Chapter 4. Submanifolds of $\mathbb{R}^n$

4.1 Definition: We restate Definition 2.1 so that it applies to functions of m variables. Let $U \subseteq \mathbb{R}^m$ be open, let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$, and write $u = (u_1, \cdots, u_m) \in U$ and $f(u) = \big(x_1(u), \cdots, x_n(u)\big) \in \mathbb{R}^n$. We say that f is $\mathcal{C}^k$ in U when all of the k^{th} order partial derivatives exist and are continuous, and we say that f is **smooth** (or that f is $\mathcal{C}^\infty$) when f is $\mathcal{C}^k$ for all $k \in \mathbb{Z}^+$. Recall that when f is $\mathcal{C}^1$ it is also differentiable, and its derivative matrix (or its Jacobian matrix) is given by

$$Df = \left(\tfrac{\partial f}{\partial u_1}, \cdots, \tfrac{\partial f}{\partial u_m}\right) = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_m} \end{pmatrix}$$

We say that f is **regular** (or f is an **immersion**) when f is $\mathcal{C}^1$ and its derivative matrix is injective, that is when the columns of the derivative matrix are linearly independent. Recall that a **homeomorphism** is a bijective continuous function whose inverse is continuous.

4.2 Definition: Let $M \subseteq \mathbb{R}^n$ and let $p = (p_1, p_2, \cdots, p_n) \in M$. We say that M is **locally equal to the graph** of a function of m variables, **near the point** p, when there exists a choice of m of the variables $x_1, \cdots, x_n$, say the variables x_k with $k \in K$ where K is the index set $K = (k_1, k_2, \cdots, k_m)$ with $1 \le k_1 < k_2 < \cdots < k_m \le n$, with the remaining $n-m$ variables being x_ℓ with $\ell \in L$ where L is the complementary index set $L = (\ell_1, \ell_2, \cdots, \ell_{n-m})$ with $1 \le \ell_1 < \ell_2 < \cdots < \ell_{n-m} \le n$ and with $K \cup L = \{1, 2, \cdots, n\}$, and there exist $\epsilon, \delta > 0$, and there exists a function $f : A \to B$ where $A = \big\{(x_{k_1}, \cdots, x_{k_m}) \big| \, |x_{k_i} - p_{k_i}| < \delta\big\}$ and $B = \big\{(x_{\ell_1}, \cdots, x_{\ell_{n-m}}) \,\big|\, |x_{\ell_i} - p_{\ell_i}| < \epsilon\big\}$ such that for the open rectangle $R = A \times B \subseteq \mathbb{R}^n$, that is for $R = \big\{x \in \mathbb{R}^n \big| (x_{k_1}, \cdots, x_{k_m}) \in A, (x_{\ell_1}, \cdots, x_{\ell n-m}) \in B\big\}$ we have

$$M \cap R = \text{Graph}\,(f) = \Big\{(x_1, \cdots, x_n) \in R \,\Big|\, (x_{\ell_1}, \cdots, x_{\ell_{n-m}}) = f(x_{k_1}, \cdots, x_{k_m})\Big\}.$$

Note that the function $g : A \to \text{Graph}\,(f)$, given by $g(u_1, \cdots, u_m) = (x_1(u), \cdots, x_n(u))$ with $x_{k_i}(u) = u_i$ and $\big(x_{\ell_1}(u), \cdots, x_{\ell_{n-m}}(u)\big) = f(u)$, is a homeomorphism from A to $\text{Graph}\,(f)$ (the inverse of g is the projection $p(x_1, \cdots, x_n) = (x_{k_1}, \cdots, x_{k_m})$, which is continuous).

4.3 Definition: An m-dimensional (smooth regular) **submanifold** of $\mathbb{R}^n$ is a set $M \subseteq \mathbb{R}^n$ which is locally equal to the graph of a smooth function of m variables, near every $p \in M$.

4.4 Example: For a smooth function $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^\ell$, where U is open, the graph $\text{Graph}\,(f) = \big\{(x, y) \,\big|\, x \in U \subseteq \mathbb{R}^m, y \in \mathbb{R}^\ell, y = f(x)\big\}$ is a smooth regular m-dimensional submanifold of $\mathbb{R}^{m+\ell}$ (given $p = (a, b) \in \text{Graph}\,(f)$ we have $f(a) = b$, and given $\epsilon > 0$, since f is continuous we can choose $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - b| < \epsilon$, and since U is open we can choose this δ to small enough so that the box given by $|x_i - a_i| < \delta$ lies inside U). Any open set $U \subseteq \mathbb{R}^m$ can be considered as a smooth regular m-dimensional submanifold of $\mathbb{R}^m$ (it is the graph of the zero function $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^0 = \{0\}$).

4.5 Example: The unit sphere $\mathbb{S}^{n-1} = \big\{x \in \mathbb{R}^n \,\big|\, |x| = 1\big\}$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Indeed, if we let $U = \big\{u \in \mathbb{R}^{n-1} \,\big|\, |u| < 1\big\}$ and define $f_k, g_k : U \to \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ by

$$f_k(u) = \big(u_1, \cdots, u_{k-1}, \sqrt{1 - |u|^2}, u_k, \cdots, u_n\big)$$
$$g_k(u) = \big(u_1, \cdots, u_{k-1}, -\sqrt{1 - |u|^2}, u_k, \cdots, u_n\big)$$

then every point $p \in \mathbb{S}^{n-1}$ is on the graph of one of the functions f_k, g_k (and if we want, we can restrict the domain of f_k or g_k to an open rectangle in U as in definition 4.2).

4.6 Theorem: *(The Inverse Function Theorem) Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ where $U \subseteq \mathbb{R}^n$ is open with $a \in U$. Suppose that f is $\mathcal{C}^1$ in U and that $Df(a)$ is invertible. Then there exists an open set $U_0 \subseteq U$ with $a \in U_0$ such that the set $V_0 = f(U_0)$ is open in $\mathbb{R}^n$ and the restriction $f : U_0 \to V_0$ is bijective, and its inverse $g = f^{-1} : V_0 \to U_0$ is $\mathcal{C}^1$ in V_0. In this case we have $Dg\big(f(a)\big) = Df(a)^{-1}$. If f is $\mathcal{C}^k$ (or $\mathcal{C}^\infty$) then so is f^{-1}.*

Proof: This theorem is usually stated without proof in MATH 237 and a full proof is usually given in MATH 147. The proof is not easy. A proof is included in Appendix 2.

4.7 Theorem: *(The Implicit Function Theorem) Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$ where U is open with $p \in U$. Suppose that f is $\mathcal{C}^1$ in U and that $Df(p)$ has rank ℓ and let $c = f(p)$. Then the level set $f^{-1}(c) = \big\{x \in U \,\big|\, f(x) = c\big\}$ is locally the graph of a $\mathcal{C}^1$ function h of $n-\ell$ variables near the point p. If f is $\mathcal{C}^k$ (or $\mathcal{C}^\infty$) then so is h.*

