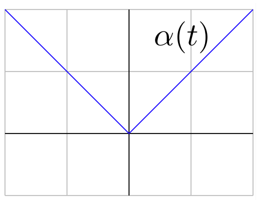
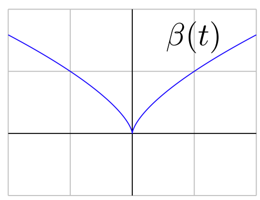
Chapter 1. Curves

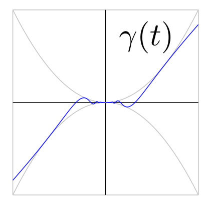
Curves in $\mathbb{R}^n$

1.1 Definition: A (parametrized) **curve** in $\mathbb{R}^n$ is a continuous map $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where I is a nonempty interval. We can write $\alpha(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)$ where each $x_k : I \subseteq \mathbb{R} \to \mathbb{R}$ is continuous. When $a \in I$ and $\alpha'(a) = \big({x_1}'(a), \cdots, {x_n}'(a)\big)$ exists, $\alpha'(a)$ is called the **tangent vector** to α at $t = a$. We say that α is $\mathcal{C}^k$ when the k^{th} order derivative of α exists and is continuous on I, we say that α is **smooth** or $\mathcal{C}^\infty$ when α is $\mathcal{C}^k$ for all $k \in \mathbb{Z}^+$, and we say that α is **regular** when α is $\mathcal{C}^1$ with $\alpha'(t) \neq 0$ for all $t \in I$. Unless otherwise stated, we shall always assume curves are smooth and regular.

1.2 Example: The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = \big(t, |t|\big)$ is not regular because $\alpha'(0)$ does not exist. The curve $\beta : \mathbb{R} \to \mathbb{R}^2$ given by $\beta(t) = (t^3, t^2)$ is not regular because $\beta'(0) = 0$. The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(0) = (0,0)$ and $\gamma(t) = \big(t, t^2 \sin \frac{1}{t}\big)$ for $t \neq 0$ is differentiable but not regular because γ' is not continuous at $t = 0$.





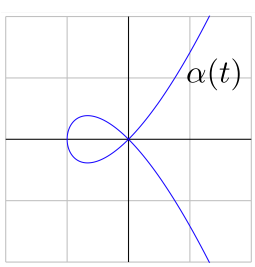


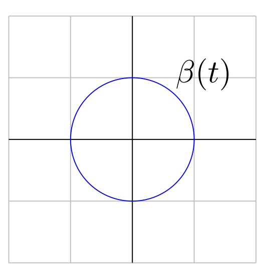
1.3 Theorem: *Every regular curve in $\mathbb{R}^n$ is locally injective.*

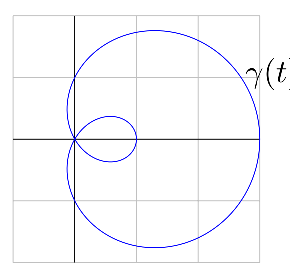
Proof: Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve, write $\alpha(t) = \big(x_1(t), \cdots, x_n(t)\big)$, and let $a \in I$. Since $\alpha'(a) \neq 0$ we have ${x_k}'(a) \neq 0$ for some index k, say ${x_k}'(a) > 0$ (the case that ${x_k}'(a) < 0$ is similar). Since α' is continuous, ${x_k}'$ is continuous. Since ${x_k}'$ is continuous and ${x_k}'(a) > 0$ we can choose $\delta > 0$ so that $|t - a| < \delta \implies {x_k}'(t) > 0$. Then x_k is increasing, hence injective, in the interval $(a - \delta, a + \delta) \cap I$, and so α is injective in the same interval.

1.4 Example: The curves $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}^2$ from Example 1.2 are not regular, but they are all injective, so a curve does not necessarily need to be regular in order to be injective.

1.5 Example: The alpha curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ which is given by $\alpha(t) = \big(t^2 - 1, t(t^2 - 1)\big)$, the circle $\beta : \mathbb{R} \to \mathbb{R}^2$ which is given by $\beta(t) = (\cos t, \sin t)$, and the limçon $\gamma : \mathbb{R} \to \mathbb{R}^2$ which is given by $\gamma(t) = \big((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t\big)$, are all regular, so they are all locally injective, but they are not (globally) injective (the alpha curve crosses itself with $\alpha(1) = \alpha(-1) = (0,0)$, the circle is periodic with $\beta(t + 2\pi k) = \beta(t)$ for all $k \in \mathbb{Z}$, and the limaçon is periodic and crosses itself).







1.6 Example: The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(0) = 0$ and $\alpha(t) = \big(t^2, t^2 \sin \frac{1}{t}\big)$ for $t \neq 0$ is differentiable, but not regular since $\alpha'(0) = 0$, and (as you can verify) it is not locally injective at $t = 0$.

1.7 Definition: For a curve $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$, the **length** of α on $[a,b]$ is

$$L = L_\alpha\big([a,b]\big) = \sup\Big\{ \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \,\Big|\, a = t_0 < t_1 < t_2 < \cdots < t_p = b \Big\}$$

(which can be infinite) and we say that α is **rectifiable** on $[a,b]$ when $L_\alpha\big([a,b]\big)$ is finite.

1.8 Theorem: *Let $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then α is rectifiable with length*

$$L = L_\alpha\big([a,b]\big) = \int_a^b \big|\alpha'(t)\big|\,dt.$$

Proof: For a partition $P = (t_0, t_1, \cdots, t_p)$, where $a = t_0 < t_1 < \cdots < t_p = b$, let us write

$$L(\alpha, P) = \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \quad \text{and} \quad S(\alpha, P) = \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(t_j - t_{j-1})$$

so $L(\alpha, P)$ is the sum which approximates Length(α) and $S(\alpha, P)$ is the Riemann sum (using right endpoints) which approximates the integral $\int_a^b |\alpha'(t)|\,dt$. First note that

$$\begin{aligned} L(\alpha,P) &= \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \le \sum_{j=1}^{p}\sum_{k=1}^{n} \big|x_k(t_j) - x_k(t_{j-1})\big| = \sum_{j=1}^{n}\sum_{k=1}^{p} \big|x_k{}'(c_{j,k})\big(t_j - t_{j-1}\big)\big| \\ &\le \sum_{j=1}^{n}\sum_{k=1}^{p} M(t_j - t_{j-1}) = \sum_{j=1}^{n} M(b-a) = nM(b-a) \end{aligned}$$

where we used the Mean Value Theorem to choose points $c_{j,k}$ between t_{j-1} and t_j such that $\big(x_k(t_j) - x_k(t_{j-1})\big) = x_k{}'(c_{j,k})(t_j - t_{j-1})$ and we let $M = \max\big\{|x_k{}'(t)|\,\big|\, 1 \le k \le n, t \in [a,b]\big\}$. This shows that $L = L_\alpha\big([a,b]\big)$ is finite.

