Chapter 5. Integration of Differential Forms

5.1 Remark: There are a number of ways of using integration in differential geometry, some of which we have already encountered. For example, in Definition 4.45, when $M \subseteq \mathbb{R}^n$ is a smooth regular submanifold of $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ is a continuous map, we defined the integral $\int_\gamma f\,dL = \int_{[a,b]} f\,dL$ of f along a smooth curve $\gamma : [a,b] \to M \subseteq \mathbb{R}^n$, and we defined the integral $\int_\sigma f\,dV = \int_R f\,dV$ of f on a closed Jordan region $R \subseteq U_\sigma$ under a chart $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$. In physics, when f represents density (mass per unit length or mass per unit volume), these integrals calculate the mass of a curve or surface. In this chapter, we wish to discuss integration of differential forms, which is related to the integral of a vector field along a curve or across a surface.

5.2 Definition: Let $m = 2$ or 3, let $U \subseteq \mathbb{R}^m$ be open, let $F : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a smooth vector field on U, and let $\gamma : [a,b] \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^m$ be a smooth regular map. Write $T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$ and $dL = |\gamma'(t)|\,dt$, and define the **integral of F along γ** to be

$$\int_\gamma F \bullet T\ dL = \int_{t=a}^b F(\gamma(t)) \bullet \gamma'(t)\,dt.$$

When $m = 2$ and $F(x,y) = \big(P(x,y), Q(x,y)\big)$ and $\gamma(t) = \big(x(t), y(t)\big)$, we also write

$$\int_\gamma F \bullet T\ dL = \int_\gamma P\,dx + Q\,dy = \int_{t=a}^b P\big(x(t),y(t)\big)\,x'(t) + Q\big(x(t),y(t)\big)\,y'(t)\ dt.$$

When $m = 3$, $F(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$ and $\gamma(t) = \big(x(t), y(t), z(t)\big)$, we also use the notation

$$\int_\gamma F \bullet T\ dL = \int_\gamma P\,dx + Q\,dy + R\,dz$$

In physics, when $\gamma(t)$ represents the position of an object which moves along a curve, and the vector field F represents the **force** at each point on the curve, the integral of F along γ measures the **work** done by the force on the object as it moves along the curve.

5.3 Definition: Let $U \subseteq \mathbb{R}^3$ be open, let $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field on U, let R be a closed Jordan region in $\mathbb{R}^2$, let $\sigma : R \subseteq \mathbb{R}^2 \to U \subseteq \mathbb{R}^3$ be a map which extends to a smooth map defined on some open set $W \subseteq \mathbb{R}^2$ with $R \subseteq W$. Write $N = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$ and $dA = |\sigma_u \times \sigma_v|\,du\,dv$, and define the **integral** (or **flux**) **of F across σ** to be

$$\int_\sigma F \bullet N\ dA = \int_R F\big(\sigma(s,t)\big) \bullet (\sigma_s \times \sigma_t)\ ds\,dt.$$

When $F(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$ and $\sigma(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$ we also write

$$\int_\sigma F \bullet N\ dA = \int_\sigma P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$$

$$= \int_R P(\sigma(u,v)) \det\begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} - Q(\sigma(u,v)) \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} + R(\sigma(u,v)) \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\ du\,dv.$$

In physics, when σ represents the shape of a surface in space, and F represents the **velocity field** of a fluid which moves through the surface σ, the integral of F across σ measures the **rate** (the volume per unit time) at which the fluid flows across the surface σ, with the sign of the integral indicating whether the fluid flows in the direction of the normal vector N or in the opposite direction.

5.4 Definition: Let $U\subseteq\mathbb{R}^m$ be open and let $R\subseteq\mathbb{R}^k$ be the closure of an open Jordan region in $\mathbb{R}^k$. A **smooth k-surface** on R in U is a map $\Phi:R\to U$ which extends to a smooth map $\Phi:W\subseteq\mathbb{R}^k\to U$ for some open set $W\subseteq\mathbb{R}^k$ with $R\subseteq W$ (we do not require that Φ is regular or injective). A 0-surface in U is a map from $\{0\}$ to U, which we can identify with a point in U.

Let $U\subseteq\mathbb{R}^m$ be open. A **smooth k-form** on U is an expression of the form

$$\alpha=\sum_I a_I\,du_I$$

where the sum is taken over all **multi-indices** $I=(i_1,i_2,\cdots,i_k)$ with each $i_j\in\{1,\cdots,m\}$, and each $a_I=a_I(u)$ is a smooth function $a_I:U\subseteq\mathbb{R}^m\to\mathbb{R}$, and we write

$$du_I=du_{i_1}\wedge du_{i_2}\wedge\cdots\wedge du_{i_k}$$

(at this stage, we have not ascribed a meaning to this expression, it is merely notation). We also use the convention that a smooth 0-form on U is a smooth function from U to $\mathbb{R}$.

When $\alpha=\sum a_I du_I$ is a smooth k-form on $U\subseteq\mathbb{R}^m$ and $\Phi:R\subseteq\mathbb{R}^k\to U\subseteq\mathbb{R}^m$ is a smooth k-surface on R in U given by $\Phi(t)=u(t)=\big(u_1(t),\cdots,u_m(t)\big)$, we define the **integral** of α on Φ to be

$$\int_\Phi\alpha=\sum_I\int_\Phi a_I(u)du_I=\sum_I\int_R a_I(u(t))\det\left(\tfrac{\partial u_I}{\partial t}(t)\right)\,dt_1\,dt_2\cdots dt_k$$

where

$$\frac{\partial u_I}{\partial t}=\frac{\partial(u_{i_1},u_{i_2},\cdots,u_{i_k})}{\partial(t_1,t_2,\cdots,t_k)}=\begin{pmatrix}\frac{\partial u_{i_1}}{\partial t_1}&\frac{\partial u_{i_1}}{\partial t_2}&\cdots&\frac{\partial u_{i_1}}{\partial t_k}\\ \vdots&&&\vdots\\ \frac{\partial u_{i_k}}{\partial t_1}&\frac{\partial u_{i_k}}{\partial t_2}&\cdots&\frac{\partial u_{i_k}}{\partial t_k}\end{pmatrix},$$

which is the matrix obtained from $Du=D\Phi$ by selecting rows $i_1,i_2,\cdots,i_k$, that is the matrix $\frac{\partial u_I}{\partial t}=Du_I=D\Phi_I$ where $\Phi_I(t)=u_I(t)=\big(u_{i_1}(t),\cdots,u_{i_k}(t)\big)$. When α is a 0-form, that is a smooth function $\alpha:U\to\mathbb{R}$, and Φ is a 0-surface, that is a point $p\in U$, we take the convention that the integral of α on Φ is $\alpha(p)$.

Notice that if two of the indices in I are equal, that is $i_j=i_\ell$ for some $j\neq\ell$, then two of the rows of the matrix $\frac{\partial u_I}{\partial t}$ are equal so that the determinant is zero. Also, notice that if a multi-index J is obtained from I by interchanging two indices, then two of the rows in the matrix $\frac{\partial u_I}{\partial t}$ are interchanged so the integral is multiplied by -1. For this reason, we make the convention that

$$du_i\wedge du_j=-du_j\wedge du_i$$

so that when J is obtained from I by interchanging two indices, we have $du_J=-du_I$. More generally, when π is a **permutation** of $\{1,2,\cdots,k\}$ (that is when π is a bijective map $\pi:\{1,\cdots,k\}\to\{1,\cdots,k\}$), if $I=(i_1,i_2,\cdots,i_k)$ and $J=\pi(I)=\big(i_{\pi(1)},i_{\pi(2)},\cdots,i_{\pi(k)}\big)$, then we have

$$du_J=(-1)^\pi du_I$$

where $(-1)^\pi$ is the **sign** of the permutation π, that is $(-1)^\pi=1$ when π is an even permutation, and $(-1)^\pi=-1$ when π is an odd permutation. With this convention, every smooth k-form on U can be written uniquely in the form

$$\alpha=\sum_{I\text{ incr}}a_I\,du_I$$

where the sum is taken over all strictly increasing multi-indices $I=(i_1,i_2,\cdots,i_k)$ with $1\le i_1<i_2<\cdots<i_k\le m$.