Proof: Since $Df(p)$ has rank ℓ, it follows that some $\ell \times \ell$ submatrix of f is invertible. By reordering the variables in $\mathbb{R}^n$, if necessary, suppose that the last ℓ of the n columns of $Df(p)$ form an invertible $\ell \times \ell$ matrix. Let $m = n - \ell$, write $p = (a, b)$ with $a = (p_1, \cdots, p_m) \in \mathbb{R}^m$ and $b = (p_{m+1}, \cdots, p_n) \in \mathbb{R}^\ell$ and write $z = f(x, y)$ with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^\ell$ and $z \in \mathbb{R}^\ell$, and write

$$Df(x,y) = \big(\tfrac{\partial z}{\partial x}(x,y), \tfrac{\partial z}{\partial y}(x,y)\big)$$

with $\frac{\partial z}{\partial y}(a,b)$ invertible. Define $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ by $F(x,y) = \big(x, f(x,y)\big) = (w, z)$. Then we have

$$DF = \begin{pmatrix} I & O \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}$$

with $DF(a,b)$ invertible. By the Inverse Function Theorem, we can choose an open set U_0 in $\mathbb{R}^n$ with $p \in U_0 \subseteq U$ such that $V_0 = F(U_0)$ is open in $\mathbb{R}^n$ and the map $F : U_0 \to V_0$ is invertible and its inverse $G : V_0 \to U_0$ is $\mathcal{C}^1$. Write $(x,y) = G(w,z) = \big(w, g(w,z)\big)$ and let $h(x) = g(x,c)$. Then, locally, we have $f^{-1}(c) = \text{Graph}\,(h)$ because

$$\begin{aligned} f(x,y) = c \iff F(x,y) = (x,c) &\iff (x,y) = G(x,c) \\ \iff (x,y) = \big(x, g(x,c)\big) &\iff (x,y) \in \text{Graph}\,(h). \end{aligned}$$

Since $G(w,z) = \big(w, g(w,z)\big)$ we have $DG(w,z) = \begin{pmatrix} I & O \\ \frac{\partial g}{\partial w} & \frac{\partial g}{\partial z} \end{pmatrix}$. Since $G = F^{-1}$ we also have

$$DG(w,z) = DF(x,y)^{-1} = \begin{pmatrix} I & O \\ -\big(\frac{\partial z}{\partial y}\big)^{-1} \frac{\partial z}{\partial x} & \big(\frac{\partial z}{\partial y}\big)^{-1} \end{pmatrix}$$

so, since $h(x) = g(x,c)$, we have

$$Dh(x) = \tfrac{\partial g}{\partial w}(x,c) = -\big(\tfrac{\partial z}{\partial y}(x, h(x))\big)^{-1} \tfrac{\partial z}{\partial x}\,(x, h(x)).$$

This formula shows that if f is $\mathcal{C}^k$ then so is h. Indeed when f is $\mathcal{C}^k$, the entries of the matrices $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are all $\mathcal{C}^{k-1}$, so the entries of the inverse matrix $\big(\frac{\partial z}{\partial y}\big)^{-1}$ are also $\mathcal{C}^{k-1}$ (by the cofactor formula for the inverse of a matrix, which shows that the entries of the inverse of a matrix are rational functions of the entries of the matrix), hence the entries of Dh are all $\mathcal{C}^{k-1}$ so that h is $\mathcal{C}^k$.

4.8 Corollary: *(Implicit Description of a Smooth Manifold) Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$ be smooth with* rank $Df(x) = \ell$ *for all $x \in U$. Then for all $c \in$* Range(f), *$f^{-1}(c) = \big\{x \in U \,\big|\, f(x) = c\big\}$ is a regular smooth $(n-\ell)$-dimensional submanifold of $\mathbb{R}^n$.*

4.9 Note: The above corollary often allows us to verify that a subset $M \subseteq \mathbb{R}^n$ is a smooth regular submanifold of $\mathbb{R}^n$ without explicitly exhibiting functions whose graphs cover M. For example, we can say that $\mathbb{S}^{n-1}$ is a smooth regular $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ because $\mathbb{S}^{n-1} = \big\{x \in \mathbb{R}^n \,\big|\, |x| = 1\big\} = f^{-1}(1)$ where $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is given by $f(x) = |x|^2 = {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2$, and $Df(x) = \big(2x_1, \cdots, 2x_n\big)$ so that $\operatorname{rank} Df(x) = 1$ for all $0 \neq x \in \mathbb{R}^n$.

4.10 Theorem: *(The Parametric Function Theorem) Let $U \subseteq \mathbb{R}^m$ be open with $a \in U$, let $\sigma : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be $\mathcal{C}^1$, and suppose that $D\sigma(a)$ has rank m. Then there is an open set $U_0 \subseteq U$ with $a \in U_0$ such that the image $\sigma(U_0)$ is equal to the graph of a $\mathcal{C}^1$ function f and the map $\sigma : U_0 \to \sigma(U_0)$ is a homeomorphism. If σ is $\mathcal{C}^k$ (or $\mathcal{C}^\infty$) then so is f.*

Proof: Since $D\sigma(a)$ has rank m, it follows that $m \leq n$ and some $m \times m$ submatrix of $D\sigma(a)$ is invertible. By reordering the variables in $\mathbb{R}^n$, if necessary, suppose that the top m rows of $D\sigma(a)$ form an invertible $m \times m$ submatrix. Write $\sigma(u) = \big(x(u), y(u)\big)$, where $x(u) = \big(x_1(u), \cdots, x_m(u)\big)$ and $y(u) = \big(y_1(u), \cdots, y_{n-m}(u)\big)$, so that we have

$$D\sigma(u) = \begin{pmatrix} Dx(u) \\ Dy(u) \end{pmatrix}$$

with $Dx(a)$ invertible. By the Inverse function Theorem, the map $x = x(u)$ is locally invertible: we can choose an open set $U_0 \subseteq U$ with $a \in U_0$ such that the image $V_0 = x(U_0)$ is open, $x : U_0 \to V_0$ is bijective, and the inverse $u = u(x)$ is $\mathcal{C}^1$. Let $f(x) = y\big(u(x)\big)$. Then $\sigma(U_0) = \operatorname{Graph}(f)$ because if $(x, y) \in \operatorname{Graph}(f)$ and we choose $u = u(x)$ then we have $(x, y) = \big(x, f(x)\big) = \big(x(u), f(x(u))\big) = \big(x(u), y(u)\big) \in \sigma(U_0)$ and if $(x, y) \in \sigma(U_0)$, say $(x, y) = \big(x(u), y(u)\big)$, then $u = u(x)$ so that $y(u) = y\big(u(x)\big) = f(x)$ hence $(x, y) = \big(x(u), y(u)\big) = \big(x, f(x)\big) \in \operatorname{Graph}(f)$. Note that the map $\sigma : U_0 \to f(U_0) = \operatorname{Graph}(f)$ is a homeomorphim with the inverse $\sigma^{-1} : \operatorname{Graph}(f) \to U_0$ given by $\sigma^{-1}(x, y) = u(x)$, which is continuous.

Suppose $\sigma(u)$ is $\mathcal{C}^k$ so $x(u)$ and $y(u)$ are $\mathcal{C}^k$ and also $u(x)$ is $\mathcal{C}^k$ (by the Inverse Function Theorem). Since $f(x) = y(u(x))$ we have $Df(x) = Dy(u(x))Du(x) = Dy(u(x))Dx(u(x))^{-1}$. Since $Dy(u)$ is $\mathcal{C}^{k-1}$ and $u(x)$ is $\mathcal{C}^k$, it follows that the composite $Dy(u(x))$ is $\mathcal{C}^{k-1}$. Similarly $Dx(u(x))$ is $\mathcal{C}^{k-1}$ and hence the inverse $Dx(u(x))^{-1}$ is also $\mathcal{C}^{k-1}$ (by the cofactor formula for the inverse of a matrix, which shows that the entries of the inverse of a matrix are rational functions of the entries of the matrix). Thus $Df(x) = Dy(u(x))Dx(u(x))^{-1}$ is $\mathcal{C}^{k-1}$ and hence $f(x)$ is $\mathcal{C}^k$.

