mathbb{R}^m$. In this case, $\left\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_m}\right\}$ is another notation for the standard basis for $\mathbb{R}^m$ and $\{du_1, \cdots, du_m\}$ is the dual basis for $T_a^*U = (\mathbb{R}^m)^*$ (so that in fact we have $\frac{\partial}{\partial u_i} = e_i$ and $du_j = e_j^T$), and $\Lambda T_a^*U = \Lambda(\mathbb{R}^m)^*$ is the space of alternating k-linear maps $\alpha : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}$, so each element $\alpha \in \Lambda^k T_a^*U = \Lambda^k(\mathbb{R}^m)^*$ can be written uniquely in the form $\alpha = \sum\limits_{I \text{ incr}} a_I \, du_I$.

Note that the notation used in T_aU is consistent with the notation used in T_pM when we consider $U \subseteq \mathbb{R}^m$ to be a smooth regular submanifold of $\mathbb{R}^m$ and use the identity map $\sigma : U \to U$ (given by $\sigma(u) = u$) as the chart.

6.24 Note: By Definition 6.16, keeping in mind that for $X \in T_pM$ and $\alpha \in T_p^*M$ we have $X(\alpha) = \alpha(X)$, when $\alpha_1, \cdots, \alpha_k \in T_p^*M$ and $X_1, \cdots, X_k \in T_pM$ we have

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)\big(X_1, \cdots, X_k\big) = \det \begin{pmatrix} \alpha_1(X_1) & \cdots & \alpha_1(X_k) \\ \vdots & & \vdots \\ \alpha_k(X_1) & \cdots & \alpha_k(X_k) \end{pmatrix}.$$

In particular,

$$(du_{i_1} \wedge \cdots \wedge du_{i_k})\big(X_1, \cdots, X_k\big) = \det \begin{pmatrix} du_{i_1}(X_1) & \cdots & du_{i_1}(X_k) \\ \vdots & & \vdots \\ du_{i_k}(X_1) & \cdots & du_{i_k}(X_k) \end{pmatrix},$$

and if we write $X_j = \sum\limits_{i=1}^m X_{j,i} \frac{\partial}{\partial u_i}$ then we have $du_\ell(X_j) = X_{j,\ell}$ and hence

$$(du_{i_1} \wedge \cdots \wedge du_{i_k})\big(X_1, \cdots, X_k\big) = \det \begin{pmatrix} X_{1,i_1} & \cdots & X_{k,i_1} \\ \vdots & & \vdots \\ X_{1,i_k} & \cdots & X_{k,i_k} \end{pmatrix}.$$

Also, as in the proof of Theorem 6.17, we have $(du_{i_1} \wedge \cdots \wedge du_{i_k})\left(\frac{\partial}{\partial u_{j_1}}, \cdots, \frac{\partial}{\partial u_{j_k}}\right) = 0$ unless the indices $i_1, \cdots, i_k$ are all distinct, and the indices $j_1, \cdots, j_k$ are a permutation of the indices $i_1, \cdots, i_k$, and when $\pi \in S_k$ and $j_1 = i_{\pi(1)}, j_2 = i_{\pi(2)}, \cdots, j_k = i_{\pi(k)}$ we have

$$(du_{i_1} \wedge \cdots \wedge du_{i_k})\left(\tfrac{\partial}{\partial u_{j_1}}, \cdots, \tfrac{\partial}{\partial u_{j_k}}\right) = (-1)^\pi.$$

6.25 Theorem: *Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold, and let $\sigma : U \to V$ be a chart on M at p. Let $\alpha = \sum\limits_{I \text{ incr}} a_I \, du_I \in \Lambda^k T_p^* M$ and let $\beta = \sum\limits_{J \text{ incr}} b_J \, du_J \in \Lambda^\ell T_p^* M$ so that $\alpha \wedge \beta = \sum\limits_{I,J \text{ incr}} a_I \, b_J \, du_I \wedge du_J \in \Lambda^{k+\ell} T_p^* M$. Then for $X_1, X_2, \cdots, X_{k+\ell} \in T_p M$,*

$$(\alpha \wedge \beta)\big(X_1, \cdots, X_{k+\ell}\big) = \sum_{\tau \in T_{k,\ell}} (-1)^\tau \, \alpha(X_{\tau(1)}, \cdots, X_{\tau(k)}) \, \beta(X_{\tau(k+1)}, \cdots, X_{\tau(k+\ell)}),$$

where $T_{k,\ell}$ is the set of all permutations $\tau \in S_{k+\ell}$ such that $\tau(1) < \tau(2) < \cdots < \tau(k)$ and $\tau(k+1) < \tau(k+2) < \cdots < \tau(k+\ell)$.