5.5 Example: A smooth 1-form on $U \subseteq \mathbb{R}^m$ is of the form $\alpha = \alpha(u) = \sum_{i=1}^{m} a_i(u)\, du_i$ where each $a_i : U \to \mathbb{R}$ is a smooth map. In particular, writing $u = (x, y, z)$, a smooth 1-form on $U \subseteq \mathbb{R}^3$ is of the form

$$\alpha = a(x,y,z)\, dx + b(x,y,z)\, dy + c(x,y,z)\, dz.$$

When $\Phi = \gamma : [r,s] \to U$ is a smooth 1-surface (that is a smooth curve) on U, and α is the smooth 1-form in U given by $\alpha = a\, dx + b\, dy + c\, dz$ and $F : U \to \mathbb{R}^3$ is the smooth vector field in U given by $F = (a,b,c)$, we have

$$\int_\Phi \alpha = \int_\gamma F \bullet T\, dL = \int_r^s a(\gamma(t))x'(t) + b(\gamma(t))y'(t) + c(\gamma(t))z'(t)\ dt.$$

A smooth 2-form on $U \subseteq \mathbb{R}^m$ is of the form $\alpha = \alpha(u) = \sum_{i<j} a_{i,j}(u)\, du_i \wedge du_j$. In particular, a smooth 2-form on $U \subseteq \mathbb{R}^3$ is of the form

$$\alpha = a(x,y,z)\, dy \wedge dz + b(x,y,z)\, dz \wedge dx + c(x,y,z)\, dx \wedge dy.$$

When $\Phi = \sigma : R \subseteq W \subseteq \mathbb{R}^2 \to U \subseteq \mathbb{R}^3$ is a smooth 2-surface in U, and α is the smooth 2-form on U given by $\alpha = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy$, and F is the smooth vector field in U given by $F = (a,b,c)$, we have

$$\begin{aligned} \int_\Phi \alpha &= \int_\sigma F \bullet N\, dA \\ &= \int_R a(\sigma(u,v)) \det \tfrac{\partial(y,z)}{\partial(u,v)} - b(\sigma(u,v)) \det \tfrac{\partial(x,z)}{\partial(u,v)} + c(\sigma(u,v)) \det \tfrac{\partial(x,y)}{\partial(u,v)}\ du\, dv. \end{aligned}$$

A smooth 3-form in $U \subseteq \mathbb{R}^m$ is of the form $\alpha = \alpha(u) = \sum_{i<j<k} a_{i,j,k}(u)\ du_i \wedge du_j \wedge du_k$. In particular, a smooth 3-form on $U \subseteq \mathbb{R}^3$ is of the form

$$\alpha = a(x,y,z)\ dx \wedge dy \wedge dz.$$

When $\Phi = \phi : R \subseteq W \subseteq \mathbb{R}^3 \to U \subseteq \mathbb{R}^3$ is a smooth 3-surface (for example, if ϕ is a smooth regular change of coordinates from W to U) and α is the smooth 3-form $\alpha = a\ dx \wedge dy \wedge dz$, we have

$$\int_\Phi \alpha = \int_R a(\phi(u,v,w)) \det D\phi(u,v,w)\ du\, dv\, dw,$$

and we note that this is similar to the change of coordinates formula for integration, but using $\det D\phi$ rather than $\big|\det D\phi\big|$.

5.6 Exercise: Let $F(x,y) = (-y, x)$, let $\gamma(t) = (\cos t, \sin t)$ for $0 \le t \le \frac{3\pi}{2}$, and let $\lambda(t) = (2-t, 1+2t)$ for $0 \le t \le 2$. Find the integrals $\int_\gamma F \bullet T\ dL$ and $\int_\lambda F \bullet T\ dL$.

5.7 Exercise: Let $F(x,y) = \left(\frac{-y}{x^2+y^2}\,,\ \frac{x}{x^2+y^2}\right)$ and let $\gamma(t) = \big(r(t)\cos\theta(t)\,,\ r(t)\sin\theta(t)\big)$ for $a \le t \le b$. Find $\int_\gamma F \bullet T\ dL$. In particular, find $\int_\lambda F \bullet T\ dL$ when λ is the line segment from $(2,1)$ to $(1,3)$.

5.8 Exercise: Let $F(x,y,z) = (-xy, z, x^2)$. Find the flux of F across the portion of the paraboloid $z = x^2 + y^2$ which lies above the square given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

5.9 Definition: Let U be an open set in $\mathbb{R}^3$, let $g : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a function and let $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $F = (P, Q, R)$. We write

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \ , \ \nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} \ , \ \nabla \bullet F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \ , \ \nabla \times F = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix} .$$

∇g is called the **gradient** of g, $\nabla \bullet F$ is called the **divergence** of F, and $\nabla \times F$ is called the **curl** of F.

5.10 Remark: We state four theorems from vector calculus, informally and without proof, and in the next section we shall formulate and prove Stokes' Theorem for Chains, which includes all four of these theorems as special cases.

5.11 Theorem: *(The Conservative Field Theorem) Let $U \subseteq \mathbb{R}^3$ be open, let $\gamma : [a, b] \to U$ be a $\mathcal{C}^1$ (or piecewise $\mathcal{C}^1$) curve in U, let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function on U. Then*

$$\int_\gamma \nabla f \bullet T \ dL = f(\gamma(b)) - f(\gamma(a)).$$

5.12 Theorem: *(Green's Theorem) Let $U \subseteq \mathbb{R}^2$ be open, let $R \subseteq U$ be the closure of an open Jordan region in $\mathbb{R}^2$, let $F = (P, Q) : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ vector field on U, and let γ be a $\mathcal{C}^1$ (or piecewise $\mathcal{C}^1$) curve in $\mathbb{R}^2$ which goes once, counterclockwise, around the boundary of R. Then*

$$\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \ dA = \int_\gamma F \bullet T \ dL.$$

5.13 Theorem: *(The Divergence Theorem, or Gauss' Theorem) Let $U \subseteq \mathbb{R}^3$ be open, let $R \subseteq U$ be the closure of an open Jordan region in $\mathbb{R}^3$, let $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field on U, let σ be a $\mathcal{C}^1$ (or piecewise $\mathcal{C}^1$) surface in $\mathbb{R}^3$ which envelops the boundary of R, wrapping once around, with the normal vector N pointing outwards. Then*

$$\int_R \nabla \bullet F \ dV = \int_\sigma F \bullet N \ dA.$$

5.14 Theorem: *(Stokes' Theorem for a Surface in $\mathbb{R}^3$) Let $U \subseteq \mathbb{R}^3$ be an open set, let $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field in U, let σ be a $\mathcal{C}^1$ surface in U, let γ be a $\mathcal{C}^1$ (or piecewise $\mathcal{C}^1$) curve in U which wraps once around the boundary of (the image of) σ in the direction compatible with the right hand rule (when the fingers of the right hand point in the direction of the tangent vector T to the curve, the thumb points in the direction of the normal vector N to the surface). Then*

$$\int_\sigma (\nabla \times F) \bullet N \ dA = \int_\gamma F \bullet T \ dL.$$

5.15 Definition: Let $U \subseteq \mathbb{R}^m$ be open. When $k \in \mathbb{Z}^+$ and α is the smooth k-form on U given by $\alpha = \sum_I a_I \, du_I$, the **exterior derivative** of α is the smooth $(k+1)$-form

$$d\alpha = d\big(\sum_I a_I \, du_I\big) = \sum_I \sum_{i=1}^m \frac{\partial a_I}{\partial u_i} \ du_i \wedge du_I = \sum_{i=1}^m \sum_I \frac{\partial a_I}{\partial u_i} \ du_i \wedge du_{i_1} \wedge \cdots \wedge du_{i_k}.$$

When $a : U \to \mathbb{R}$ is a smooth function and α is the 0-form $\alpha(u) = a(u)$, the exterior derivative of α is

$$d\alpha = da = \sum_{i=1}^m \frac{\partial a}{\partial u_i} \, du_i.$$

5.16 Note: When α is a smooth k-form in an open set $U \subseteq \mathbb{R}^3$, the exterior derivative $d\alpha$ is related to the gradient of a function or the divergence or the curl of a vector field. Let $U \subseteq \mathbb{R}^3$ be open, let $a, b, c : U \to \mathbb{R}$ be smooth functions and write $u = (x, y, z) \in U$.