4.11 Corollary: *(Parametric Description of a Smooth Manifold) An m-dimensional smooth regular submanifold of $\mathbb{R}^n$ is a set $M \subseteq \mathbb{R}^n$ such that for each point $p \in M$ there is a smooth regular homeomorphism $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M \subseteq \mathbb{R}^n$, where U is open in $\mathbb{R}^m$ and V is open in M with $p \in V$ (V is open in M means that $V = W \cap M$ for some open set $W \subseteq \mathbb{R}^n$).*

4.12 Remark: One subtle aspect of the above corollary is that, it is necessary to include the stipulation that the map σ is a homeomorphism onto its image (even though we can restrict the domain of any regular map to make it become a homeomorphism onto its image). For example (as mentioned in Example 1.5) the alpha curve $\alpha(t) = \big(t^2-1, t(t^2-1)\big)$ is a regular map, so it is locally injective, but it crosses its self at the origin when $t = \pm 1$, and the image of the alpha curve is not a smooth manifold because it is not locally equal to a graph near the origin. If we restrict the domain of the alpha curve to $t > -1$, then α becomes injective (the curve no longer crosses itself) but the image is still not a smooth manifold, because it is still not locally equal to a graph near the origin.

4.13 Example: We saw in Example 4.5 that the sphere $\mathbb{S}^{n-1}$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ which can be covered by the graphs of the $2n$ smooth functions f_k, g_k. There are many alternative ways of describing the sphere parametrically and covering the sphere by the images of smooth regular homeomorphisms. To give just one example, we can cover the sphere by the images of two such maps using **stereographic projection**. The north and south poles of $\mathbb{S}^{n-1}$ are the points $\pm e_n = \pm(0,0,\cdots,0,1)$. The stereographic projections from the north and south poles are the maps $\phi : \mathbb{S}^{n-1} \setminus \{e_n\} \to \mathbb{R}^{n-1}$ and $\psi : \mathbb{S}^{n-1} \setminus \{-e_n\} \to \mathbb{R}^{n-1}$ defined as follows. Given $x \in \mathbb{S}^{n-1} \setminus \{e_n\}$, we define $u = \phi(x)$ to be the point in $\mathbb{R}^{n-1}$ such that $\big(u,0\big)$ lies on the line through e_n and x, and given $x \in \mathbb{S}^{n-1} \setminus \{-e_n\}$, we define $v = \psi(x)$ to be the point in $\mathbb{R}^{n-1}$ such that $\big(v,0\big)$ lies on the line through $-e_n$ and x. Let us find explicit formulas for ϕ, $\sigma = \phi^{-1}$, ψ and $\rho = \psi^{-1}$. The line in $\mathbb{R}^n$ from e_n to x is given, parametrically, by

$$\alpha(t) = e_n + t(x - e_n) = \big(tx_1, tx_2, \cdots, tx_{n-1}, 1 + t(x_n - 1)\big).$$

Note that $\alpha(t)$ is of the form $(u,0)$ when $t = \frac{1}{1-x_n}$ and then

$$(u,0) = \alpha\big(\tfrac{1}{1-x_n}\big) = \big(\tfrac{x_1}{1-x_n}, \tfrac{x_2}{1-x_n}, \cdots, \tfrac{x_{n-1}}{1-x_n}, 0\big)$$

so we have

$$u = \phi(x) = \big(\tfrac{x_1}{1-x_n}, \tfrac{x_2}{1-x_n}, \cdots, \tfrac{x_{n-1}}{1-x_n}\big).$$

The line in $\mathbb{R}^n$ through e_n and $(u,0)$ where $u = \phi(x) \in \mathbb{R}^{n-1}$ is given, parametrically, by

$$\beta(t) = e_n + t((u,0) - e_n) = \big(tu_1, tu_2, \cdots, tu_{n-1}, 1 - t\big).$$

This line meets the sphere at the points e_n and x. We have

$$\big|\beta(t)\big|^2 = 1 \iff t^2|u|^2 + (1-t)^2 = 1 \iff t^2|u|^2 - 2t + t^2 = 0 \iff t = 0 \text{ or } t = \tfrac{2}{|u|^2+1}.$$

When $t = 0$ we have $\beta(t) = e_n$ so we must have

$$x = \sigma(u) = \phi^{-1}(u) = \beta\big(\tfrac{2}{|u|^2+1}\big) = \big(\tfrac{2u_1}{|u|^2+1}, \tfrac{2u_2}{|u|^2+1}, \cdots, \tfrac{2u_{n-1}}{|u|^2+1}, \tfrac{|u|^2-1}{|u|^2+1}\big).$$

The formulas for $x = \sigma(u)$ and $u = \phi(x) = \sigma^{-1}(x)$ show that σ and σ^{-1} are both continuous, so $\sigma : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \setminus \{e_n\}$ is a homeomorphism. The formula for σ shows that σ is smooth. The formula for ϕ shows that the map $\phi : \mathbb{S}^{n-1} \setminus \{e_n\} \to \mathbb{R}^{n-1}$ extends (using the same formula) to a smooth map $\phi : V \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1}$ where $V = \big\{x \in \mathbb{R}^n \,\big|\, x_n \neq 1\big\}$. Since $\phi(\sigma(u)) = u$ for all $u \in \mathbb{R}^{n-1}$, we have $D\phi(\sigma(u))D\sigma(u) = I$ for all u, and hence $D\sigma(u)$ must be injective for all u so that σ is regular. As an optional exercise, you could verify this by calculating $D\sigma$ explicitly.

A similar calculation shows that ψ and $\rho = \psi^{-1}$ are given by

$$v = \psi(x) = \big(\tfrac{x_1}{1+x_n}, \tfrac{x_2}{1+x_n}, \cdots, \tfrac{x_{n-1}}{1+x_n}\big)$$
$$x = \rho(v) = \psi^{-1}(v) = \big(\tfrac{2v_1}{1+|v|^2}, \tfrac{2v_2}{1+|v|^2}, \cdots, \tfrac{2v_{n-1}}{1+|v|^2}, \tfrac{1-|v|^2}{1+|v|^2}\big),$$

and a similar argument shows that the map $\rho : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \setminus \{-e_n\}$ is a smooth regular homeomorphism.

4.14 Definition: By Corollary 4.11, an m-dimensional smooth regular submanifold of $\mathbb{R}^n$ is a set $M \subseteq \mathbb{R}^n$ for which there exists a set $\mathcal{A}$ of smooth regular homeomorphisms $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$, where U_σ is open in $\mathbb{R}^m$ and V_σ is open in M, with $M = \bigcup_{\sigma \in \mathcal{A}} V_\sigma$. Such a set of maps $\mathcal{A}$ is called an **atlas** for M, and the maps $\sigma \in \mathcal{A}$ are called (coordinate) **charts** on M. When $p \in M$ is in the range of $\sigma \in \mathcal{A}$, we say that σ is a **chart at** p.

4.15 Theorem: *Let $\mathcal{A}$ be an atlas on smooth regular submanifold $M \subseteq \mathbb{R}^n$. When σ and ρ are two charts whose images have non-empty intersection, the map $\rho^{-1}\sigma$ is a smooth regular change of coordinates.*

Proof: Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ and $\rho : U_\rho \subseteq \mathbb{R}^m \to V_\rho \subseteq M$ be charts with $V_\sigma \cap V_\rho \neq \emptyset$. Note that $\rho^{-1}\sigma : \sigma^{-1}(V_\sigma \cap V_\rho) \to \rho^{-1}(V_\sigma \cap V_\rho)$ and $\sigma^{-1}\rho : \rho^{-1}(V_\sigma \cap V_\rho) \to \sigma^{-1}(V_\sigma \cap V_\rho)$. These two maps are continuous and they are inverses of each other, so they are homeomorphisms. We need to prove that they are smooth. Before giving a proof, let us remark that we cannot simply say "since ρ is smooth and σ^{-1} is smooth, therefore the composite $\sigma^{-1}\rho$ is smooth" because we do not know that σ^{-1} is smooth (indeed we have not yet even defined what it means for a function of the form $\sigma^{-1} : V_0 \subseteq M \to U_0 \subseteq \mathbb{R}^m$ to be smooth).