Proof: By linearity, it suffices to prove the formula in the case that $\alpha = du_I$ and $\beta = du_J$. By Note 6.24, if we write $X_j = \sum\limits_{i=1}^m X_{j,i} \frac{\partial}{\partial u_i}$ then we have

$$(du_I \wedge du_J)(X_1, \cdots, X_{k+\ell}) = \det \begin{pmatrix} X_{1,i_1} & \cdots & X_{k+\ell,i_1} \\ \vdots & & \vdots \\ X_{1,i_k} & \cdots & X_{k+\ell,i_k} \\ X_{1,j_1} & \cdots & X_{k+\ell,j_1} \\ \vdots & & \vdots \\ X_{1,j_\ell} & \cdots & X_{k+\ell,j_\ell} \end{pmatrix}$$

$$= \sum_{\pi \in S_{k+\ell}} (-1)^\pi \, X_{\pi(1),i_1} \cdots X_{\pi(k),i_k} X_{\pi(k+1),j_1} \cdots X_{\pi(k+\ell),j_\ell}$$

$$= \sum_{\tau \in T_{k,\ell}} \sum_{\mu \in S_k} \sum_{\nu \in S_\ell} (-1)^\tau (-1)^\mu (-1)^\nu X_{\tau(\mu(1)),i_1} \cdots X_{\tau(\mu(k)),i_1} \, X_{\tau(k+\nu(1)),j_1} \cdots X_{\tau(k+\nu(\ell)),j_\ell}$$

$$= \sum_{\tau \in T_{k,\ell}} (-1)^\tau \, \det \begin{pmatrix} X_{\tau(1),i_i} & \cdots & X_{\tau(k),i_1} \\ \vdots & & \vdots \\ X_{\tau(1),i_k} & \cdots & X_{\tau(k),i_k} \end{pmatrix} \det \begin{pmatrix} X_{\tau(k+1),j_1} & \cdots & X_{\tau(k+\ell),j_1} \\ \vdots & & \vdots \\ X_{\tau(k+1),j_\ell} & \cdots & X_{\tau(k+\ell),j_\ell} \end{pmatrix}$$

$$= \sum_{\tau \in T_{k,\ell}} (-1)^\tau \, du_I\big(X_{\tau(1)}, \cdots, X_{\tau(k)}\big) \, du_J\big(X_{\tau(k+1)}, \cdots, X_{\tau(k+\ell)}\big).$$

6.26 Definition: Let $M \subseteq \mathbb{R}^r$ and $N \subseteq \mathbb{R}^s$ be smooth regular submanifolds and let $f : M \to N$ be a smooth map with $f(p) = q$. Recall that the pushforward $f_* : T_p M \to T_q N$ is given by $f_*(\gamma'(0)) = \delta'(0)$ where $\gamma(0) = p$ and $\delta(t) = f(\gamma(t))$. We define the **pullback** $f^* : \Lambda^k T_q^* N \to \Lambda^k T_p^* M$ by

$$(f^* \beta)(X_1, \cdots, X_k) = \beta\big(f_* X_1, \cdots, f_* X_k\big)$$

where $\beta \in \Lambda^k T_q^* N$ and each $X_j \in T_p M$.

6.27 Theorem: *Let $M \subseteq \mathbb{R}^r$ and $N \subseteq \mathbb{R}^s$ be smooth regular submanifolds and let $f : M \to N$ be a smooth map with $f(p) = q$. Let $\sigma : U \subseteq \mathbb{R}^m \to \sigma(U) \subseteq M$ be a chart on M at p with $\sigma(a) = p$, and let $\rho : V \subseteq \mathbb{R}^n \to \rho(V) \subseteq N$ be a chart on N at q. Let $\beta = \sum\limits_{J \text{ incr}} b_J \, dv_J \in \Lambda^k T_q^* N$. Then*

$$f_* \beta = \sum_{I \text{ incr}} a_I \, du_I \quad \text{where} \quad a_I = \sum_{J \text{ incr}} b_J \, \det \tfrac{\partial v_J}{\partial x_I}(a)$$

for $v(u) = (\rho^{-1} f \sigma)(u) = (v_1(u), \cdots, v_n(u))$ so that $\frac{\partial v_J}{\partial u_I}(a) = D(\rho^{-1} f \sigma)(a)^I_J$.