When α is the smooth 0-form $\alpha = a$, and f is the smooth function $f = a$, we have

$$\begin{aligned} d\alpha &= da = \tfrac{\partial a}{\partial x}\,dx + \tfrac{\partial a}{\partial y}\,dy + \tfrac{\partial a}{\partial z}, dz, \\ \nabla f &= \nabla a = \left(\tfrac{\partial a}{\partial x}, \tfrac{\partial a}{\partial y}, \tfrac{\partial a}{\partial z}\right). \end{aligned}$$

When α is the 1-form $\alpha = a\,dx + b\,dy + c\,dz$ and F is the vector field $F = (a, b, c)$ we have

$$\begin{aligned} d\alpha &= \left(\tfrac{\partial c}{\partial y} - \tfrac{\partial b}{\partial z}\right) dy \wedge dz + \left(\tfrac{\partial a}{\partial z} - \tfrac{\partial c}{\partial x}\right) dz \wedge dx + \left(\tfrac{\partial b}{\partial x} - \tfrac{\partial a}{\partial y}\right) dx \wedge dy, \\ \nabla \times F &= \left(\tfrac{\partial c}{\partial y} - \tfrac{\partial b}{\partial x}\,,\ \tfrac{\partial a}{\partial z} - \tfrac{\partial c}{\partial x}\,,\ \tfrac{\partial b}{\partial x} - \tfrac{\partial a}{\partial y}\right). \end{aligned}$$

When α is the smooth 2-form $\alpha = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$ and F is the smooth vector field $F = (a, b, c)$, we have

$$\begin{aligned} d\alpha &= \left(\tfrac{\partial a}{\partial x} + \tfrac{\partial b}{\partial y} + \tfrac{\partial c}{\partial z}\right) dx \wedge dy \wedge dz, \\ \nabla \cdot F &= \tfrac{\partial a}{\partial x} + \tfrac{\partial b}{\partial y} + \tfrac{\partial c}{\partial z}\,. \end{aligned}$$

5.17 Note: By comparing the formulas in the above note to the statements of the four theorems from vector calculus, one sees that the conclusions of all four vector calculus theorems can be written in the form

$$\int_\Phi d\alpha = \int_{\partial\Phi} \alpha$$

where Φ is a curve, or a surface, or a region in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $\partial\Phi$ denotes the boundary of Φ (which we have not yet formally defined), and α is an appropriately chosen k-form. For example, for Green's Theorem we use $\alpha = P\,dx + Q\,dy$ with $d\alpha = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$.

5.18 Exercise: Let γ be a curve which goes once around the circle $x^2 + y^2 = 1$, let R be the disc $R = \left\{(x, y)\middle| x^2 + y^2 \leq 1\right\}$, and let $F(x, y) = (x^2 y, -xy^2)$. Verify that the conclusion of Green's Theorem holds.

5.19 Exercise: Let R be the tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$, let σ be the boundary surface of R (which consists of four triangles), and let $f(x, y, z) = xy + z^2$. Verify the conclusion of Gauss' Theorem.

5.20 Exercise: Let γ be a curve whose image is given by $z = x^2$ and $x^2 + y^2 = 1$, let σ be a surface whose image is given by $z = x^2$ with $x^2 + y^2 \leq 1$, and let $F(x, y, z) = (y, -x, z^2)$. Verify the conclusion of Stokes' Theorem.

5.21 Exercise: Find the integral of F along γ when $F(x, y) = \left(x - y^3\,,\ x^3 + y^3\right)$ and γ is the boundary curve of the quarter-disc given by $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$.

5.22 Exercise: Find the flux of F across σ when $F(x, y, z) = \left(xy^2\,,\ x^2 y\,,\ (x^2+y^2)z^2\right)$ and σ is the boundary surface of the cylinder given by $(x, y, z) = \left(\sin t, 0, \cos t\right)$ for $0 \leq t \leq 2\pi$.

5.23 Exercise: Find the integral of F along γ when F is the vector field given by $F(x, y, z) = \left(x^2 z + \sqrt{x^3 + x^2 + 2}\,,\ xy\,,\ xy + \sqrt{z^3 + z^2 + 2}\right)$ and γ goes once around the circle given by $y = 0$ and $x^2 + z^2 = 1$.

5.24 Definition: Let $a_0, a_1, \cdots, a_k \in \mathbb{R}^m$. The **convex hull** of $\{a_0, a_1, \cdots, a_k\}$ is the set

$$[a_0, a_1, \cdots, a_k] = \left\{ \sum_{i=0}^{k} s_i a_i \,\middle|\, \text{each } s_i \geq 0\,,\ \sum_{i=0}^{k} s_i = 1 \right\}.$$

Note that if we let $u_i = a_i - a_0$ for $1 \leq i \leq k$ then

$$[a_0, a_1, \cdots, a_k] = a_0 + \text{Span}\{u_1, \cdots, u_k\} = \left\{ a_0 + \sum_{i=1}^{k} t_i u_i \,\middle|\, \text{each } t_i \geq 0\,,\ \sum_{i=1}^{k} t_i \leq 1 \right\}.$$

We say that the ordered $(k+1)$-tuple $(a_0, a_1, \cdots, a_k)$ is **affinely independent** when the k-tuple $(u_1, u_2, \cdots, u_k)$ is linearly independent. In this case, verify that the coefficients s_i (or the coefficients t_i) for an element $u \in [a_0, a_1, \cdots, a_k]$ are uniquely determined.

An **affine map** from $\mathbb{R}^k$ to $\mathbb{R}^\ell$ is a map $F : \mathbb{R}^k \to \mathbb{R}^\ell$ of the form $F(x) = p + Ax$ for some $p \in \mathbb{R}^\ell$ and some $\ell \times k$ matrix A. Verify, as an exercise, that when $(a_0, a_1, \cdots, a_k)$ is affinely independent in $\mathbb{R}^k$ and $b_0, b_1, \cdots, b_k \in \mathbb{R}^\ell$, there is a unique affine map $F : \mathbb{R}^k \to \mathbb{R}^\ell$ with $F(a_i) = b_i$ for all i, namely the map given by $F\big(\sum_{i=0}^k s_i a_i\big) = \sum_{i=1}^k s_i b_i$ where each $s_i \geq 0$ with $\sum_{i=0}^k s_i = 1$.

A k-**simplex** in $\mathbb{R}^m$ is a set of the form $[a_0, a_1, \cdots, a_k]$ for some affinely independent $(k+1)$-tuple $(a_0, a_1, \cdots, a_k)$ of elements $a_i \in \mathbb{R}^m$. The **standard** k-**simplex** in $\mathbb{R}^k$ is the simplex

$$\Delta^k = [e_0, e_1, \cdots, e_k] \subseteq \mathbb{R}^k$$

where $e_0 = 0$ and e_j is the j^{th} standard basis vector in $\mathbb{R}^k$ for $1 \leq j \leq k$.