Let us show that $\sigma^{-1}\rho$ is smooth in a neighbourhood of any point in $\rho^{-1}(V_\sigma \cap V_\rho)$. Let $b \in \sigma^{-1}(V_\sigma \cap V_\rho)$, let $p = \rho(b)$ and let $a = \sigma^{-1}(p)$. Using the notation of the proof of the Parametric Function Theorem, after reordering the variables in $\mathbb{R}^n$ if necessary, we can write $\sigma(u) = \big(x(u), y(u)\big)$ and choose $U_0 \subseteq \sigma^{-1}(V_\sigma \cap V_\rho)$ open with $a \in U_0$ such that $x : U_0 \to V_0$ invertible with smooth inverse $u = u(x)$, and then $\sigma(U_0)$ is equal to the graph of the smooth function $f(x) = y\big(u(x)\big)$, and the inverse $\sigma^{-1} : \sigma(U_0) \to U_0$ is given by $\sigma^{-1}(x, y) = u(x)$. If we then write $\rho(v) = \big(\rho_1(v), \rho_2(v)\big)$, then the map $\sigma^{-1}\rho$ is given by $\sigma^{-1}(\rho(v)) = u(\rho_1(v))$, which is a smooth function of v for $v \in {\rho_1}^{-1}(V_0)$, as required.

4.16 Remark: The above theorem shows that when we defined a 2-dimensional smooth regular submanifold of $\mathbb{R}^n$ in Chapter 3, the requirement in Part 2 of Definition 3.22 (that $\rho^{-1}\sigma$ and $\sigma^{-1}\rho$ are smooth) is superfluous.

4.17 Remark: The above theorem also shows that a smooth regular submanifold $M \subseteq \mathbb{R}^n$ cannot have two different dimensions (it cannot be both m-dimensional and ℓ-dimensional when $m \neq \ell$). Indeed if $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ and $\rho : U_\rho \subseteq \mathbb{R}^\ell \to V_\rho \subseteq M$ are two charts whose images intersect, then since the maps $\rho^{-1}\sigma$ and $\sigma^{-1}\rho$ are smooth invertible maps, for $\sigma(a) = p = \rho(b)$ the matrices $D(\rho^{-1}\sigma)(a)$ and $D(\sigma^{-1}\rho)(b)$ are inverses of eachother, so they must be square matrices, and $D(\rho^{-1}\sigma)(a)$ is an $\ell \times m$ matrix. Thus the **dimension** of a smooth regular submanifold $M \subseteq \mathbb{R}^n$ is well-defined, and we denote it by $\dim(M)$.

4.18 Remark: The parametric description of a manifold given in Corollary 4.11, and the definition of an atlas and charts given in Definition 4.14, and the result of the above theorem (that the composite of one chart with the inverse of another is a smooth change of coordinates), are combined together with some topology, to make a more general and abstract definition of a smooth manifold (not necessarily contained in Euclidean space).

4.19 Remark: A smooth regular submanifold $M \subseteq \mathbb{R}^n$ can be given many different atlases, but it has a unique **maximal atlas** which consists of all possible smooth regular homeomorphisms $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$, where U is open in $\mathbb{R}^m$ and V is open in M.

4.20 Example: When M is an m-dimensional submanifold of $\mathbb{R}^k$ and N is an n-dimensional submanifold of $\mathbb{R}^\ell$, the **cartesian product** $M \times N = \big\{(x, y) \,\big|\, x \in M, y \in N\big\}$ is an $(m+n)$-dimensional submanifold of $\mathbb{R}^{k+\ell}$. Indeed if $\mathcal{A}$ is an atlas for M and $\mathcal{B}$ is an atlas for N, we can construct an atlas $\mathcal{C}$ for $M \times N$ as follows. Given $\sigma \in \mathcal{A}$ and $\rho \in \mathcal{B}$, say $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ and $\rho : U_\rho \subseteq \mathbb{R}^\ell \to V_\rho \subseteq N$, we let $U_{\sigma\times\rho} = U_\sigma \times U_\rho \subseteq \mathbb{R}^{m+\ell}$ and $V_{\sigma\times\rho} = V_\sigma \times V_\rho \subseteq M \times N$ and define $\sigma\times\rho : U_{\sigma\times\rho} \subseteq \mathbb{R}^{k+\ell} \to V_{\sigma\times\rho} \subseteq M \times N$ by $(\sigma\times\rho)(u, v) = \big(\sigma(u), \rho(v)\big)$. Note that $D(\sigma\times\rho) = \left(\begin{smallmatrix} D\sigma & 0 \\ 0 & D\rho \end{smallmatrix}\right)$ so $\sigma\times\rho$ is smooth and regular, and the inverse of $(\sigma\times\rho)$ is given by $(\sigma\times\rho)(x, y) = \big(\sigma^{-1}(x), \rho^{-1}(y)\big)$ which is continuous, hence $\sigma\times\rho$ is a homeomorphism. Thus the set $\mathcal{C} = \big\{\sigma\times\rho, \,\big|\, \sigma \in \mathcal{A}, \rho \in \mathcal{B}\big\}$ is an atlas for $M \times N$. For example, the n**-torus** $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a submaifold of $\mathbb{R}^{2n}$.

Smooth Maps

4.21 Definition: Let $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^\ell$ be two (smooth regular) submanifolds, and let $f : M \to N$ be continuous. We say that f is **differentiable** at $p \in M$ when the composite $\rho^{-1} f \sigma$ is differentiable at $\sigma^{-1}(p)$ for every chart σ on M at p and every chart ρ on N at $f(p)$. Note that by Theorem 4.15, if $\rho^{-1} f \sigma$ is differentiable at $\sigma^{-1}(p)$ for one chart σ on M at p and one chart ρ on N at $f(p)$, then $\mu^{-1} f \tau$ is differentiable at $\tau^{-1}(p)$ for every chart τ on M at p and every chart μ on N at $f(p)$ because $\mu^{-1} f \tau = (\mu^{-1}\rho)(\rho^{-1} f \sigma)(\sigma^{-1}\tau)$. We say that f is **differentiable** (on M) when f is differentiable at every point $p \in M$, equivalently when $\rho^{-1} f \sigma$ is differentiable (in its domain) for every chart σ on M and every chart ρ on N (the empty function with the empty domain is differentiable, vacuously). We say that f is **smooth** or $\mathcal{C}^\infty$ (on M) when $\rho^{-1} f \sigma$ is smooth for every chart σ on M and every chart ρ on N. We say that f is a (smooth) **diffemorphism** when f is bijective and both f and f^{-1} are smooth.