Proof: Let $\alpha = f^*\beta = \sum\limits_{I \text{ incr}} a_I\, du_I$. For $I = (i_1, \cdots, i_k)$, the coefficient a_I is given by

$$\begin{aligned} a_I &= (f^*\beta)\big(\tfrac{\partial}{\partial u_{i_1}}, \cdots, \tfrac{\partial}{\partial u_{i_k}}\big) \\ &= \beta\big(f_*\tfrac{\partial}{\partial u_{i_1}}, \cdots, f_*\tfrac{\partial}{\partial u_{i_k}}\big) \\ &= \sum_{J \text{ incr}} b_J\, dv_J\Big(\sum_{\ell_1=1}^{n} \tfrac{\partial v_{\ell_1}}{\partial u_{i_1}}\, \tfrac{\partial}{\partial v_{\ell_1}}, \cdots, \sum_{\ell_k=1}^{n} \tfrac{\partial v_{\ell_k}}{\partial u_{i_k}}\, \tfrac{\partial}{\partial v_{\ell_k}}\Big) \\ &= \sum_{J \text{ incr}} \sum_{\text{all } L} b_J \tfrac{\partial v_{\ell_1}}{\partial u_{i_1}} \cdots \tfrac{\partial v_{\ell_k}}{\partial u_{i_k}}\, dv_J\big(\tfrac{\partial}{\partial v_{\ell_1}}, \cdots, \tfrac{\partial}{\partial v_{\ell_k}}\big) \end{aligned}$$

Since $dv_J\big(\frac{\partial}{\partial v_{\ell_1}}, \cdots, \frac{\partial}{\partial v_{\ell_k}}\big) = 0$ unless $\ell_1, \cdots, \ell_k$ is a permutation of $i_1, \cdots, i_k$, in which case it is equal to the sign of the permutation, we have

$$\begin{aligned} a_I &= \sum_{J \text{ incr}} \sum_{\pi \in S_k} (-1)^\pi b_J \frac{\partial v_{j_{\pi(1)}}}{\partial u_{i_1}} \cdots \frac{\partial v_{j_{\pi(k)}}}{\partial u_{i_k}} \\ &= \sum_{J \text{ incr}} b_J \det \tfrac{\partial v_J}{\partial u_I}. \end{aligned}$$

6.28 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. A **smooth k-form** on M is a map $\omega : M \to \bigcup\limits_{p \in M} \Lambda^k T_p^* M$ such that for each chart $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$ on M, when we write ω (restricted to V) in the form $\omega(p) = \sum\limits_{I \text{ incr}} w(p)\, du_I$, each of the coefficient functions $w_I : V \subseteq M \to \mathbb{R}$ is smooth (as a map between manifolds), or equivalently, when we write the pullback $\alpha = (\sigma^{-1})^*\omega$, which is a map $\alpha : U \to (\mathbb{R}^m)^*$, in the form $\alpha(u) = \sum\limits_{I \text{ incr}} a_I(u)\, du_I$, each of the coefficient functions $a_I : U \subseteq \mathbb{R}^m \to \mathbb{R}$ is smooth.

6.29 Note: When $\mathcal{A}$ is an atlas on M, a smooth k-form ω on M determines, and is determined by, the smooth k-forms $\sigma^*\omega$ where $\sigma \in \mathcal{A}$: when $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$, the restriction of ω to V, given by $\omega(p) = \sum\limits_{I \text{ incr}} w_I(p)\, du_I$, and the pullback $\alpha = \sigma^*\omega$, given by $\alpha(u) = \sum\limits_{I \text{ incr}} a_I(u)\, du_I$, are related by the formula $a_I(u) = w_I(\sigma(u))$. This is in agreement with our previous Definition 5.36.

6.30 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. When ω is a smooth k-form on M and $X_1, \cdots, X_k$ are smooth vector fields on M, $\omega(X_1, \cdots, X_k)$ is the smooth function on M given by $\omega(X_1, \cdots, X_k)(p) = \omega(p)\big(X_1(p), \cdots, X_k(p)\big)$. When ω is a smooth k-form on M and τ is a smooth ℓ-form on M, the **exterior product** (or the **wedge product**) $\omega \wedge \tau$ is the smooth $(k+\ell)$-form on M given by $(\omega \wedge \tau)(p) = \omega(p) \wedge \tau(p)$. When $f : M \to N$ is a smooth map between manifolds and τ is a smooth k-form on N, the **pullback** of τ by f is the smooth k-form $f^*\beta$ on M given by $(f^*\tau)(p) = f^*(\tau(f(p)))$. We note that Theorem 6.27 shows that the pullback is given by the same formula which we used earlier to define the pullback in Definition 5.33. When ω is a smooth k-form on M, we define the **exterior derivative** of ω to be the smooth $(k+1)$-form $d\omega$ on M such that, for each chart $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$, if we write $\sigma^*\omega = \sum\limits_{I \text{ incr}} a_I(u)\, du_I$ then we have

$$\sigma^*(d\omega) = \sum_{I \text{ incr}} \sum_{i=1}^{m} \tfrac{\partial a_I}{\partial u_i}\, du_i \wedge du_I$$

(in agreement with our earlier Definition 5.15). Note that these definitions are all consistent with our previous definitions, from Chapter 5, and so Stokes' Theorem still holds using these new definitions.