Note that if $P = (t_0, t_1, \cdots, t_p)$ is a partition of $[a,b]$, and Q is a partition which is obtained by adding one more point, say $Q = (t_0, t_1, \cdots, t_{j-1}, s, t_j, \cdots, t_p)$, then we have $L(\alpha, P) \le L(\alpha, Q)$ because $\big|\alpha(t_j) - \alpha(t_{j-1}\big) \le \big|\alpha(t_j) - \alpha(s)\big| + \big|\alpha(s) - \alpha(t_{j-1})\big|$. It follows (by induction) that when Q is any partition with $P \subseteq Q$ we have

$$L(\alpha, P) \le L(\alpha, Q) \le L.$$

Also note that for any partition P, with $c_{j,k}$ chosen as above, we have

$$\begin{aligned} \Big|L(\alpha,P) - S(\alpha,P)\Big| &= \Big| \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| - \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(t_j - t_{j-1}) \Big| \\ &= \Big| \sum_{j=1}^{p} \Big|\big(x_1(t_j) - x_1(t_{j-1}), \cdots, x_n(t_j) - x_n(t_{j-1}\big)\Big| - \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(t_j - t_{j-1}) \Big| \\ &= \Big| \sum_{j=1}^{p} \Big|\big(x_1{}'(c_{j,1}), \cdots, x_n{}'(c_{j,n})\big)\Big|(t_j - t_{j-1}) - \sum_{j-1}^{p} \Big|\big(x_1{}'(t_j), \cdots, x_n{}'(t_j)\big)\Big|(t_j - t_{j-1}) \Big| \\ &\le \sum_{j=1}^{p} \Big| \big|\big(x_1{}'(c_{j,1}), \cdots, x_n{}'(c_{j,n})\big)\big| - \big|\big(x_1{}'(t_j), \cdots, x_n{}'(t_j)\big)\big| \Big| (t_t - t_{j-1}) \\ &\le \sum_{j=1}^{p} \Big|\big(x_1{}'(c_{j,1}) - x_1{}'(t_j), x_n{}'(c_{j,n}) - x_n{}'(t_j)\big)\Big|(t_j - t_{j-1}) \\ &\le \sum_{j=1}^{p}\sum_{k=1}^{n} \Big|x_k{}'(c_{j,k}) - x_k{}'(t_j)\Big|(t_j - t_{j-1}). \end{aligned}$$

Let $\epsilon > 0$. Since each $x_k{}'$ is continuous (hence uniformly continuous) on $[a,b]$ and since $|\alpha'|$ is continuous (hence Riemann integrable) on $[a,b]$, we can choose $\delta > 0$ such that for all $s,t\in[a,b]$ with $|s-t|<\delta$ we have $\left|x_k{}'(s)-x_k{}'(t)\right|<\frac{\epsilon}{3n(b-a)}$ for all k , and such that for every partition $P=(t_0,t_1,\cdots,t_p)$ with $|P|<\delta$ we have $\left|\int_a^b|\alpha'(t)|\,dt-S(\alpha,P)\right|<\frac{\epsilon}{3}$ where $|P|$ is the size of the partition P, that is $|P|=\max\left\{t_j-t_{j-1}\,\middle|\,1\le j\le p\right\}$. Choose a partition P_1 with $|P_1|<\delta$ and choose a partition P_2 such that $\left|L-L(\alpha,P_2)\right|<\frac{\epsilon}{3}$ then let $P=P_1\cup P_2$. Since $P_2\subseteq P$ we have $L(\alpha,P_2)\le L(\alpha,P)\le L$ so that $\left|L-L(\alpha,P)\right|\le\left|L-L(\alpha,P_2)\right|<\frac{\epsilon}{3}$. Since $|P|\le|P_1|<\delta$ we have $\left|\int_a^b|\alpha'(t)|\,dt-S(\alpha,P)\right|<\frac{\epsilon}{3}$. Also since $|P|<\delta$, for all of the points $c_{j,k}$ we have $|c_{j,k}-t_j|<\delta$ so that $\left|x_k{}'(c_{j,k})-x_k{}'(t_j)\right|<\frac{\epsilon}{3n(b-a)}$ and hence (as shown above) $\left|L(\alpha,P)-S(\alpha,P)\right|\le\sum_{j=1}^{p}\sum_{k=1}^{n}\left|x_k{}'(c_{j,k})-x_k{}'(t_j)\right|(t_j-t_{j-1})<\frac{\epsilon}{3}$. Thus

$$\begin{aligned}\left|L-\int_a^b|\alpha'(t)|\,dt\right|&\le\left|L-L(\alpha,P)\right|+\left|L(\alpha,P)-S(\alpha,P)\right|+\left|S(\alpha,P)-\int_a^b|\alpha'(t)|\,dt\right|\\&<\tfrac{\epsilon}{3}+\tfrac{\epsilon}{3}+\tfrac{\epsilon}{3}=\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, it follows that $L=\int_a^b|\alpha'(t)|\,dt$, as required.

1.9 Example: A curve which is differentiable, but not $\mathcal{C}^1$, can have infinite length. For example, consider the curve $\alpha:[0,1]\to\mathbb{R}^2$ given by $\alpha(t)=\big(x(t),y(t)\big)$ where $x(t)=t$ $y(t)=t^2\cos\frac{\pi}{t^2}$ when $t\neq0$ with $y(0)=0$. Note that $x(y)$ and $y(t)$ are both differentiable (with $y'(0)=0$) but $y'(t)$ is not continuous at 0 (as you can check).

Let P be the partition $P=(t_0,t_1,\cdots t_p)$ with $t_0=0$ and $t_j=\frac{1}{\sqrt{p-j+1}}$, that is let $P=\big(0,\frac{1}{\sqrt{p}},\frac{1}{\sqrt{p-1}},\cdots,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{2}},1\big)$. We have $y(t_j)=\frac{1}{p-j+1}\cos(p-j+1)\pi=\frac{(-1)^{p-j+1}}{p-j+1}$ for $1\le j\le p$, and hence $\left|y(t_j)-y(t_{j-1})\right|=\left|\frac{1}{p-j+1}+\frac{1}{p-j+2}\right|\ge\frac{2}{p-j+2}$ for $2\le j\le p$. Letting $\ell=p-j+2$ we have

$$\sum_{j=1}^{p}\left|\alpha(t_j)-\alpha(t_{j-1}\right|\ge\sum_{j=2}^{p}\left|y(t_j)-y_(t_{j-1})\right|\ge\sum_{j=2}^{p}\frac{2}{p-j+2}=\sum_{\ell=2}^{p}\frac{2}{\ell}.$$

Since $\sum\frac{2}{\ell}$ diverges, it follows that $L_\alpha\big([a,b]\big)=\infty$.