5.25 Definition: Let $U \subseteq \mathbb{R}^m$ be an open set. A **smooth** k-**simplex** in U is a smooth k-surface on Δ^k in U, that is a smooth map $\Phi : \Delta^k \subseteq \mathbb{R}^k \to U \subseteq \mathbb{R}^m$ which extends to a smooth map $\Phi : W \subseteq \mathbb{R}^k \to U \subseteq \mathbb{R}^m$ for some open set $W \subseteq \mathbb{R}^k$ with $\Delta^k \subseteq W$. A **smooth** k-**chain** in U is a formal finite sum

$$\Psi = \sum_{i=1}^{\ell} c_i \Phi_i$$

where each $c_i \in \mathbb{Z}$ and each Φ_i is a smooth k-simplex. If two of the k-simplices are equal, say if $\Phi_i = \Phi_j = \Phi$ with $i \neq j$, we can write $c_i\Phi_i + c_j\Phi_j$ as $(c_i + c_j)\Phi$. If the smooth k-simplices Φ_i are all distinct, then the coefficients c_i in the sum are uniquely determined. We add smooth k-chains in the natural way: if $\Psi = \sum_{i=1}^{\ell} c_i \Phi_i$ and $\Theta = \sum_{i=1}^{\ell} d_i \Phi_i$ then $\Psi + \Theta = \sum_{i=1}^{\ell} (c_i + d_i)\, \Phi_i$ (where, if the set of smooth k-chains Φ_i which occur in the sum which represents Ψ is not the same as the set of smooth k-chains which occurs in Θ, we simply take the union of the two sets of k-chains and represent both Ψ and Θ in terms of the k-chains in the union, with some of the coefficients being zero). We remark that students familiar with free abelian groups will recognize that the set of all smooth k-chains is the **free abelian group** generated by the set of smooth k-simplices.

5.26 Definition: When $\alpha = \sum a_I \, dx_I$ is a smooth k-form on an open set $U \in \mathbb{R}^m$ and $\Psi = \sum_{i=1}^{\ell} c_i \Phi_i$ is a smooth k-chain in U (where each Φ_i is a smooth k-simplex in U), we define the **integral** of α on Ψ to be

$$\int_\Psi \alpha = \sum_{i=1}^{\ell} c_i \int_{\Phi_i} \alpha\,.$$

5.27 Definition: For $0 \leq j \leq k+1$, the j^{th} **face map** on Δ^k (or in Δ^{k+1}) is the affine map

$$F_j : \Delta^k \subseteq \mathbb{R}^k \to \Delta^{k+1} \subseteq \mathbb{R}^{k+1}$$

with $F_j(e_i) = e_i$ for $i < j$ and $F_j(e_i) = e_{i+1}$ for $i \geq j$. Note that F_j sends the standard simplex $\Delta^k = [e_0, e_1, \cdots, e_k] \subseteq \mathbb{R}^k$ to the simplex

$$F_j(\Delta^k) = [e_0, \cdots, \widehat{e_j}, \cdots, e_{k+1}] = [e_0, e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_{k+1}] \subseteq \Delta^{k+1} \subseteq \mathbb{R}^{k+1},$$

where the **hat** symbol in the term $\widehat{e_j}$ indicates that the entry e_j is omitted.

For $k \geq 0$, the **boundary** of the smooth $(k+1)$-simplex $\Phi : \Delta^{k+1} \to U \subseteq \mathbb{R}^m$ is the smooth k-chain

$$\partial \Phi = \sum_{j=0}^{k+1} (-1)^j \Phi F_j$$

and the **boundary** of the smooth $(k+1)$-chain $\Psi = \sum_{i=1}^{\ell} c_i \Phi_i$ is the k-chain $\partial \Psi = \sum_{i=1}^{\ell} c_i \, \partial \Phi_i$.

5.28 Lemma: *Let $U \subseteq \mathbb{R}^m$ be open, let $\Phi : \Delta^{k+1} \to U \subseteq \mathbb{R}^m$ be a smooth $(k+1)$-simplex, write $\Phi(t) = u(t) = \big(u_1(t), \cdots, u_m(t)\big)$, and let $I = (i_1, \cdots, i_k)$ be a multi-index. Then*

$$\sum_{j=1}^{k+1} (-1)^{j+1} \tfrac{\partial}{\partial t_j} \left(\det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})} \right) = 0.$$

Proof: The determinant $\det \frac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}$ is a sum of terms of the form

$$\pm \tfrac{\partial u_{i_1}}{\partial t_{\pi(1)}} \tfrac{\partial u_{i_2}}{\partial t_{\pi(2)}} \cdots \tfrac{\partial u_{i_k}}{\partial t_{\pi(k)}}$$

where π is a bijective map from $\{1, 2, \cdots, k\}$ to $\{1, \cdots, \widehat{j}, \cdots, k+1\}$, and so the derivative $\frac{\partial}{\partial t_j} \det \frac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}$ is a sum of terms of the form

$$\pm \tfrac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell} \cdot \tfrac{\partial u_{i_1}}{\partial t_{\pi(1)}} \cdots \widehat{\tfrac{\partial u_{i_n}}{\partial t_{\pi(n)}}} \cdots \tfrac{\partial u_{i_k}}{\partial t_{\pi(k)}}$$

where $\pi(n) = \ell$. When the sum $\sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial}{\partial t_j} \det \frac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}$ is expanded, the terms involving $\frac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell}$ occur in $(-1)^{j+1} \frac{\partial}{\partial t_j} \det \frac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}$ and $(-1)^{\ell+1} \frac{\partial}{\partial t_\ell} \det \frac{\partial(u_{i_1}, \cdots, u_{i_\ell})}{\partial(t_1, \cdots, \widehat{t_\ell}, \cdots, t_{k+1})}$. Fix j, ℓ with $j < \ell$. Since $j < \ell$, the $(\ell-1)^{\text{st}}$ column of $\frac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}$ involves $\frac{\partial u_{i_n}}{\partial t_\ell}$, and by expanding the derivative along this column gives

$$\begin{aligned}
(-1)^{j+1} \tfrac{\partial}{\partial t_j} \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})} &= (-1)^{j+1} \tfrac{\partial}{\partial t_j} \left(\sum_{n=1}^{k} (-1)^{n+\ell-1} \tfrac{\partial u_{i_n}}{\partial t_\ell} \det \tfrac{\partial(u_{i_1}, \cdots, \widehat{u_{i_n}}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, \widehat{t_\ell}, \cdots, t_{k+1}} \right) \\
&= \sum_{n=1}^{k} (-1)^{j+\ell+n} \tfrac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell} \det \tfrac{\partial(u_{i_1}, \cdots, \widehat{u_{i_n}}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, \widehat{t_\ell}, \cdots, t_{k+1})} + E
\end{aligned}$$

where $E = \sum_{n=1}^{k} (-1)^{j+\ell+n} \frac{\partial u_{i_n}}{\partial t_\ell} \frac{\partial}{\partial t_j} \det \frac{\partial(u_{i_1}, \cdots, \widehat{u_{i_n}}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, \widehat{t_\ell}, \cdots, t_{k+1})}$, which does not involve $\frac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell}$. Similarly, expanding the determinant $\det \frac{\partial(u_{i_1}, \cdots, u_{i_\ell})}{\partial(t_1, \cdots, \widehat{t_\ell}, \cdots, t_{k+1})}$ along the j^{th} column shows that

$$(-1)^{\ell+1} \tfrac{\partial}{\partial t_\ell} \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_\ell}, \cdots, t_{k+1})} = \sum_{n=1}^{k} (-1)^{j+\ell+n+1} \tfrac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell} \det \tfrac{\partial(u_{i_1}, \cdots, \widehat{u_{i_n}}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, \widehat{t_\ell}, \cdots, t_{k+1})} + F$$

where F does not involve $\frac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell}$, and so all the terms which involve $\frac{\partial^2 u_{i_n}}{\partial t_j \partial t_\ell}$ cancel.