4.22 Note: Let $M \subseteq \mathbb{R}^n$ be a (smooth, regular) submanifold and let σ be a chart on M, say $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$. Note that $U \subseteq \mathbb{R}^m$ is an m-dimensional submanifold of $\mathbb{R}^m$ with an atlas consisting of one chart, namely the inclusion map $I : U \to \mathbb{R}^m$ given by $I(u) = u$. Also note that $V \subseteq \mathbb{R}^n$ is an m-dimensional submanifold of $\mathbb{R}^n$ with an atlas consisting of one chart, namely the map $\sigma : U \to V$. By Definition 4.21, the map $\sigma : U \to V$ is smooth (rather trivially) because $\sigma^{-1}\sigma I = I$, which is smooth. Likewise, the map $\sigma^{-1} : V \to U$ is smooth (again rather trivially) because $I^{-1}\sigma^{-1}\sigma = I^{-1} = I$, which is smooth. Thus the chart σ is a diffeomorhism from $U \subseteq \mathbb{R}^m$ to $V = \sigma(U) \subseteq \mathbb{R}^n$.

4.23 Note: Let $M \subseteq \mathbb{R}^n$ be a (smooth, regular) submanifold, and consider the inclusion map $J : M \to \mathbb{R}^n$ given by $J(p) = p$. Note that $\mathbb{R}^n$ is a submanifold of itself using an atlas which consists of one chart, namely the identity map $I : \mathbb{R}^n \to \mathbb{R}^n$. By Definition 4.21, the inclusion map $J : M \to \mathbb{R}^n$ is smooth (rather trivially) because for every chart $\sigma : U \subseteq \mathbb{R}^m \to V \subseteq M$ on M we have $I^{-1} J \sigma = \sigma$, which is smooth.

4.24 Note: When one of the two manifolds $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^\ell$ is equal to $\mathbb{R}^k$ or $\mathbb{R}^\ell$, the definitions of differentiability and smoothness can be simplified a little. For example, a map $f : M \subseteq \mathbb{R}^k \to \mathbb{R}^\ell$ is smooth when $f\sigma$ is smooth for every chart σ on M, and a map $f : \mathbb{R}^k \to N \subseteq \mathbb{R}^\ell$ is smooth when $\rho^{-1} f$ is smooth for every chart ρ on N.

4.25 Note: The composite of smooth maps between manifolds is smooth. Indeed, let L, M and N be smooth manifolds (in various Euclidean spaces) and let $f : L \to M$ and $g : M \to N$ be smooth maps, and consider the composite $gf : L \to N$. Let $\tau : U_\tau \to V_\tau \subseteq L$ be any chart on L and let $\rho : U_\rho \to V_\rho \subseteq N$ be any chart on N. We need to show that the composite $\rho^{-1} g f \tau$ is smooth (assuming its domain is not empty). Let a be in the domain, that is let $a \in \tau^{-1}\big(f^{-1}(g^{-1}(V_\rho))\big)$. Choose a chart σ on M at the point $p = f(\tau(a))$. Then in an open set containing a, we have $\rho^{-1} g f \tau = (\rho^{-1} g \sigma)(\sigma^{-1} f \tau)$, which is smooth.

4.26 Note: By Theorem 4.15, differentiability and smoothness do not depend on the choice of atlas.

4.27 Theorem: *Let $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^\ell$ be smooth, regular submanifolds, and let $f : M \to N$ be continuous.*

(1) Define $g : M \to \mathbb{R}^\ell$ by $g(p) = f(p)$ for all $p \in M$. Then f is smooth if and only if g is smooth.

(2) Let $W \subseteq \mathbb{R}^k$ be an open set with $M \subseteq W$ and let $g : W \to \mathbb{R}^\ell$ be a continuous extension of f to W (this means that $g(p) = f(p)$ for all $p \in M$). If g is smooth then so is f.

Proof: Let us prove Part 1. Note that if f is smooth then so is g, because g is the composite $g = Jf$ where $J : N \to \mathbb{R}^\ell$ is the inclusion (which is smooth, by Note 4.23). Suppose that g is smooth. Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ be a chart on M and let $\rho : U_\rho \subseteq \mathbb{R}^n \to V_\rho \subseteq N$ be a chart on N. We need to show that $\rho^{-1} f \sigma$ is smooth, and it suffices to show that it is smooth in an open neighbourhood of any point. Say $a \in U_\sigma$, $p = \sigma(a) \in M$, $q = f(p) = g(p) \in N$ and $b = \rho^{-1}(q) \in U_\rho$. Using the notation of the proof of the Parametric Function Theorem (applied to the chart ρ), after reordering the variables in $\mathbb{R}^\ell$ if necessary, we can write $\rho(u) = \big(x(u), y(u)\big)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{\ell-n}$, and choose $U_0 \subseteq U_\rho$ open with $b \in U_0$ such that $x : U_0 \to V_0$ invertible with smooth inverse $u = u(x)$, and then $\rho(U_0)$ is equal to the graph of the smooth function $h : V_0 \to \mathbb{R}^{\ell-n}$ given by $h(x) = y\big(u(x)\big)$, and the inverse $\rho^{-1} : \rho(U_0) \to U_0$ is given by $\rho^{-1}(x, y) = u(x)$. We have $\rho(U_0) = \text{Graph}\,(h) = \big\{(x, h(x)) \,\big|\, x \in V_0\big\} \subseteq V_0 \times \mathbb{R}^{\ell-n}$ Notice that we can extend the map $\rho^{-1} : \rho(U_0) \to U_0$ to a map $\psi : V_0 \times \mathbb{R}^{\ell-n} \to U_0$ using the same formula $\psi(x, y) = u(x)$. Note that $g^{-1}\big(V_0 \times \mathbb{R}^{\ell-n}\big)$ is an open subset of W which contains p, and $\sigma^{-1}\big(g^{-1}\big(V_0 \times \mathbb{R}^{\ell-n}\big)\big)$ is an open subset of U_σ which contains a. On this set we have $\rho^{-1} f \sigma = \psi g \sigma$, which is smooth, as required.

Part 2 follows easily from Part 1. Indeed, suppose that $g : W \subseteq \mathbb{R}^k \to \mathbb{R}^\ell$ is a smooth extension of f to an open set $W \subseteq \mathbb{R}^k$ with $M \subseteq W$. By Part 1, the map $g : W \to N$ is smooth, and given any chart σ on M and any chart ρ on N we have $\rho^{-1} f \sigma = \rho^{-1} g \sigma$.

4.28 Example: Define $f : \mathbb{S}^2 \to \mathbb{R}$ by $f(x, y, z) = z \sin(x+y)$. Then f is smooth on $\mathbb{S}^2$, by Part 2 of the above theorem $\big($using $W = \mathbb{R}^3$ and $g(x, y, z) = z \sin(x+y)\big)$.

4.29 Example: Define $h : \mathbb{R}^3 \to \mathbb{R}$ by

$$h(x, y, z) = \begin{cases} \frac{x^2+y^2}{1-z} & \text{when } z \neq 1 \\ 2 & \text{when } z = 1. \end{cases}$$

Show that h is not smooth on $\mathbb{R}^3$ but the restriction of h to $\mathbb{S}^2$ is smooth.

Solution: The function h is not smooth on $\mathbb{R}^3$, indeed it is not even continuous at points of the form $(x, y, 1)$: for $\alpha(t) = (x, y, t)$ we have $\lim\limits_{t \to 1} \alpha(t) = (x, y, 1)$, so if h was continuous we would have $\lim\limits_{t \to 1} h(\alpha(t)) = h(x, y, 1) = 2$, but instead we have $\lim\limits_{t \to 1} h(\alpha(t)) = \lim\limits_{t \to 1} \frac{x^2+y^2}{1-t^2}$, which does not exist when $(x, y) \neq (0, 0)$ and which is equal to 0 when $(x, y) = (0, 0)$.