1.10 Definition: When $\alpha:I\subseteq\mathbb{R}\to\mathbb{R}^n$ is a continuous curve and $s:I\subseteq\mathbb{R}\to J\subseteq\mathbb{R}$ is a homeomorphism with inverse $t=t(s)$, the curve $\beta:J\subseteq\mathbb{R}\to\mathbb{R}^n$ defined by $\beta(s)=\alpha\big(t(s)\big)$ is called a **reparameterisation** of α, and the map s is called a **change of parameter** (or a **change of coordinates**). When s is $\mathcal{C}^1$ with $s'(t)\neq0$ for all t, we say that s is **regular**. By the Inverse Function Theorem, if $s=s(t)$ is smooth (or $\mathcal{C}^k$) and regular then so is its inverse $t=t(s)$. When $s'(t)>0$ for all t we say s **preserves direction** and when $s'(t)<0$ for all t we say s **reverses direction**. When α and s are both smooth (or $\mathcal{C}^k$) and regular, so is β, and for $t=t(s)$ we have $\beta'(s)=\alpha'\big(t(s)\big)t'(s)=\frac{\alpha'(t)}{s'(t)}$. When $|\beta'(s)|=1$ for all $s\in J$, we say that β is **parameterised by arclength**. Unless otherwise stated, we shall assume that any change of coordinates is smooth and regular.

1.11 Theorem: *Every regular curve can be reparameterised by arclength, using a regular direction-preserving change of coordinates.*

Proof: Let $\alpha:I\subseteq\mathbb{R}\to\mathbb{R}^n$ be a regular curve. Let $a\in I$ and define $s(t)=\int_a^t\left|\alpha'(r)\right|dr$. Note that $s'(t)=\left|\alpha'(t)\right|>0$ so $s(t)$ is regular and strictly increasing, and it maps the interval I to an interval J, and if α is $\mathcal{C}^k$ then so is $s=s(t)$. By the inverse function $t=t(s)$ satisfies $t'(s)=\frac{1}{s'(t)}=\frac{1}{|\alpha'(t)|}$. The reparameterised curve $\beta:J\to\mathbb{R}^n$ given by $\beta(s)=\alpha\big(t(s)\big)$ satisfies $\beta'(s)=\alpha'\big(t(s)\big)t'(s)=\frac{\alpha'(t(s))}{|\alpha'(t(s))|}$ so that $\left|\beta'(s)\right|=1$ for all $s\in J$.

Curves in $\mathbb{R}^2$

1.12 Definition: Let $\beta : J \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve parameterised by arclength. For a vector $u = (x, y) \in \mathbb{R}^2$, write $u^\times = (-y, x)$ and note that $|u^\times| = |u|$. The **unit tangent vector** and the **unit normal vector** of β at s are the vectors

$$T(s) = T_\beta(s) = \beta'(s),$$
$$N(s) = N_\beta(s) = T(s)^\times.$$

Since β is parametrized by arclength, $|T(s)| = |\beta'(s)| = 1$ and $|N(s)| = |\beta'(s)^\times| = 1$ for all s. For all s we have $\beta'(s) \cdot \beta'(s) = \big|\beta'(s)\big|^2 = 1$. By differentiation both sides we obtain $\frac{d}{ds}\big(\beta'(s) \cdot \beta'(s)\big) = 0$, that is $2\,\beta'(t) \cdot \beta''(t) = 0$. Thus $\beta''(s)$ is orthogonal to $\beta'(s) = T(s)$, and so $\beta''(s)$ lies in the span of $T(s)^\times = N(s)$. We define the **signed curvature** of β at s to be the real number $k(s) = k_\beta(s)$ such that

$$\beta''(s) = k(s)\, N(s) = k_\beta(s)\, N_\beta(s).$$

Since $|N_\beta(s)| = 1$ we have $|\beta''(s)| = \big|k_\beta(s)\big|$. The **scalar curvature** of β at s is

$$\kappa(s) = \kappa_\beta(s) = \big|k(s)\big| = \big|\beta''(s)\big|.$$

When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ is a smooth regular curve we first reparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$, and then we define $T(t) = T_\alpha(t) = T_\beta(s(t))$, $N(t) = N_\alpha(t) = N_\beta(s(t))$, $k(t) = k_\alpha(t) = k_\beta(s(t))$ and $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$, and we call these the unit tangent vector, the unit normal vector, the signed curvature, and the scalar curvature, of α at t. The following theorem shows that these are well-defined, that is they do not depend on the choice of $a \in I$.

1.13 Theorem: *For a smooth regular curve $\alpha = \alpha(t)$ we have*

$$T = \frac{\alpha'}{|\alpha'|}\ ,\ N = \left(\frac{\alpha'}{|\alpha'|}\right)^\times$$
$$k = \frac{\det_2(\alpha', \alpha'')}{|\alpha'|^3} = \frac{(\alpha' \times \alpha'') \cdot e_3}{|\alpha'|^3} = \frac{\det_3(\alpha', \alpha'', e_3)}{|\alpha'|^3}$$
$$\kappa = \frac{\big|\det_2(\alpha', \alpha'')\big|}{|\alpha'|^3} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

where $\det_2(\alpha', \alpha'')$ is the determinant of the 2×2 matrix with columns $\alpha', \alpha'' \in \mathbb{R}^2$, and where we identify $\alpha', \alpha'' \in \mathbb{R}^2$ with $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ so that $\alpha' \times \alpha''$ is the cross product of two vectors $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ and $\det_3(\alpha', \alpha'', e_3)$ is the determinant of the 3×3 matrix whose first two columns are $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ and whose last column is the 3^{rd} standard basis vector e_3.

Proof: First verify (easily) that when we identify $u, v \in \mathbb{R}^2$ with $\binom{u}{0}, \binom{v}{0} \in \mathbb{R}^3$ we have

$$u^\times \cdot v = \det_2(u, v) = (u \times v) \cdot e_3 = \det_3(u, v, e_3)$$

and $\big|\det_2(u, v)\big| = \big|u \times v\big|$.

Peparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$. We have $T_\beta(s) = \beta'(s)$ and $N_\beta(s) = \beta'(s)^\times$. Let us find formulas for $k_\beta(s)$ and $\kappa_\beta(s)$. By definition, $k_\beta(s)\beta'(s)^\times = k_\beta(s) N_\beta(s) = \beta''(s)$. Take the dot product of both sides with $\beta'(s)^\times$ to get

$$k_\beta(s) = \beta'(s)^\times \cdot \beta''(s)$$
$$\kappa_\beta(s) = \big|k_\beta(s)\big| = \big|\beta'(s) \times \beta''(s)\big|.$$

Now let us find formulas for $T(t) = T_\alpha(t) = T_\beta(s(t))$, $N(t) = N_\alpha(t) = N_\beta(s(t))$, $k(t) = k_\alpha(t) = k_\beta(s(t))$ and $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$. We have $\alpha(t) = \beta(s(t))$ so that $\alpha'(t) = \beta'(s(t))s'(t)$. Since $|\beta'(s(t))| = 1$ and $s'(t) > 0$, it follows that $\big|\alpha'(t)\big| = s'(t)$. Since $\beta''(s) = k_\beta(s)N_\beta(s)$ and $|N_\beta(s)| = |T_\beta(s)| = 1$, we have $\big|\beta''(s)\big| = \big|k_\beta(s)\big| = \kappa_\beta(s)$. Since $\beta''(s)$ is orthogonal to $\beta'(s)$ (see Definition 1.12) we have $\kappa_\beta(s) = \big|\beta''(s)\big| = \big|\beta'(s)\times\beta''(s)\big|$. We have

$$\alpha(t) = \beta(s(t))\ ,\ \alpha'(t) = \beta'(s(t))s'(t)\ ,\ \alpha''(t) = \beta''(s(t))s'(t)^2 + \beta'(s(t))s''(t),$$
$$\alpha'(t)\times\alpha''(t) = \big(\beta'(s(t))\times\beta''(s(t))\big)(s')^3.$$

Since $\big|\alpha'(t)\big| = s'(t)$ this gives

$$\frac{\alpha'(t)}{|\alpha'(t)|} = \beta'(s(t)) = T_\beta(s(t)) = T(t)\ \ ,\ \ \Big(\frac{\alpha'(t)}{|\alpha'(t)|}\Big)^\times = T_\beta(s(t))^\times = N_\beta(s(t)) = N(t),$$
$$\frac{\alpha'(t)\times\alpha''(t)}{|\alpha'(t)|^3} = \beta'(s(t))\times\beta''(s(t)),$$
$$\frac{\big|\alpha'(t)\times\alpha''(t)\big|}{|\alpha'(t)|^3} = \Big|\beta'(s(t))\times\beta''(s(t))\Big| = \kappa_\beta(s(t)) = \kappa(t)\,.$$

Finally note that since $\beta'' = k_\beta N_\beta = k_\beta(\beta')^\times$ we have $\beta'\times\beta'' = k_\beta\,\beta'\times(\beta')^\times = k_\beta\,e_3$ and so $k_\beta = (\beta'\times\beta'')\cdot e_3$.

1.14 Theorem: *For a smooth regular curve α in $\mathbb{R}^2$, the curvature of α is identically zero if and only if (the image of) α lies on a straight line.*

Proof: Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve. Choose $a \in I$ and reparametrize α by arclength by setting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(t)|\,dt$. Suppose that $\kappa(t) = 0$ for all t. Then we have $0 = \kappa(t(s)) = \big|\beta''(s)\big|$ for all s so that $\beta''(s) = 0$ for all s. By integrating once we obtain $\beta'(s) = u$ for some $u \in \mathbb{R}^2$ since $|\beta'(s)| = 1$, u is a unit vector) and by integrating again we obtain $\beta(s) = p + su$ for some $p \in \mathbb{R}^2$. Thus $\alpha(t) = p + s(t)u$ for all t so that α lies on the line through p in the direction of u.

Suppose, conversely, that (the image of) α lies on a straight line, say the line $p + su$ where $p, u \in \mathbb{R}^2$ and $|u| = 1$. Then for every $t \in I$ there is a (unique) $s = s(t)$ such that $\alpha(t) = p + s(t)u$. We remark that taking the dot product with u gives $s(t) = (\alpha(t) - p)\cdot u$ for all t so we see that $s(t)$ is smooth. Since $\alpha(t) = p + s(t)u$, we have $\alpha'(t) = s'(t)u$ and $\alpha''(t) = s''(t)u$ so that $\alpha(t)\times\alpha''(t) = s'(t)s''(t)\,u\times u = 0$ and kence $\kappa(t) = 0$ for all t.

1.15 Definition: Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve, let $a \in I$, and suppose that $\kappa(a) \neq 0$. We define the **osculating circle** (or the **best-fit circle**) of α at $t = a$ as follows. Let $p = \alpha(a)$, $T = T(a)$, $N = N(a)$, $k = k(a)$ and $\kappa = \kappa(a)$. Reparametrize by arclength, letting $\beta(s) = \alpha(t(s)$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$ so that we have $\beta(0) = p$. $\beta'(0) = T$ and $\beta''(0) = kN$. The osculating circle at $t = a$ is the circle given by

$$\begin{aligned}\sigma(s) &= \big(p + \tfrac{1}{k}\,N\big) - \tfrac{1}{k}\cos(ks)N + \tfrac{1}{k}\sin(ks)T\\ \sigma'(s) &= \sin(ks)N + \cos(ks)T\\ \sigma''(s) &= k\cos(ks)N - k\sin(ks)T\end{aligned}$$

which is the circle of radius $R = \frac{1}{\kappa}$ centered at $p + \frac{1}{k}N$, parametrized by arclength (since $|\sigma'(s)| = 1$ for all s), such that $\sigma(0) = p = \beta(0)$, $\sigma'(0) = T = \beta'(0)$ and $\sigma''(0) = kN = \beta''(0)$.

1.16 Note: When α is a smooth regular curve, the scalar curvature at $t = a$ is equal to the reciprocal of the radius of the best-fit circle at $t = a$.

1.17 Theorem: *(Polar Coordinates) Let $I \subseteq \mathbb{R}$ be an interval with $a \in I$, let $p \in \mathbb{R}^2$, and let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a continuous curve in $\mathbb{R}^2$ with $\alpha(t) \neq p$ for any $t \in I$. Let $r_0 = \big|\alpha(a) - p\big|$ and choose $\theta_0 \in \mathbb{R}$ such that $\alpha(a) - p = r_0(\cos\theta_0, \sin\theta_0)$ (θ_0 is unique up to an integer multiple of 2π). Then there exist unique continuous functions $r, \theta : I \to \mathbb{R}$ with $r(a) = r_0$ and $\theta(a) = \theta_0$ such that*

$$\alpha(t) = p + r(t)\big(\cos\theta(t), \sin\theta(t)\big)$$

for all $t \in I$. Moreover, if α is smooth (or $\mathcal{C}^k$) then so are the functions r and θ.