5.29 Theorem: *(Stoke's Theorem for Chains in $\mathbb{R}^m$) Let α be a smooth k-form on $U \subseteq \mathbb{R}^m$ and let Ψ be a smooth $(k+1)$-chain in U. Then*

$$\int_\Psi d\alpha = \int_{\partial\Psi} \alpha.$$

Proof: By linearity, it suffices to consider a k-form of the form $\alpha = \alpha(u) = a(u)\, du_I$ for some fixed multi-index $I = (i_1, i_2, \cdots, i_k)$ and a $(k+1)$-chain of the form $\Psi = \Phi$ where Φ is a single smooth $(k+1)$- simplex. Writing $\Phi(t) = u(t) = (u_1(t), \cdots, u_m(t))$, expanding the determinant along the top row, and making use of the above lemma, we have

$$\begin{aligned}
\int_\Psi d\alpha &= \int_\Phi \sum_{i=1}^m \tfrac{\partial a}{\partial u_i}\, du_i \wedge du_{i_1} \wedge \cdots \wedge du_{i_k} \\
&= \int_{\Delta^{k+1}} \sum_{i=1}^m \tfrac{\partial a}{\partial u_i}(\Phi(t)) \cdot \det \tfrac{\partial(u_i, u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, t_2, \cdots, t_{k+1})}(t)\; dt_1 dt_2 \cdots dt_{k+1} \\
&= \int_{\Delta^{k+1}} \sum_{i=1}^m \tfrac{\partial a}{\partial u_i}(\Phi(t)) \cdot \sum_{j=1}^{k+1} (-1)^{j+1} \tfrac{\partial u_i}{\partial t_j}(t)\; \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(t)\; dt_1 dt_2 \cdots dt_{k+1} \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \int_{\Delta^{k+1}} \sum_{i=1}^m \tfrac{\partial a}{\partial u_i}(\Phi(t))\, \tfrac{\partial u_i}{\partial t_j}(t) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(t)\; dt_1 dt_2 \cdots dt_{k+1} \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \int_{\Delta^{k+1}} \tfrac{\partial(a\Phi)}{\partial t_j}(t) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(t)\; dt_1 dt_2 \cdots dt_{k+1} \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \int_{\Delta^{k+1}} \tfrac{\partial}{\partial t_j}\Big((a\Phi) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}\Big)(t)\; dt_1 dt_2 \cdots dt_{k+1} \text{ (by the lemma)} \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \int\limits_{(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1}) \in \Delta^k} \left[\Big((a\Phi) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}\Big)(t)\right]_{t_j=0}^{1-\sum_{i\neq j} t_i} dt_1 \cdots \widehat{dt_j} \cdots dt_{k+1} \\
&= A + B
\end{aligned}$$

where

$$\begin{aligned}
A &= \sum_{j=1}^{k+1} (-1)^{j+1} \int_{r\in\Delta^k} \Big((a\Phi) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}\Big)(r_1, \cdots, r_{j-1}, 1-\Sigma r_i\,, r_j \cdots, r_k)\; dr_1 \cdots dr_k \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \int_{s\in\Delta^k} \Big((a\Phi) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}\Big)(1-\Sigma s_i\,,\, s_1, s_2, \cdots, s_k)\; ds_1 \cdots ds_k,
\end{aligned}$$

since for $r = \phi(s) = \big(1 - \Sigma s_i\,,\, s_1, \cdots, s_{j-2}, \widehat{s_{j-1}}, s_j, \cdots, s_k\big)$ we have $\det D\phi = \pm 1$, and

$$B = \sum_{j=1}^{k+1} (-1)^j \int_{s\in\Delta^k} \Big((a\Phi) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}\Big)(s_1, \cdots, s_{j-1}, 0, s_j \cdots, s_k)\; ds_1 \cdots ds_k.$$

Note that in the above integrals, $\frac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}$ denotes the $k \times k$ matrix obtained from the $m \times (k+1)$ matrix $D\Phi$ by selecting the rows $i_1, i_2, \cdots, i_k$ and removing the j^{th} column.

On the other hand, we have

$$\int_{\partial\Psi} \alpha = \sum_{j=0}^{k+1} (-1)^j \int_{\Phi F_j} \alpha = \int_{\Phi F_0} \alpha + \sum_{j=1}^{k+1} (-1)^j \int_{\Phi F_j} \alpha\,.$$

To complete the proof, we shall show that $\displaystyle\int_{\Phi F_0} \alpha = A$ and $\displaystyle\sum_{j=1}^{k+1} \int_{\Phi F_j} \alpha = B$.

Writing $\Phi F_0(s) = v(s) = \big(v_1(s), \cdots, v_m(s)\big)$, we have

$$\int_{\Phi F_0} \alpha = \int_{s\in\Delta^k} a((\Phi F_0)(s)) \cdot \det \tfrac{\partial(v_{i_1}, v_{i_2}, \cdots, v_{i_k})}{\partial(s_1, s_2, \cdots, s_k)}(s)\ ds_1 \cdots ds_k.$$

The map F_0 is the affine map which sends $[e_0, \cdots, e_k]$ to $[e_1, e_2, \cdots, e_{k+1}]$, which is given by

$$F_0(s_1, \cdots, s_k) = e_1 + \sum_{j=1}^{k} s_j(e_{j+1} - e_1) = \big(1 - \Sigma s_i\ ,\ s_1, s_2, \cdots, s_k\big),$$

and its Jacobian matrix is $DF_0 = \left(\begin{smallmatrix} -1 \cdots -1 \\ I \end{smallmatrix}\right)$ where I is the $k\times k$ identity matrix. The matrix $\frac{\partial(v_{i_1}, \cdots, v_{i_k})}{\partial(s_1, \cdots, s_k)}$ is obtained from the matrix $D(\Phi F_0)$ by selecting rows $i_1, i_2, \cdots, i_k$, so it is equal to the matrix $D(\Phi_I F_0)$ where write $\Phi_I(t) = u_I(t) = \big(u_{i_1}(t), \cdots u_{i_k}(t)\big)$. Thus, denoting the columns of the matrix $D\Phi_I$ by $\frac{\partial \Phi_I}{\partial t_j}$, and using the fact that the determinant is a linear function of the columns, we have

$$\begin{aligned}
\det \tfrac{\partial(v_{i_1}, \cdots, v_{i_k})}{\partial(s_1, \cdots, s_k)}(s) &= \det D(\Phi_I F_0)(s) = \det\Big(D\Phi_I(F_0 s) \cdot DF_0(s)\Big) \\
&= \det\Big(\tfrac{\partial \Phi_I}{\partial t_1}, \tfrac{\partial \Phi_I}{\partial t_2}, \cdots, \tfrac{\partial \Phi_I}{t_{k+1}}\Big)(F_0 s) \cdot \begin{pmatrix} -1 & \cdots & -1 \\ & I & \end{pmatrix} \\
&= \det\Big(\tfrac{\partial \Phi_I}{\partial t_2} - \tfrac{\partial \Phi_I}{\partial t_1}\ ,\ \tfrac{\partial \Phi_I}{\partial t_3} - \tfrac{\partial \Phi_I}{\partial t_1}\ ,\ \cdots\ ,\ \tfrac{\partial \Phi_I}{\partial t_{k+1}} - \tfrac{\partial \Phi_I}{\partial t_1}\Big)(F_0 s) \\
&= \det\Big(\tfrac{\partial \Phi_I}{\partial t_2}, \tfrac{\partial \Phi_I}{\partial t_3}, \cdots, \tfrac{\partial \Phi_I}{\partial t_{k+1}}\Big) - \sum_{j=2}^{k+1} \det\Big(\tfrac{\partial \Phi_I}{\partial t_2}, \cdots, \tfrac{\partial \Phi_I}{\partial t_{j-1}}, \tfrac{\partial \Phi_I}{\partial t_1}, \tfrac{\partial \Phi_I}{\partial t_{j+1}}, \cdots, \tfrac{\partial \Phi_I}{t_{k+1}}\Big) \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \det\Big(\tfrac{\partial \Phi_I}{\partial t_1}, \cdots, \widehat{\tfrac{\partial \Phi_I}{\partial t_j}}, \cdots, \tfrac{\partial \Phi_I}{\partial t_{k+1}}\Big)(F_0 s) \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(F_0 s).
\end{aligned}$$