But we claim that the restriction of h to $\mathbb{S}^2$ is smooth. Let $f : \mathbb{S}^2 \to \mathbb{R}$ be the restriction of h. For all $(x, y, z) \in \mathbb{S}^2$ we have $x^2 + y^2 + z^2 = 1$ so that $x^2 + y^2 = 1 - z^2$. When $z \neq 1$ we have $h(x, y, z) = \frac{x^2+y^2}{1-z} = \frac{1-z^2}{1-z} = 1 + z$, and at the point $(x, y, z) = (0, 0, 1)$ we have $h(x, y, z) = 2 = 1 + z$, and so $f(x, y, z) = h(x, y, z) = 1 + z$ for all $(x, y, z) \in \mathbb{S}^2$. Thus f is smooth on $\mathbb{S}^2$, by Part 2 of the above theorem $\big($using $W = \mathbb{R}^3$ and $g(x, y, z) = 1 + z\big)$.

4.30 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular m-dimensional submanifold and let $p \in M$. A **tangent vector** on M at p is a vector of the form $X_p = \gamma'(0) \in \mathbb{R}^n$ for some smooth map $\gamma : J \in \mathbb{R} \to M \subseteq \mathbb{R}^n$ with $\gamma(0) = p$ where J is an open interval with $0 \in J$. Given a chart $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ at p, with say $\sigma(a) = p$, if we let $\alpha(t) = \sigma^{-1}(\gamma(t))$ so that $\gamma(t) = \sigma(\alpha(t))$ for all $t \in I$ where I is an open interval with $0 \in I$, then we have $\gamma'(t) = D\sigma(\alpha(t))\alpha'(t)$ so that $\gamma'(0) = D\sigma(a)\alpha'(0)$. This shows that every tangent vector X_p on M at p lies in the range of the linear map $D\sigma(a)$ (equivalently X_p lies in the column space of the matrix $D\sigma(a)$). On the other hand, given any vector $A \in \mathbb{R}^m$, we can choose a smooth curve $\alpha : I \subseteq \mathbb{R} \to U\sigma \subseteq \mathbb{R}^m$ with $\alpha(0) = a$ and $\alpha'(0) = A$ (for example, we can let $\alpha(t) = a + tA$) and let $\gamma(t) = \sigma(\alpha(t))$ for all $t \in J$ where J is an open interval with $0 \in J$, and then we have $\gamma'(0) = D\sigma(a)\, A$. Thus the set of all tangent vectors on M at p is equal to the range of the linear map $D\sigma(a)$, which is a vector space. We define the **tangent space** of M at p, denoted by T_pM, to be the vector space of all tangent vectors on M at p. When σ is a chart on M at p with $\sigma(a) = p$ we have

$$T_pM = \text{Range}\, D\sigma(a).$$

Since σ is regular so that $D\sigma(a)$ is injective, the linear map $D\sigma(a)$ is a bijective linear map (that is a vector space isomorphism) from $\mathbb{R}^m$ to T_pM. In particular, the dimension of the tangent space T_pM is equal to the dimension of M.

4.31 Example: When $U \subseteq \mathbb{R}^m$ is open, so U is an m-dimensional submanifold of $\mathbb{R}^m$ with an atlas consisting of a single chart, namely the inclusion map $\sigma : U \to \mathbb{R}^m$ given by $\sigma(a) = a$, for all $a \in U$ we have $D\sigma(a) = I$, so that

$$T_aU = \text{Range } D\sigma(a) = \mathbb{R}^m\,.$$

4.32 Example: For every smooth map $\gamma : I \subseteq \mathbb{R} \to \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ with $\gamma(0) = p$, we have $\gamma(t) \cdot \gamma(t) = 1$ for all t, and differentiating gives $\gamma'(t) \cdot \gamma(t) = 0$ for all t. In particular, $\gamma'(0) \cdot p = 0$. Thus $T_p\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional space in $\mathbb{R}^n$ orthogonal to p, that is

$$T_p\mathbb{S}^{n-1} = \text{Span}\{p\}^\perp = \ker(p^T).$$

4.33 Remark: Let $M \subseteq \mathbb{R}^n$ be an m-dimensional smooth regular submanifold with $m < n$. When $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ is a chart on M at p with $\sigma(a) = p$, the map $\sigma : U_\sigma \to V_\sigma$ is invertible with inverse $\sigma^{-1} : V_\sigma \to U_\sigma$, but V_σ is not an open set in $\mathbb{R}^n$ (it is only an open set in M) and so it does not have a Jacobian matrix (we do not write $D\sigma^{-1}$). By the Parametric Function Theorem, as described in the proof of Theorem 4.27, we can restrict σ to an open subset $U_0 \subseteq U_\sigma$ and extend the inverse $\sigma^{-1} : \sigma(U_0) \to U_0$ to a smooth function $\phi : V_0 \times \mathbb{R}^{n-m} \to U_0$. This extended map ϕ is a one-sided inverse of σ (we have $\phi\sigma = I$ but $\sigma\phi \neq I$) and it has a Jacobian matrix which is a one-sided inverse of $D\sigma(a)$ $\big($we have $D\phi(p)D\sigma(a) = I$ but $D\sigma(a)D\phi(p) \neq I\big)$. The matrix $D\sigma(a)$ is an $n \times m$ matrix and the matrix $D\phi(p)$ is an $m \times n$ matrix with $m < n$. The linear map $D\sigma(a) : \mathbb{R}^m = T_aU \to T_pM$ is a vector space isomorphism and its inverse is the linear map $D\phi(p) : T_pM \subseteq \mathbb{R}^n \to \mathbb{R}^m = T_aU$. We could, logically speaking, denote this inverse map by $D\sigma(a)^{-1}$, but we avoid using this notation because it would give the impression that $D\sigma(a)$ is an invertible matrix.

4.34 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold, and let $p \in M$. A vector $X_p \in T_pM$ determines a differential operator on the space of all functions $f : M \to \mathbb{R}$ which are differentiable at the point p as follows: choose a smooth map $\gamma : J \subseteq \mathbb{R} \to M \subseteq \mathbb{R}^n$ with $\gamma(0) = p$ and $\gamma'(0) = X_p$, define $g : I \subseteq \mathbb{R} \to \mathbb{R}$ by $g(t) = f(\gamma(t))$, and define the (directional) **derivative** of f at p with respect to X_p to be

$$X_p(f) = g'(0)\,.$$

Note that although we have $g(t) = f(\gamma(t))$, it does not make sense (in general) to write $Dg(t) = Df(\gamma(t))\gamma'(t)$ because the function f is not defined in an open set in $\mathbb{R}^n$ (it is only defined on $M \subseteq \mathbb{R}^n$). Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ be any chart on M at p, say $\sigma(a) = p$, define $\alpha : I \subseteq \mathbb{R} \to U_\sigma \subseteq \mathbb{R}^m$ by $\alpha(t) = \sigma^{-1}(\gamma(t))$ so that we have $\gamma(t) = \sigma(\alpha(t))$, and let $A = \alpha'(0) \in T_aU = \mathbb{R}^m$. Since f is differentiable at $p \in M$ it follows (from the definition of differentiability) that $f\sigma$ is differentiable at a and we have $g(t) = f(\sigma(\alpha(t)))$ so that $g'(t) = D(f\sigma)(\alpha(t))\alpha'(t)$, hence

$$X_p(f) = g'(0) = D(f\sigma)(\alpha(0))\alpha'(0) = D(f\sigma)(a)\,A.$$

4.35 Note: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. Let $p \in M$ and let $X_p \in T_pM$. Let $\gamma : J \subseteq \mathbb{R} \to M \subseteq \mathbb{R}^n$ be a smooth map with $\gamma(0) = p$ and $\gamma'(0) = X_p$. Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ and $\rho : U_\rho \subseteq \mathbb{R}^m \to V_\rho \subseteq M$ be two charts at p on M with say $\sigma(a) = p = \rho(b)$, and recall that $\rho^{-1}\sigma$ is a regular change of coordinates. Let $\alpha(t) = \sigma^{-1}(\gamma(t))$ and $\beta(t) = \rho^{-1}(\gamma(t))$, and let $A = \alpha'(0) \in T_aU_\sigma = \mathbb{R}^m$ and $B = \beta'(0) \in T_bU_\rho = \mathbb{R}^m$, so that we have $X_p = D\sigma(a)A = D\rho(b)B$. Then we have $\beta(t) = \rho^{-1}(\gamma(t)) = \rho^{-1}(\sigma(\alpha(t)))$ so that $\beta'(t) = D(\rho^{-1}\sigma)(\alpha(t))\alpha'(t)$. Thus the vectors A and B are related by

$$B = \beta'(0) = D(\rho^{-1}\sigma)(\alpha(0))\alpha'(0) = D(\rho^{-1}\sigma)(a)A.$$