Proof: We omit the proof, but we remark that it is surprisingly involved.

1.18 Definition: For a continuous curve $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^2$ with $\alpha(t) \neq p$ for any t, we define the **winding number** $\text{Wind}(\alpha, p)$ of α about p as follows. We let $r_0 = \big|\alpha(a) - p\big|$ and choose $\theta_0 \in [0, 2\pi)$ so that $\alpha(a) = p + r_0(\cos\theta_0, \sin\theta_0)$, then we let $r, \theta : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be the unique continuous maps such that $\alpha(t) = p + r(t)\big(\cos\theta(t), \sin\theta(t)\big)$ for all $t \in [a,b]$, and then we define

$$\text{Wind}(\alpha, p) = \tfrac{1}{2\pi}\big(\theta(b) - \theta(a)\big).$$

When α is regular, we define the **turning number** of α to be

$$\text{Turn}(\alpha) = \text{Wind}\big(\alpha', 0).$$

1.19 Theorem: *Let $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$ and write $\alpha(t) = \big(x(t), y(t)\big)$.*
(1) If α is a $\mathcal{C}^1$ curve with $\alpha(t) \neq 0$ for any $t \in [a,b]$ then

$$\text{Wind}(\alpha, 0) = \tfrac{1}{2\pi}\int_a^b \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2}\, dt.$$

(2) If α is $\mathcal{C}^2$ regular curve then

$$\text{Turn}(\alpha) = \tfrac{1}{2\pi}\int_a^b k_\alpha(t)\,\big|\alpha'(t)\big|\, dt.$$

Proof: To prove Part 1, write α in polar coordinates as $\alpha(t) = r(t)\big(\cos\theta(t), \sin\theta(t)\big)$, that is write $x = r\cos\theta$ and $y = r\sin\theta$ where $r = r(t)$ and $\theta = \theta(t)$ are continuous with $r(t) > 0$ for all $t \in [a,b]$ and $\theta(a) \in [0, 2\pi)$. Then

$$\begin{aligned}\int_a^b \frac{x\,y' - y\,x'}{x^2 + y^2}\, dt &= \int_a^b \frac{(r\cos\theta)\big(r'\sin\theta + r\cos\theta\,\theta'\big) - (r\sin\theta)\big(r'\cos\theta - r\sin\theta\,\theta'\big)}{r^2\cos^\theta + r^2\sin^2\theta}\, dt \\ &= \int_a^b \frac{r^2\cos^2\theta\,\theta' + r^2\sin^2\theta\,\theta'}{r^2}\, dt = \int_a^b \theta'\, dt \\ &= \theta(b) - \theta(a) = 2\pi\,\text{Wind}(\alpha, 0).\end{aligned}$$

To prove Part 2, write $\alpha'(t)$ in polar coordinates as $\alpha'(t) = |\alpha'(t)|\big(\cos\theta(t), \sin\theta(t)\big)$ with $\theta(a) \in [0, 2\pi)$. Since α is $\mathcal{C}^2$ and regular, we note that α' is $\mathcal{C}^1$ with $\alpha'(t) \neq 0$ for all $t \in [a,b]$. Reparametrize α by arclength letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(t)|\, dt$, then write $\beta'(s)$ in polar coordinates as $\beta'(s) = \big|\beta'(s)\big|\big(\cos\phi(s), \sin\phi(s)\big)$ with $\phi(0) \in [0, 2\pi)$. Since $|\beta'(s)| = 1$ we have $\big(\cos\phi(s(t)), \sin\phi(s(t))\big) = \beta'(s(t)) = \frac{\alpha'(t)}{|\alpha'(t)|} = \big(\cos\theta(t), \sin\theta(t)\big)$ for all $t \in [a,b]$, and hence $\phi(s(t)) = \theta(t)$ for all $t \in [a,b]$ (by the uniqueness of the polar representation). Since $\beta'(s) = \big(\cos\phi(s), \sin\phi(s)\big)$, we have

$$\beta''(s) = \big(-\sin\phi(s)\,\phi'(s), \cos\phi(s)\,\phi'(s)\big) = \phi'(s)\big(-\sin\phi(s), \cos\phi(s)\big) = \phi'(s)\,\beta'(s)^\times$$

By the definition of $k_\beta(s)$ we see that $k_\beta(s)=\phi'(s)$. Thus

$$\begin{aligned}\int_a^b k_\alpha(t)\big|\alpha'(t)\big|\,dt &= \int_a^b k_\beta(s(t))s'(t)\,dt = \int_{s(a)}^{s(b)} k_\beta(s)ds = \int_{s(a)}^{s(b)} \phi'(s)\,ds \\ &= \phi(s(b))-\phi(s(a)) = \theta(b)-\theta(a) = 2\pi\,\mathrm{Wind}(\alpha',0) = 2\pi\,\mathrm{Turn}(\alpha).\end{aligned}$$

1.20 Theorem: *(The Fundamental Theorem for Plane Curves) Let $I\subseteq\mathbb{R}$ be an interval with $a\in I$, let $p,u\in\mathbb{R}^2$ with $|u|=1$, and let $\ell:I\subseteq\mathbb{R}\to\mathbb{R}$ be a smooth function. Then there exists a unique smooth regular curve $\beta:I\subseteq\mathbb{R}\to\mathbb{R}^2$ with $|\beta'(s)|=1$ for all $s\in I$ such that $\beta(a)=p$ and $\beta'(a)=u$ and $k(s)=\ell(s)$ for all $s\in I$.*

Proof: Suppose that such a curve β exists. Since $|\beta'(s)|=1$ for all s, we can write β' in polar coordinates as $\beta'(s)=\big(\cos\theta(s),\sin\theta(s)\big)$ with $\theta(a)\in[0,2\pi)$. Then we have $\beta''(s)=\big(-\sin\theta(s)\theta'(s),\cos\theta(s)\theta'(s)\big)=\theta'(s)\beta'(s)^\times$ so that $\theta'(s)=k(s)=\ell(s)$. We can integrate to get $\theta(s)=\theta(a)+\int_a^s\ell(t)\,dt$. Since $\beta'(s)=\big(\cos\theta(s),\sin\theta(s)\big)$ we can integrate again to get

$$\beta(s)=p+\Big(\int_a^s\cos\theta(t)\,dt\,,\,\int_a^s\sin\theta(t)\,dt\Big).$$

This shows that $\beta(s)$ is uniquely determined and gives us a formula for $\beta(s)$.