Thus we have

$$\int_{\Phi F_0} \alpha = \int_{s\in\Delta^k} (a\Phi)(F_0 s) \cdot \sum_{j=1}^{k+1} (-1)^{j+1} \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(F_0 s)\ ds_1 \cdots ds_k = A.$$

Fix j with $1 \le j \le k+1$ and write $\Phi F_j(s) = v(s) = \big(v_1(s), \cdots, v_m(s)\big)$. Then we have

$$\int_{\Phi F_j} \alpha = \int_{s\in\Delta^k} a(\Phi F_j(s)) \cdot \det \tfrac{\partial(v_{i_1}, \cdots, v_{i_k})}{\partial(s_1, \cdots, s_k)}(s)\ ds_1 \cdots ds_k.$$

The map F_j is the affine map which sends $[e_0, \cdots, e_k]$ to $[e_0, \cdots, \widehat{e_j}, \cdots, e_{k+1}]$, which is given by

$$F_j(s_1, \cdots, s_k) = \big(s_1, \cdots, s_{j-1}, 0, s_j, \cdots, s_k\big),$$

and its Jacobian matrix $DF_j(s)$ is the matrix obtained from the $(k+1)\times(k+1)$ identity matrix by removing the j^{th} column. Multiplying a matrix on the right by $DF_j(s)$ removes the j^{th} column from the matrix and so, writing $\Phi_I(t) = \big(u_{i_1}(t), \cdots, u_{i_k}(t)\big)$, we have

$$\tfrac{\partial(v_{i_1}, \cdots, v_{i_k})}{\partial(s_1, \cdots, s_k)} = D(\Phi_I F_j)(s) = D\Phi_I(F_j s) \cdot DF_j(s) = \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(F_j s).$$

Thus

$$\sum_{j=1}^{k+1} \int_{\Phi F_j} \alpha = \sum_{j=1}^{k+1} \int_{s\in\Delta^k} (a\Phi)(F_j s) \cdot \det \tfrac{\partial(u_{i_1}, \cdots, u_{i_k})}{\partial(t_1, \cdots, \widehat{t_j}, \cdots, t_{k+1})}(F_j s)\ ds_1 \cdots ds_k = B.$$

5.30 Notation: For an $n\times m$ matrix $A\in M_{n\times m}(\mathbb{R})$ and for multi-indices $I=(i_1,\cdots,i_k)$ and $J=(j_1,\cdots,j_\ell)$, write A_I for the $k\times m$ matrix obtained from A by selecting the rows $i_1,\cdots,i_k$, and write A^J for the $n\times\ell$ matrix obtained from A by selecting the columns $j_1,\cdots,j_\ell$, and let A_I^J be the $k\times\ell$ matrix $A_I^J=(A_I)^J=(A^J)_I$. We remark that if $R_I=(e_{i_1},\cdots,e_{i_k})^T\in M_{k\times n}(\mathbb{R})$ and $C_J=(e_{j_1},\cdots,e_{j_\ell})\in M_{m\times\ell}(\mathbb{R})$, then we have $A_I=R_IA$, $A^J=AC_J$ and $A_I^J=R_IAC^J$.

5.31 Theorem: *(The Cauchy-Binet Determinant Formula) Let $k,m\in\mathbb{Z}^+$ with $k\le m$, let $A\in M_{k\times m}(\mathbb{R})$ and let $B\in M_{m\times k}(\mathbb{R})$. Then*

$$\det(AB)=\sum_{J\ \text{incr}}\det A^J\cdot\det B_J$$

where the sum is taken over all stricly increasing multi-indices $J=(j_1,\cdots,j_k)$.

Proof: Let P be the set of bijective maps $\pi:\{1,2,\cdots,k\}\to\{1,2,\cdots,k\}$, and for $\pi\in P$ let $(-1)^\pi$ be the sign of π. Then

$$\begin{aligned}\det(AB)&=\sum_{\pi\in P}(-1)^\pi(AB)_{1,\pi(1)}(AB)_{2,\pi(2)}\cdots(AB)_{k,\pi(k)}\\&=\sum_{\pi\in P}\Big(\sum_{i_1=1}^m A_{1,i_1}B_{i_1,\pi(1)}\Big)\Big(\sum_{i_2=1}^m A_{2,i_2}B_{i_2,\pi(2)}\Big)\cdots\Big(\sum_{i_k=1}^m A_{k,i_k}B_{i_k,\pi(k)}\Big)\\&=\sum_{\text{all }I}\big(A_{1,i_1}A_{2,i_2}\cdots A_{k,i_k}\big)\sum_{\pi\in P}(-1)^\pi\big(B_{i_1,\pi(1)}B_{i_2,\pi(2)}\cdots B_{i_k,\pi(k)}\big)\\&=\sum_{\text{all }I}\big(A_{1,i_1}A_{2,i_2}\cdots A_{k,i_k}\big)\cdot\det B_I\end{aligned}$$

When two of the entries of a multi-index I are equal, we have $\det B_I=0$, so the sum can be taken over all mult-indices with distict entries. And each multi-index $I=(i_1,\cdots,i_k)$ with distinct entries, is (uniquely) of the form $I=\pi(J)=(j_{\pi(1)},\cdots,j_{\pi(k)})$ for some strictly increasing multi-index $J=(j_1,\cdots,j_k)$ and some permutation $\pi\in P$. Thus

$$\begin{aligned}\det(AB)&=\sum_{\text{all }I}\big(A_{1,i_1}A_{2,i_2}\cdots A_{k,i_k}\big)\cdot\det B_I\\&=\sum_{J\ \text{incr}}\sum_{\pi\in P}(-1)^\pi\big(A_{1,j_{\pi(1)}}A_{2,j_{\pi(2)}}\cdots A_{k,j_{\pi(k)}}\big)\cdot\det B_J\\&=\sum_{J\ \text{incr}}\det A^J\cdot\det B_J.\end{aligned}$$

5.32 Remark: In the case $k=m$, the above theorem gives $\det(AB)=\det A\cdot\det B$.

5.33 Definition: Let $U\subseteq\mathbb{R}^m$ and $V\subseteq\mathbb{R}^n$ be open, let $f:U\to V$ be smooth, and let $\beta=\beta(v)=\sum\limits_{J\ \text{incr}}b_J(v)\,dv_J$ be a smooth k-form on V. We define the **pullback** of β by f to be the smooth k-form $\alpha=f^*\beta$ on U given by

$$\alpha=\alpha(u)=\sum_{I\ \text{incr}}a_I(u)\,du_I\ \text{ where }\ a_I(u)=\sum_{J\ \text{incr}}b_J(v(u))\cdot\det\tfrac{\partial v_J}{\partial u_I}(u)$$

where $v(u)=f(u)=\big(v_1(u),\cdots,v_n(u)\big)$ so that $\frac{\partial v_J}{\partial u_I}(u)=Df(u)_J^I$.

5.34 Remark: The three properties of the pullback proven below in the following theorem are precisely the properties that we need to use in order to define smooth k-forms on manifolds and extend Stokes' Theorem so that it applies in this more general situation.

5.35 Theorem: *(Pullback Formulas) Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^\ell$ be open, and let $f : U \to V$ and $g : V \to W$ be smooth.*
(1) When γ is a smooth k-form on W, we have $(gf)^*\gamma = f^*(g^*\gamma)$.
(2) When β is a smooth k-form on V and $\Phi : \Delta^k \to U$ is a smooth k-surface in U, we have

$$\int_{f\Phi} \beta = \int_\Phi f^*\beta.$$

(3) When β is a smooth k-form on V we have $df^*\beta = f^*d\beta$.