Thus a tangent vector $X_p \in T_pM$ determines, and is determined by, a tangent vector $A \in T_aU_\sigma = \mathbb{R}^m$ for each chart σ on M at at p, and the vectors for different charts are related by the above formula.

4.36 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. A **vector field** on M is a function $X : M \to \bigcup_{p\in M} T_pM$ such that $X_p = X(p) \in T_pM$ for all $p \in M$. When X is a vector field on M, given a chart $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$, for each $u \in U_\sigma$ there is a unique vector $A = A_\sigma(u) \in T_uU_\sigma = \mathbb{R}^m$ such that $X(\sigma(u)) = D\sigma(u)A_\sigma(u)$. We say that X is **continuous** when the function $A_\sigma : U_\sigma \to \mathbb{R}^m$ is continuous for every chart σ, and we say that X is **smooth** when A_σ is smooth for every chart σ.

4.37 Example: When $U \subseteq \mathbb{R}^m$ is open, so U is an m-dimensional submanifold of $\mathbb{R}^m$ with an atlas consisting of the inclusion map, a smooth (or continuous) vector field on U is simply a smooth (or continuous) map $X : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$.

4.38 Remark: By the change of coordinates formula in Note 4.35, the definitions of continuity and smoothness do not depend on a choice of atlas.

4.39 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. A smooth vector field on M determines a differential operator on the space of all smooth functions $f : M \to \mathbb{R}$ as follows: given a smooth vector field $X : M \to \bigcup\limits_{p\in M} T_pM$ and a smooth function $f : M \to \mathbb{R}$, we define the (directional) **derivative** of f with respect to the vector field X to be the map $X(f) : M \to \mathbb{R}$ given by $X(f)(p) = X_p(f)$. Note that this map $X(f)$ is smooth. Indeed by the definition of smoothness, the map $X(f)$ is smooth provided that the composite $X(f)\sigma$ is smooth for every chart σ on M, and when σ is a chart we have $X(f)(\sigma(u)) = D(f\sigma)(u)A_\sigma(u)$, which is a smooth function of u.

4.40 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold, and let X be a smooth vector field on M. An **integral curve** of X on M at p is a smooth map $\gamma : J \subseteq \mathbb{R} \to M$ with $\gamma(0) = p$ (where J is an open interval with $0 \in J$) such that

$$\gamma'(t) = X(\gamma(t)) \ \text{ for all } \ t \in J.$$

Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ be a chart at p with $\sigma(a) = p$. Let $\alpha : I \subseteq \mathbb{R} \to U_\sigma \subseteq \mathbb{R}^m$ be a smooth map with $\alpha(0) = a$, and let $\gamma(t) = \sigma(\alpha(t))$. Let A_σ be the smooth vector field on U_σ such that $X(\sigma(u)) = D\sigma(u)A_\sigma(u)$ for all $u \in U_\sigma$. Note that $\gamma'(t) = D\sigma(\alpha(t))\alpha'(t)$ and that $X(\gamma(t)) = X(\sigma(\alpha(t))) = D\sigma(\alpha(t))\, A_\sigma(\alpha(t))$. Since σ is regular so that $D\sigma(\alpha(t))$ is injective, we see that

$$\begin{aligned}
\gamma \text{ is an integral curve of } X \text{ on } M \text{ at } p &\iff \gamma'(t) = X(\gamma(t)) \text{ for all } t \in I \\
&\iff D\sigma(\alpha(t))\alpha'(t) = D\sigma(\alpha(t))\, A_\sigma(\alpha(t)) \text{ for all } t \in I \\
&\iff \alpha'(t) = A_\sigma(\alpha(t)) \text{ for all } t \in I \\
&\iff \alpha \text{ is an integral curve of } A_\sigma \text{ on } U_\sigma \text{ at } a.
\end{aligned}$$

4.41 Remark: When $M \subseteq \mathbb{R}^n$ is a smooth regular submanifold, X be a smooth vector field on M and $p \in M$, using existence and uniqueness theorems for differential equations one can show that there is a unique integral curve $\gamma : I \subseteq \mathbb{R} \to M$ of X on M at p defined on a maximal open interval $I \subseteq \mathbb{R}$ with $0 \in I$.

4.42 Definition: Let $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^\ell$ be smooth regular submanifolds, and let $f : M \to N$ be a smooth map. When $X_p \in T_pM$, the **pushforward** of the vector X_p by the map f is the vector $f_*X_p \in T_{f(p)}N$ defined as follows: choose a smooth curve $\gamma : I \subseteq \mathbb{R} \to M$ with $0 \in I$, $\gamma(0) = p$ and $\gamma'(0) = X_p$ and define $f_*X_p = \delta'(0) \in T_{f(p)}N$ where $\delta(t) = f(\gamma(t))$. Let σ be a chart on M at p, say $\sigma(a) = p$, let $A_\sigma \in T_aU = \mathbb{R}^m$ be the unique vector such that $D\sigma(a)A_\sigma = X_p$, and let $\alpha : I \subseteq \mathbb{R} \to U_\sigma$ be given by $\alpha(t) = \sigma^{-1}(\gamma(t))$ so that we have $\gamma(t) = \sigma(\alpha(t))$ for $t \in I$. Then $X_p = \gamma'(0) = D\sigma(a)\alpha'(0)$ so that $\alpha'(0) = A_\sigma$, and since $\delta(t) = f(\sigma(\alpha(t)))$, we have

$$f_*X_p = \delta'(0) = D(f\sigma)(a)\alpha'(0) = D(f\sigma)(a)A_\sigma.$$

This gives a formula for f_*X_p in terms of the local coordinates, and it shows that the vector f_*X_p does not depend on the choice of the curve $\gamma(t)$. In the case that f is a diffeomorphism with inverse $g : N \to M$, and X is a vector field on M (so the we have a vector $X(p) = X_p \in T_pM$ for every point $p \in M$), the **pushforward** of X by f is the vector field f_*X on N given by $(f_*X)(f(p)) = f_*X_p \in T_{f(p)}N$. Verify, as an exercise, that if the vector field X is smooth (and the function f is smooth) then the vector field f_*X is smooth.

4.43 Example: When $U \subseteq \mathbb{R}^m$ (so U is an m-dimensional submanifold of $\mathbb{R}^m$ with an atlas consisting of the inclusion map) and $f : U \subseteq \mathbb{R}^m \to N \subseteq \mathbb{R}^\ell$ is a smooth map, and $A \in T_aU = \mathbb{R}^m$, we have $f_*A = Df(a)A \in T_{f(a)}N \subseteq \mathbb{R}^\ell$.