Conversely, we can choose $\theta_0\in[0,2\pi)$ so that $(\cos\theta_0,\sin\theta_0)=u$, and then define $\theta(s)=\theta_0+\int_a^s\ell(t)\,dt$ so that $\theta(a)=\theta_0$ and $\theta'(s)=\ell(s)$ for all $s\in I$, and then define $\beta(s)=p+\Big(\int_a^s\cos\theta(t)\,dt\,,\,\int_a^s\sin\theta(t)\,dt\Big)$ so that $\beta'(s)=\big(\cos\theta(s),\sin\theta(s)\big)$ for all $s\in I$. Then $|\beta'(s)|=1$ for all s and $\beta(a)=p$ and $\beta'(a)=\big(\cos\theta(a),\sin\theta(a)\big)=(\cos\theta_0,\sin\theta_0)=u$ and $\beta''(s)=\theta'(s)\beta'(s)^\times$ so that $k(s)=\theta'(s)=\ell(s)$ for all $s\in I$, as required.

Curves in $\mathbb{R}^3$

1.21 Definition: Let $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$ parametrized by arclength (so $|\beta'(s)| = 1$ for all $s \in I$). The **unit tangent vector** of β at s is the unit vector $T(s) = T_\beta(s) = \beta'(s)$. The vector $\beta''(s)$ iis called the **curvature vector** of β at s. The **scalar curvature** of β at s is given by $\kappa(s) = \kappa_\beta(s) = \big|\beta''(s)\big|$.

If $\beta''(s) \neq 0$ then we define the **principal normal vector** of β at s to be the unit vector $P(s) = P_\beta(s) = \frac{\beta''(s)}{|\beta''(s)|}$, and we define the **binormal vector** of β at s to be the unit vector $B(s) = B_\beta(s) = T(s) \times P(s)$. Note that $\big\{T(s), P(s), B(s)\big\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$. Since $B = T \times P$ and $P = \frac{T'}{|T'|}$, we have

$$B' = T' \times P + T \times P' = |T'|\, P \times P + T \times P' = T \times P'.$$

Notice that B' is orthogonal to both T and B $\big($it is orthogonal to T because $B' = T \times P'$ and it is orthogonal to B because we have $B(s) \cdot B(s) = |B(s)|^2 = 1$ for all s so taking the derivative on both sides gives $2\, B' \cdot B = 0\big)$. Since $\{T, P, B\}$ is an orthonormal basis for $\mathbb{R}^3$ and B' is orthogonal to both T and B, we have $B' = (B' \cdot P)P$. We define the **torsion** of β at s to be $\tau(s) = \tau_\beta(s) = -B'(s) \cdot P(s)$ so that $B'(s) = -\tau(s)P(s)$ for all s (the negative sign is included so that the torsion of the right-handed helix is positive).

To summarize the above definitions, when $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is a smooth regular curve, parametrized by arclength, with non-zero curvature vector $\beta''(s) \neq 0$, the **unit tangent vector**, the **principal normal vector,** the **binormal vector**, the **scalar curvature** and the **torsion** of β at s are given by

$$\begin{aligned}
T(s) &= T_\beta(s) = \beta'(s), \\
P(s) &= P_\beta(s) = \tfrac{\beta''(s)}{|\beta''(s)|}, \\
B(s) &= B_\beta(s) = T(s) \times P(s), \\
\kappa(s) &= \kappa_\beta(s) = \big|\beta''(s)\big|, \\
\tau(s) &= \tau_\beta(s) = -\, B'(s) \cdot P(s).
\end{aligned}$$

and $\{T, (s)P(s), B(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ for every $s \in I$. From the definition of P and κ we have $T' = \kappa P$, and as explained above, we defined $\tau = -B' \cdot P$ so that $B' = -\tau P$. Since $P = B \times T$ we also have

$$P' = B' \times T + B \times T' = -\tau P \times T + \kappa B \times P = \tau B - \kappa T.$$

Thus the derivatives T', P' and B' satisfy the following matrix identity which gives a system of three equations called the **Frenet-Seret Formulas**

$$\begin{pmatrix} T' \\ P' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ P \\ B \end{pmatrix}.$$

When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is a smooth regular curve in $\mathbb{R}^3$, we reparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$, then we define the **unit tangent vector** of α at t to be $T(t) = T_\alpha(t) = T_\beta(s(t))$, and if $\beta''(s(t)) \neq 0$, we define the **principal normal vector**, the **binormal vector**, the **scalar curvature** and the **torsion** of α at t to be given by $P(t) = P_\alpha(t) = P_\beta(s(t))$, $B(t) = B_\alpha(t) = B_\beta(s(t))$, $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$ and $\tau(t) = \tau_\alpha(t) = \tau_\beta(s(t))$. The following theorem shows that these are all well-defined, that is they do not depend on the choice of $a \in I$.

1.22 Theorem: *Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve. For all $t \in I$ for which $\alpha'(t)\times\alpha''(t) \neq 0$, we have*

$$T = \frac{\alpha'}{|\alpha'|} \quad , \quad P = \frac{T'}{|T'|} \quad , \quad B = T\times P$$
$$\kappa = \frac{|\alpha'\times\alpha''|}{|\alpha'|^3} \quad , \quad \tau = \frac{\det_3(\alpha',\alpha'',\alpha''')}{\left|\alpha'\times\alpha''\right|^2}.$$

Proof: Choose $a \in I$ and let $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$. Then $\alpha(t) = \beta(s(t))$ and so for all $t \in I$

$$\begin{aligned}
\alpha'(t) &= \beta'(s(t))s'(t), \\
\alpha''(t) &= \beta''(s(t))s'(t)^2 + \beta'(t)s''(t), \\
\alpha'''(t) &= \beta'''(s(t))s'(t)^3 + 3\,\beta''(s(t))s'(t)s''(t) + \beta'(t)s'''(t), \\
\alpha'(t)\times\alpha''(t) &= \big(\beta'(s(t)) \times \beta''(s(t))\big)s'(t)^3, \\
(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t) &= \Big(\big(\beta'(s(t))\times\beta''(s(t))\big)\cdot\beta'''(s(t))\Big)s'(t)^6.
\end{aligned}$$

Since $\alpha'\times\alpha'' = (\beta'\times\beta'')(s')^3$, we have $\alpha'\times\alpha'' = 0 \iff \beta'\times\beta'' = 0$. Since $|\beta'(s)| = 1$ for all s, it follows $\big($by taking the derivative of $1 = \beta'(s)\cdot\beta'(s)\big)$ that β' and β'' are orthogonal, and so we have $|\beta'\times\beta''| = |\beta'|\,|\beta''| = |\beta''|$ so that $\beta'\times\beta'' = 0 \iff \beta'' = 0$. Since $T_\alpha(t) = \beta'(s(t))$ we have $T_\alpha{}'(t) = \beta''(s(t))s'(t)$ so that $T_\alpha{}'(t) = 0 \iff \beta''(s(t)) = 0$. Thus

$$\alpha'(t)\times\alpha''(t) = 0 \iff \beta'(s(t))\times\beta''(s(t)) = 0 \iff \beta''(s(t)) = 0 \iff T_\alpha{}'(t) = 0.$$