Proof: To prove Part 1, let $\gamma = \sum_{K \text{ incr}} c_K \, dw_K$ be a smooth k-form on W. Then we have

$$\begin{aligned}
f^*(g^*\gamma) &= f^*\Big(\sum_{J,K \text{ incr}} c_K(gv) \det \tfrac{\partial g_K}{\partial v_J}(v) \, dv_J \Big) \\
&= \sum_{I,J,K \text{ incr}} c_K(gfu) \det \tfrac{\partial g_K}{\partial v_J}(fu) \det \tfrac{\partial v_J}{\partial u_I}(u) \, du_I \\
&= \sum_{I,K \text{ incr}} c_K(gfu) \sum_{J \text{ incr}} \det Dg(fu)^J_K \det Df(u)^I_J \, du_I \\
&= \sum_{I,K \text{ incr}} c_K(gfu) \det Dg(fu)_K Df(u)^I \; du_I \text{ (by Theorem 5.31)} \\
&= \sum_{I,K \text{ incr}} c_K(gfu) \det D(gf)^I_K \, du_I = \sum_{I,K \text{ incr}} c_K(gfu) \det \tfrac{\partial (gf)_K}{\partial u_I}(u) \, du_I \\
&= (gf)^*\gamma.
\end{aligned}$$

To prove Part 2, let $\beta = \sum_{J \text{ incr}} b_J(v) \, dv_J$ be a smooth k-form on V and let $\Phi : \Delta^k \to U$ be a smooth k-surface in U. Then $f^*\beta(u) = \sum_{I,J \text{ incr}} b_J(fu) \det \frac{\partial v_J}{\partial u_I}(u) \, du_I$ and so

$$\begin{aligned}
\int_\Phi f^*\beta &= \sum_{I,J \text{ incr}} \int_{\Delta^k} b_J(f\Phi t) \det \tfrac{\partial v_J}{\partial u_I}(\Phi t) \det \tfrac{\partial u_I}{\partial t}(t) \, dt_1 \cdots dt_k \\
&= \sum_{J \text{ incr}} \int_{\Delta^k} b_J(f\Phi t) \Big(\sum_{I \text{ incr}} \det Df(\Phi t)^I_J \det D\Phi(t)_I \Big) \, dt_1 \cdots dt_k \\
&= \sum_{J \text{ incr}} \int_{\Delta^k} b_J(f\Phi t) \det \big(Df(\Phi t)_J D\Phi(t) \big) \, dt_1 \cdots dt_k \quad \text{(by Theorem 5.31)} \\
&= \sum_{J \text{ incr}} \int_{\Delta^k} b_J(f\Phi t) \det D(f\Phi)_J(t) \, dt_1 \cdots dt_k = \sum_{J \text{ incr}} \int_{\Delta^k} b_J(f\Phi t) \det \tfrac{\partial v_J}{\partial t}(t) \, dt_1 \cdots dt_k \\
&= \int_{f\Phi} \beta \, .
\end{aligned}$$

To prove Part 3, let $\beta = \sum_{J \text{ incr}} b_J(v) \, dv_J$ be a smooth k-form on V. Write $L = (\ell_1, \cdots, \ell_{k+1})$. Expanding the determinant along the first row, we have

$$\begin{aligned}
f^*d\beta &= f^*\Big(\sum_{j=1}^n \tfrac{\partial b_J}{\partial v_j}(v) \, dv_j \wedge dv_J \Big) \\
&= \sum_L \sum_{j=1}^n \tfrac{\partial b_J}{\partial v_j}(fu) \det \tfrac{\partial (v_j, v_{j_1}, \cdots, v_{v_k})}{\partial (u_{\ell_1}, \cdots, u_{\ell_{k+1}})}(u) \; du_L \\
&= \sum_L \sum_{j=1}^n \tfrac{\partial b_J}{\partial v_j}(fu) \sum_{i=1}^{k+1} (-1)^{i+1} \tfrac{\partial v_j}{\partial u_{\ell_i}}(u) \det \tfrac{\partial (v_{j_1}, \cdots, v_{j_k})}{\partial (u_{\ell_1}, \cdots, \widehat{u_{\ell_i}}, \cdots, u_{\ell_{k+1}})}(u) \; du_L .
\end{aligned}$$

Note that the proof of Lemma 5.28 (with small alterations) shows that

$$\sum_{i=1}^{k+1}(-1)^{i+1}\tfrac{\partial}{\partial u_{\ell_i}}\ \det \tfrac{\partial(v_{j_1},\cdots,v_{j_k})}{\partial(u_{\ell_1},\cdots,\widehat{u_{\ell_i}},\cdots,u_{\ell_{k+1}})} = 0.$$

Writing $I=(i_1,\cdots,i_k)$ and $L=(\ell_1,\cdots,\ell_{k+1})$, we have

$$\begin{aligned} df^*\beta &= d\Big(\sum_I b_J(fu)\ \det \tfrac{\partial v_J}{\partial u_I}(u)\ du_I\Big) \\ &= \sum_I \sum_{\ell\notin I} \tfrac{\partial}{\partial u_\ell}\big(b_J(fu)\ \det \tfrac{\partial v_J}{\partial u_I}(u)\big)\ du_\ell\wedge du_I \\ &= \sum_L \sum_{i=1}^{k+1} \tfrac{\partial}{\partial u_{\ell_i}}\Big(b_J(fu)\ \det \tfrac{\partial(v_{j_1},\cdots,v_{j_k})}{\partial(u_{\ell_1},\cdots,\widehat{u_{\ell_i}},\cdots,u_{\ell_{k+1}})}(u)\Big)\ du_{\ell_i}\wedge du_{(\ell_1,\cdots,\widehat{\ell_i},\cdots,\ell_{k+1})} \\ &= \sum_L \sum_{i=1}^{k+1}(-1)^{i+1} \tfrac{\partial}{\partial u_{\ell_i}}\Big(b_J(fu)\ \det \tfrac{\partial(v_{j_1},\cdots,v_{j_k})}{\partial(u_{\ell_1},\cdots,\widehat{u_{\ell_i}},\cdots,u_{\ell_{k+1}})}(u)\Big)\ du_L \\ &= \sum_L \sum_{i=1}^{k+1}(-1)^{i+1} \tfrac{\partial b_J}{\partial u_{\ell_i}}(fu)\ \det \tfrac{\partial(v_{j_1},\cdots,v_{j_k})}{\partial(u_{\ell_1},\cdots,\widehat{u_{\ell_i}},\cdots,u_{\ell_{k+1}})}(u)\ du_L \quad \text{(by Lemma 5.28)} \\ &= \sum_L \sum_{i=1}^{k+1}\sum_{j=1}^{n}(-1)^{i+1} \tfrac{\partial b_J}{\partial v_j}(fu)\ \tfrac{\partial v_j}{\partial u_{\ell_i}}(u)\det \tfrac{\partial(v_{j_1},\cdots,v_{j_k})}{\partial(u_{\ell_1},\cdots,\widehat{u_{\ell_i}},\cdots,u_{\ell_{k+1}})}(u)\ du_L\,. \end{aligned}$$

5.36 Definition: Let $M\subseteq\mathbb{R}^r$ be a smooth regular submanifold with atlas $\mathcal{A}$. A **smooth k-form** ω on M consists of a smooth k-form ω_σ on $U_\sigma\subseteq\mathbb{R}^m$ for each chart $\sigma:U_\sigma\to V_\sigma$ in $\mathcal{A}$ such that whenever $\sigma,\rho\in\mathcal{A}$ are two charts with $V_\sigma\cap V_\rho\neq 0$, we have $\omega_\sigma=(\rho^{-1}\sigma)^*\omega_\rho$; to be precise, if we let α be the restriction of ω_σ to $\sigma^{-1}(V_\sigma\cap V_\rho)$ and we let β be the restriction of ω_ρ to $\rho^{-1}(V_\sigma\cap V_\rho)$ and we let f be the smooth change of coordinates map $f=\rho^{-1}\sigma:\sigma^{-1}(V_\sigma\cap V_\rho)\to\rho^{-1}(V_\sigma\cap V_\rho)$ then we have $\alpha=f^*\beta$.