4.44 Example: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold and let $\sigma : U_\sigma \to V_\sigma \subseteq M$ be a chart on M at p with say $\sigma(a) = p$. Recall that σ is a diffeomorphism from $U_\sigma \subseteq \mathbb{R}^m$ to $V_\sigma \subseteq M$. Let $\phi = \sigma^{-1} : V_\sigma \to U_\sigma$. Let X be a smooth vector field on M, and let X_σ be the restriction of X to the open set $V_\sigma \subseteq M$. For each $u \in U_\sigma$, let $A_\sigma(u)$ be the unique vector in $T_uU_\sigma = \mathbb{R}^m$ such that $D\sigma(u)A_\sigma(u) = X(\sigma(u)) = X_\sigma(\sigma(u))$. Then we have $A_\sigma = \phi_*X_\sigma$ and $X_\sigma = \sigma_*A_\sigma$.

4.45 Definition: Let $M \subseteq \mathbb{R}^n$ be a smooth regular m-dimensional submanifold of $\mathbb{R}^n$. Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ be a chart on M. The **Riemannian metric** on U_σ induced by σ is the smooth map $g = g_\sigma : U_\sigma \to M_{m\times m}(\mathbb{R})$ given by

$$g(u) = D\sigma(u)^T D\sigma(u).$$

The Riemannian metric gives an inner product, hence also a norm, on $T_uU_\sigma = \mathbb{R}^m$ at each point $u \in U_\sigma$: for $u \in U_\sigma$ and $A, B \in T_uU_\sigma = \mathbb{R}^m$ we have

$$\langle A, B\rangle = B^T g(u) A \quad \text{and} \quad \|A\| = \sqrt{\langle A, A\rangle} = \sqrt{A^T g(u) A}.$$

When $\alpha : I \subseteq \mathbb{R} \to U_\alpha \subseteq \mathbb{R}^m$ is a differentiable curve and $\gamma : I \subseteq \mathbb{R} \to M \subseteq \mathbb{R}^n$ is given by $\gamma(t) = \sigma(\alpha(t))$, we have $\gamma'(t) = D\sigma(\alpha(t))\alpha'(t)$ hence

$$|\gamma'(t)| = \sqrt{\gamma'(t)^T\gamma'(t)} = \sqrt{\alpha'(t)^T g(\alpha(t))\alpha'(t)} = \|\alpha'(t)\|.$$

When $[a, b] \subseteq I$, the **length** of α on $[a, b]$ with respect to the metric g on U_σ is equal to the length of γ on $[a, b]$ with respect to the standard metric in $\mathbb{R}^n$, that is

$$L_\gamma\big([a, b]\big) = \int_{[a,b]} dL = \int_a^b |\gamma'(t)|\, dt = \int_a^b \|\alpha'(t)\|\, dt\,.$$

More generally, for a continuous function $f : M \to \mathbb{R}$, we define the integral f along γ on $[a, b]$ to be

$$\int_{[a,b]} f\; dL = \int_a^b f(\gamma(t))\, |\gamma'(t)|\, dt = \int_a^b (f\sigma)(\alpha(t))\, \|\alpha'(t)\|\, dt.$$

When $R \subseteq U_\sigma$ is a closed Jordan region, we define the **volume** of R with respect to the metric g_σ to be

$$\mathrm{Vol}_\sigma(R) = \int_R dV = \int_R \sqrt{\det g(u)}\; du_1\, du_2 \cdots du_m\,.$$

More generally, for a continuous function $f : M \to \mathbb{R}$ we define the integral of f on R under σ to be

$$\int_R f\, dV = \int_R (f\sigma)(u)\sqrt{\det g(u)}\; du_1\, du_2 \cdots du_m\,.$$

For some motivation behind the above definition, review Remark 2.9 and have a look at Theorem 1.2 in Appendix 1.

4.46 Note: Let $M \subseteq \mathbb{R}^n$ be a smooth regular submanifold. Let $\sigma : U_\sigma \subseteq \mathbb{R}^m \to V_\sigma \subseteq M$ and $\rho : U_\rho \subseteq \mathbb{R}^m \to V_\rho \subseteq M$ be two charts with intersecting images, and recall that the map $\phi = \rho^{-1}\sigma$ is a smooth regular change of coordinates with inverse $\psi = \sigma^{-1}\rho$. Since $\rho = \sigma\sigma^{-1}\rho = \sigma\psi$ we have $D\rho = D\sigma D\psi$ and so

$$g_\rho = D\rho^T D\rho = (D\sigma D\psi)^T(D\sigma D\psi) = D\psi^T D\sigma^T D\sigma D\psi = D\psi^T g_\sigma D\psi.$$

It follows that $\det d_\rho = \det g_\sigma (\det D\psi)^2$ and so if R is a closed Jordan region in U_σ and Q is a closed Jordan region in U_ρ) such that $\sigma(R) = \rho(Q)$ then (by the change of variables formula for integration) we have

$$\begin{aligned} A_\sigma(R) &= \int_R \sqrt{\det g_\sigma(u)}\; du_1 \cdots du_m = \int_Q \sqrt{\det g_\sigma(\psi(v))}\big|\det D\psi(v)\big|\; dv_1 \cdots dv_m \\ &= \int_Q \sqrt{\det g_\rho(v)}\; dv_1 \cdots dv_m = A_\rho(Q). \end{aligned}$$

4.47 Remark: If $\gamma : I \subseteq \mathbb{R} \to M \subseteq \mathbb{R}^n$ is a smooth curve and $[a,b] \subseteq I$, then the image $\gamma\big([a,b]\big)$ might not be contained in the image of a single chart. In this case, we can choose a partition $a = a_0 < a_1 < \cdots < a_\ell = b$ such that the image under γ of each subinterval is contained in the image of a chart, say $\gamma\big([x_{k-1}, x_k]\big) \subseteq V_k$ where $\sigma_k : U_k \subseteq \mathbb{R}^m \to V_k \subseteq M$ is a chart. Then for $\alpha_k(t) = \sigma_k^{-1}(\gamma(t))$, we can calculate the integral of f along γ in terms of local coordinates

$$\int_{[a,b]} f\, dL = \int_a^b f(\gamma(t))\, |\gamma'(t)|\, dt = \sum_{k=1}^{\ell} \int_{a_{k-1}}^{a_k} (f\sigma_k)(\alpha_k(t))\, \|\alpha_k{}'(t)\|\, dt.$$

In the same way, we can calculate the integral of a continuous function $f : M \to \mathbb{R}$ on a region $S \subseteq M$ which does not lie in the image of a single chart. It is difficult to explain precisely how this can be done and to prove rigorously that the resulting integral is well-defined, but let us give an informal description. Suppose that we can cut the region S into ℓ subregions of the form $\sigma_k(R_k)$ where each $\sigma_k : U_k \subseteq \mathbb{R}^m \to V_k \subseteq M$ is a chart and R_k is a closed Jordan region in U_k, and suppose that the subregions $\sigma_k(R_k)$ only overlap along their boundaries. Then the integral of f on S is the sum

$$\int_S f\ dA = \sum_{k=1}^{\ell} \int_{R_k} f\, dV = \sum_{k=1}^{n} \int_{R_k} (f\sigma_k)(u) \sqrt{\det g_{\sigma_k}(u)}\ du_1\, du_2 \cdots du_m\, .$$

In particular, if the entire submanifold $M \subseteq \mathbb{R}^n$ can be cut into finitely many such regions $\sigma_k(R_k)$, then we can calculate the integral $\int_M f\ dA$ of a continuous function f on the manifold, and we can calculate the volume of the manifold $V = \int_M dA$.