Suppose that $\alpha'(t)\times\alpha''(t) \neq 0$. Since $T_\alpha{}'(t) = \beta''(s(t))s'(t)$ and $s'(t) = |\alpha'(t)| > 0$ we have

$$\frac{T_\alpha{}'(t)}{\left|T_\alpha{}'(t)\right|} = \frac{\beta''(s(t))s'(t)}{\left|\beta''(s(t))s'(t)\right|} = \frac{\beta''(s(t))}{\left|\beta''(s(t))\right|} = P_\beta(s(t) = P_\alpha(t) \text{ and}$$
$$B_\alpha(t) = B_\beta(s(t)) = T_\beta(s(t))\times P_\beta(s(t)) = T_\alpha(t)\times P_\alpha(t).$$

Since $\alpha'\times\alpha'' = (\beta'\times\beta'')(s')^3$ and $|\beta'\times\beta''| = |\beta''|$ and $s' = |\alpha'|$, we have

$$\frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3} = \left|\beta''(s(t))\right| = \kappa_\beta(s(t)) = \kappa_\alpha(t).$$

To simplify notation, write β for $\beta(s(t))$ and similarly for β' and β'', and write T for $T_\beta(s(t)) = T_\alpha(t)$ and similarly for P and B, and write κ for $\kappa_\beta(s(t)) = \kappa_\alpha(t)$ and similarly for τ. Since $\beta' = T$, using the Frenet-Serre formulas and the fact that $\{T, P, B\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$, we have

$$\begin{aligned}
(\beta'\times\beta'')\cdot\beta'' &= (T\times T')\cdot T'' = (T\times(\kappa P))\cdot(\kappa P)' = (T\times(\kappa P))\cdot(\kappa' P + \kappa P') \\
&= \kappa^2(T\times P)\cdot P' = \kappa^2(T\times P)\cdot(-\kappa T + \tau B) = \kappa^2\tau.
\end{aligned}$$

Since we have $\det_3(\alpha',\alpha'',\alpha''') = (\alpha'\times\alpha'')\cdot\alpha''' = \big((\beta'\times\beta'')\cdot\beta'''\big)(s')^6 = \kappa^2\tau\,|\alpha'|^6$ and we have $|\alpha'\times\alpha''| = \kappa\,|\alpha'|^3$, it follows that

$$\frac{\det_3\big(\alpha'(t),\alpha''(t),\alpha'''(t)\big)}{\left|\alpha'(t)\times\alpha''(t)\right|^2} = \frac{\kappa_\alpha(t)^2\tau_\alpha(t)\,|\alpha'(t)|^6}{\kappa_\alpha(t)^2|\alpha'(t)|^6} = \tau_\alpha(t).$$

1.23 Example: The curve $\alpha:\mathbb{R}\to\mathbb{R}^3$ given by $\alpha(t)=\big(a\cos t,a\sin t,bt\big)$ is called a (right-handed) **helix**. We have

$$\begin{aligned}\alpha'(t)&=\big(-s\sin t,a\cos t,b\big),\\ \alpha''(t)&=\big(-a\cos t,-a\sin t,0\big),\\ \alpha'''(t)&=\big(a\sin t,-a\cos t,0\big)\text{ and}\\ \alpha'(t)\times\alpha''(t)&=\big(ab\sin t,-ab\cos t,a^2,\big)\end{aligned}$$

and so

$$\begin{aligned}\big|\alpha'(t)\big|&=(a^2+b^2)^{1/2},\\ \big|\alpha'(t)\times\alpha''(t)\big|&=a(a^2+b^2)^{1/2}\text{ and}\\ (\alpha'(t)\times\alpha''(t))\bullet\alpha'''(t)&=a^2b,\end{aligned}$$

and hence

$$\begin{aligned}\kappa(t)&=\frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3}=\frac{a}{a^2+b^2}\text{ and}\\ \tau(t)&=\frac{(\alpha'(t)\times\alpha''(t))\bullet\alpha'''(t)}{|\alpha'(t)|^6}=\frac{b}{a^2+b^2}.\end{aligned}$$

We note that the scalar curvature and the torsion of the helix are constant.

1.24 Theorem: *Let $\alpha:I\subseteq\mathbb{R}\to\mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$.*
(1) The curvature of α is identically zero if and only if (the image of) α lies on a line.
(2) If α has non-vanishing curvature (so its torsion is defined) then the torsion of α is identically zero if and only if (the image of) α lies in a plane.

Proof: The proof of Part 1 is the same as the proof of the analogous theorem for plane curves (Theorem 1.14). To prove part 2, suppose that $\kappa_\alpha(t)\neq 0$ for all $t\in I$. Choose $a\in I$ and let $\beta(s)=\alpha(t(s))$ where $s(t)=\int_a^t|\alpha'(r)|\,dr$.

Suppose $\tau_\alpha(t)=0$ for all t. Then $\tau_\beta(s)=\tau_\alpha(t(s))=0$ for all s. Write $\tau(s)=\tau_\beta(s)$. We have $B'(s)=-\tau(s)P(s)=0$ for all s, so $B(s)$ is constant, say $B(s)=b\in\mathbb{R}^3$ for all s and note that $|b|=|B(s)|=1$. Note that $\frac{d}{ds}(\beta(s)\bullet b)=\beta'(s)\bullet b=T(s)\bullet B(s)=0$ for all s, and so $\beta(s)\bullet b$ is constant, say $\beta(s)\bullet b=c\in\mathbb{R}$. Thus we have $\alpha(t)\bullet b=\beta(s(t))\bullet b=c$ for all t and so (the image of) α lies on the plane in $\mathbb{R}^3$ given by $x\bullet b=c$.