5.37 Example: An open set $U\subseteq\mathbb{R}^m$ is a submanifold of $\mathbb{R}^m$ with an atlas consisting of a single chart, namely the inclusion map $\sigma:U\to\mathbb{R}^m$. In this trivial case, a smooth k-form on the manifold $M=U$ is the same thing as a smooth k-form on the open set $U\subseteq\mathbb{R}^m$.

5.38 Definition: Let $N\subseteq\mathbb{R}^s$ and $M\subseteq\mathbb{R}^r$ be two (smooth regular) submanifolds, let $f:N\to M$ be a smooth map, and let ω be a smooth k-form on M. The **pullback** of ω by f is the smooth k-form $\lambda=f^*\omega$ on N defined as follows: given a chart $\nu:U_\nu\to V_\nu$ on N, for $c\in U_\nu$ and $q=\nu(c)\in N$, we choose a chart σ on M at $p=f(q)$ and define λ_ν in a neighbourhood of c by $\lambda_\nu=(\sigma^{-1}f\nu)^*\omega_\sigma$. By Part 1 of Theorem 5.35, this does not depend on the choice of chart: if ρ is another chart on M at p then since $\omega_\sigma=(\rho^{-1}\sigma)^*\omega_\rho$, we have $(\sigma^{-1}f\nu)^*\omega_\sigma=(\sigma^{-1}f\nu)^*(\rho^{-1}\sigma)^*\omega_\rho=(\rho^{-1}\sigma\sigma^{-1}f\nu)^*\omega_\rho=(\rho^{-1}f\nu)^*\omega_\rho$.

5.39 Example: Let $M\subseteq\mathbb{R}^m$ be a smooth regular submanifold and let ω be a smooth k-form on M. When $\sigma:U_\sigma\subseteq\mathbb{R}^m\to V_\sigma\subseteq M$ is a chart on M, the k-form ω_σ on U_σ is the same thing as the pullback $\lambda=\sigma^*\omega$. Indeed, in the formula $\lambda_\nu=(\sigma^{-1}f\nu)^*\omega_\sigma$, we use $f=\sigma$ and we take ν to be the trivial chart so that $\sigma^{-1}f\nu=\sigma^{-1}\sigma\nu$ is the identity map. For this reason, it is common to write ω_σ as $\sigma^*\omega$.

5.40 Definition: Let $M\subseteq\mathbb{R}^n$ be a smooth regular submanifold and let ω be a smooth k-form on M. We define the **exterior derivative** of ω to be the smooth $(k+1)$-form $d\omega$ on M defined by $(d\omega)_\sigma=d(\omega_\sigma)$ for each chart σ on M. By Part 3 of Theorem 5.35: this does define a $(k+1)$-form on M because when σ and ρ are two charts with $V_\sigma\cap V_\rho\neq\emptyset$, we have $(\rho^{-1}\sigma)^*(d\omega)_\rho=(\rho^{-1}\sigma)^*(d\,\omega_\rho)=d\big((\rho^{-1}\sigma)^*\omega_\rho\big)=d(\omega_\sigma)=(d\omega)_\sigma$.

5.41 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. A **smooth k-simplex** on M is a map $\Theta : \Delta^k \to M$ which extends to a smooth map $\Theta : W \to M$ for some open set $W \subseteq \mathbb{R}^k$ with $\Delta^k \subseteq W$. A **smooth k-chain** on M is a formal finite sum $\chi = \sum_{j=1}^{\ell} c_j \Theta_j$ where each $c_j \in \mathbb{Z}$ and each Θ_j is a smooth k-simplex on M. When Θ is a smooth $(k+1)$-simplex on M, the **boundary** of Θ is the smooth k-chain $\chi = \sum_{j=0}^{k+1} (-1)^j \Theta F_j$ where $F_j : \Delta^k \to \Delta^{k+1}$ is the j^{th} face map. When χ is the smooth $(k+1)$-chain $\chi = \sum_{j=1}^{\ell} c_j \Theta_j$, the **boundary** of χ is the smooth k-chain $\partial\chi = \sum_{j=1}^{\ell} c_j \partial\Theta_j$.

5.42 Definition: When Θ is a smooth k-simplex on M and ω is a smooth k-form on M, we define the **integral** of ω on Θ to be

$$\int_\Theta \omega = \int_{\Delta^k} \Theta^*\omega = \int_J \Theta^*\omega$$

where $J : \Delta^k \to \mathbb{R}^k$ is the inclusion. When $\chi = \sum_{j=1}^{\ell} c_j \Theta_j$, we define $\displaystyle\int_\chi \omega = \sum_{j=1}^{\ell} c_j \int_{\Theta_i} \omega$.

5.43 Theorem: *(Stoke's Theorem for Smooth Chains on Submanifolds of $\mathbb{R}^n$) Let $M \subseteq \mathbb{R}^n$ be a regular smooth submanifold, let ω be a smooth k-form on M, and let χ be a smooth $(k+1)$-chain on M. Then*

$$\int_\chi d\omega = \int_{\partial\chi} \omega\,.$$

Proof: By linearity, we may assume that $\chi = \Theta$ where Θ is a single smooth $(k+1)$-simplex on M. Say Θ extends to a smooth map $\Theta : W \to M$ where W is open in $\mathbb{R}^{k+1}$ with $\Delta^{k+1} \subseteq W$. Assume that the image $\Theta(W)$ is contained in the range V_σ of a single chart $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ (in general, we can subdivide that standard simplex Δ^{k+1} into small subsimplices Δ_i contained in open sets W_i with each image $\Theta(W_i)$ contained in the range of a single chart σ_i). Since $\sigma : U_\sigma \to V_\sigma$ is a diffeomorphism, the map $\Phi = \sigma^{-1}\Theta$ is smooth, so $\Phi : \Delta^{k+1} \subseteq W \to U_\sigma \subseteq \mathbb{R}^m$ is a smooth $(k+1)$-simplex in U_σ. As in Definition 5.38 and Example 5.39, the smooth $(k+1)$-form $\Theta^* d\omega$ on $W \subseteq \mathbb{R}^{k+1}$ given by $\Theta^* d\omega = (\sigma^{-1}\Theta)^*(d\omega)_\sigma = \Phi^* d\omega_\sigma$, and the smooth k-form $\Theta^*\omega$ on W is given by $\Theta^*\omega = (\sigma^{-1}\Theta)^*\omega_\sigma = \Phi^*\omega_\sigma$. By Stokes' Theorem for Chains in $\mathbb{R}^m$, and by Parts 1 and 2 of Theorem 5.35, we have

$$\begin{aligned}\int_\Theta \omega &= \int_{\Delta^{k+1}} \Theta^*\omega = \int_{\Delta^{k+1}} \Phi^* d\omega_\sigma = \int_\Phi d\omega_\sigma = \int_{\partial\Phi} \omega_\sigma = \sum_{j=0}^{k+1} (-1)^j \int_{\Phi F_j} \omega_\sigma \\ &= \sum_{j=0}^{k+1} (-1)^j \int_{\Delta^k} F_j^*\Phi^*\omega_\sigma = \sum_{j=0}^{k+1} (-1)^j \int_{\Delta^k} F_j^*\Theta^*\omega = \sum_{j=0}^{k+1} \int_{\Theta F_j} \omega = \int_{\partial\Theta} \omega.\end{aligned}$$