Suppose, conversely, that (the image of) α lies on a plane in $\mathbb{R}^3$, say $\alpha(t)\bullet b=c$ for all $t\in I$ where $b,c\in\mathbb{R}^3$ with $|b|=1$. Then $\beta(s)\bullet b=\alpha(t(s))\bullet b=c$ for all s. Take the derivative to get $\beta'(s)\bullet b=0$ and $\beta''(s)\bullet b=0$ for all s, that is $T(s)\bullet b=0$ and $\kappa(s)P(s)\bullet b=0$ for all s. Since we are assuming that $\kappa_\alpha(t)\neq 0$ for all t, hence $\kappa(s)=\kappa_\beta(s)\neq 0$ for all s, it follows that $P(s)\bullet b=0$ for all s. Since $\{T(s),P(s),B(s)\}$ is orthonormal and $T(s)\bullet b=P(s)\bullet b=0$ and $|b|=1$, it follows that $B(s)=\pm b$ for all s. Since $B(s)$ is continuous, either we have $B(s)=b$ for all s or we have $B(s)=-b$ for all s and, in either case, $B'(s)=0$ for all s. Since $0=B'(s)=-\tau(s)P(s)$ with $|P(s)|=1$, we have $\tau(s)=0$, that is $\tau_\beta(s)=0$, for all s, and hence $\tau_\alpha(t)=\tau_\beta(s(t))=0$ for all t.

1.25 Definition: Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$, let $a \in I$, and suppose that $\kappa(a) \neq 0$ (and hence $\tau(a)$ is defined). We define the **osculating plane** of α at $t = a$ to be the plane through $\alpha(a)$ parallel to $T(a)$ and $P(a)$, that is the plane $(x-\alpha(a))\bullet B(a) = 0$. We define the **osculating circle** (or the **best-fit circle**) of α at $t = a$ as we did for a planar curve (in Definition 1.15). Let $p = \alpha(a)$, $T = T(a)$, $P = P(a)$, and $\kappa = \kappa(a)$. Reparametrize by arclength, letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$ so that we have $\beta(0) = p$. $\beta'(0) = T$ and $\beta''(0) = \kappa P$. The osculating circle at $t = a$ is the circle given by

$$\begin{aligned}\sigma(s) &= \left(p + \tfrac{1}{\kappa}\, P\right) - \tfrac{1}{\kappa}\cos(\kappa s)P + \tfrac{1}{\kappa}\sin(\kappa s)T \\ \sigma'(s) &= \sin(\kappa s)P + \cos(\kappa s)T \\ \sigma''(s) &= \kappa\cos(ks)P - \kappa\sin(\kappa s)T\end{aligned}$$

which is the circle of radius $R = \frac{1}{\kappa}$ centered at $p + \frac{1}{\kappa}P$, parametrized by arclength (since $|\sigma'(s)| = 1$ for all s), such that $\sigma(0)=p=\beta(0)$, $\sigma'(0)=T=\beta'(0)$ and $\sigma''(0)=\kappa P=\beta''(0)$.

1.26 Note: When α is a smooth regular curve, the scalar curvature at $t = a$ is equal to the reciprocal of the radius of the osculating circle at $t = a$.

1.27 Theorem: *(The Fundamental Theorem for Space Curves) Given $p, u, v \in \mathbb{R}^3$ with $|u| = |v| = 1$ and $u\bullet v = 0$, and given smooth functions $c, d : I \subseteq \mathbb{R} \to \mathbb{R}$ where I is an interval with $0 \in I$ and $c(s) > 0$ for all $s \in I$, there exists a unique smooth regular curve $\beta : I \to \mathbb{R}^3$ with $\beta(0)=p$, $T(0)=u$, $P(0)=v$ and $\kappa(s)=c(s)$ and $\tau(s)=d(s)$ for all $s\in I$.*

Proof: We want to have $T' = \kappa P$, $P' = -\kappa T + \tau B$ and $B' = -\tau P$, so we solve the system of linear first order differential equations

$$\begin{aligned}X' &= cY \\ Y' &= -cX + dZ \\ Z' &= -dY\end{aligned}$$

with the initial conditions $X(0) = u$, $Y(0) = v$ and $Z(0) = u\times v$ (such a system always has a unique solution). We claim that $\{X(s), Y(s), Z(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ for all s (this is true when $s = 0$ from the initial conditions). Write $X_1 = X$, $X_2 = Y$ and $X_3 = Z$ and define $F_{k,\ell} : I \to \mathbb{R}$ by $F_{k,\ell}(s) = X_k(s)\bullet X_\ell(s)$ for $1 \leq k \leq \ell \leq 3$. Then the functions $F_{k,\ell}$ satisfy the system of differential equations

$$\begin{aligned}
\tfrac{d}{ds}F_{1,1} &= \tfrac{d}{ds}(X\bullet X) = 2\,X'\bullet X = 2\,(cY)\bullet X = 2c\,F_{1,2} \\
\tfrac{d}{ds}F_{1,2} &= \tfrac{d}{ds}(X\bullet Y) = X'\bullet Y + X\bullet Y' = c\,Y\bullet Y + X\bullet(-cX+dZ) = -c\,F_{1,1} + d\,F_{1,3} + c\,F_{2,2} \\
\tfrac{d}{ds}F_{1,3} &= \tfrac{d}{ds}(X\bullet Z) = X'\bullet Z + X\bullet Z' = c\,Y\bullet Z + X\bullet(-dY) = -d\,F_{1,2} + c\,F_{2,3} \\
\tfrac{d}{ds}F_{2,2} &= \tfrac{d}{ds}(Y\bullet Y) = 2\,Y'\bullet Y = 2\,(-cX+dZ)\bullet Y = -2c\,F_{1,2} + 2d\,F_{2,3} \\
\tfrac{d}{ds}F_{2,3} &= \tfrac{d}{ds}(Y\bullet Z) = Y'\bullet Z + Y\bullet Z' = (-cX+dZ)\bullet Z + Y\bullet(-dY) = -c\,F_{1,3} - d\,F_{2,2} + d\,F_{3,3} \\
\tfrac{d}{ds}F_{3,3} &= \tfrac{d}{ds}(Z\bullet Z) = 2\,Z'\bullet Z = 2\,(-dY)\bullet Z = -2\,F_{2,3}
\end{aligned}$$

with the initial conditions $F_{k,k}(0) = 1$ and $F_{k,\ell}(0) = 0$ when $k \neq \ell$. Again, such a system has a unique solution, and the unique solution to this system is easily seen to be given by the constant functions $F_{k,k}(s) = 1$ and $F_{k,\ell}(s) = 0$ for all $s \in I$ and all $k \neq \ell$. Thus $\{X(s), Y(s), Z(s)\}$ is an orthonormal system for all $s \in I$, as claimed. To get $\beta'(s) = T(s) = X(s)$ with $\beta(0) = p$ we must choose $\beta(s) = p + \int_0^s X(t)\, dt$. Then we have $T = X$ and $\kappa(s) = |\beta''(s)| = |T'| = |X'| = |cY| = c$ and $P = \frac{1}{\kappa}T' = \frac{1}{c}X' = \frac{1}{c}(cY) = Y$ and $B = T\times P = X\times Y = Z$ and $\tau = -B'\bullet P = -Z'\bullet Y = (dY)\bullet Y = d$, as required.