Appendix 2. The Generalized Cross Product

2.1 Definition: Given vectors $u_1, u_2, \cdots, u_k \in \mathbb{R}^n$, we define the **parallelotope** on $u_1, \cdots, u_k$ to be the set

$$P(u_1, \cdots, u_k) = \Big\{ \sum_{j=1}^{k} t_i u_i \Big| 0 \le t_i \le 1 \text{ for all } i \Big\}.$$

We define the **volume** of this parallelotope, denoted by $V(u_1, \cdots, u_k)$, recursively by $V(u_1) = |u_1|$ and

$$V(u_1, \cdots, u_k) = V(u_1, \cdots, u_{k-1}) \big| \mathrm{Proj}_{U^\perp}(u_k) \big|$$

where $U = \mathrm{Span}\{u_1, \cdots, u_{k-1}\}$.

2.2 Theorem: *Let $u_1, \cdots, u_k \in \mathbb{R}^n$ and let $A = (u_1, \cdots, u_k) \in M_{n\times k}(\mathbb{R})$. Then*

$$V(u_1, \cdots, u_k) = \sqrt{\det(A^T A)}.$$

Proof: We prove the theorem by induction on k. Note that when $k = 1$, $u_1 \in \mathbb{R}^n$ and $A = u_1 \in M_{n\times 1}(\mathbb{R})$, we have $V(u_1) = |u_1| = \sqrt{u_1 \cdot u_1} = \sqrt{u_1{}^T u_1} = \sqrt{A^T A}$, as required. Let $k \ge 2$ and suppose, inductively, that when $A = (u_1, \cdots, u_{k-1}) \in M_{n\times k-1}$ we have $\det(A^T A) > 0$ and $V(u_1, \cdots, u_{k-1}) = \sqrt{\det(A^T A)}$. Let $B = (u_1, \cdots, u_k) = (A, u_k)$. Let $U = \mathrm{Span}\{u_1, \cdots, u_{k-1}\} = \mathrm{Col}(A)$. Let $v = \mathrm{Proj}_U(u_k)$ and $w = \mathrm{Proj}_{U^\perp}(u_k)$. Note that $v \in U = \mathrm{Col}(A)$ and $w \in U^\perp = \mathrm{Null}(A^T)$. Then we have $u_k = v + w$ so that $B = (A, v + w)$. Since $v \in \mathrm{Col}(A)$, the matrix B can be obtained from the matrix (A, w) by performing elementary column operations of the type $C_k \mapsto C_k + tC_i$. Let E be the product of the elementary matrices corresponding to these column operations, and note that $B = (A, v + w) = (A, w)E$. Since the row operations $C_k \mapsto C_k + tC_i$ do not alter the determinant, E is a product of elementary matrices of determinant 1, so we have $\det(E) = 1$. Since $\det(E) = 1$ and $w \in \mathrm{Null}(A^T)$ we have

$$\begin{aligned}
\det(B^T B) &= \det\Big(E^T (A, w)^T (A, w) E\Big) = \det\left(\begin{pmatrix} A^T \\ w^T \end{pmatrix} \begin{pmatrix} A & w \end{pmatrix}\right) \\
&= \det\begin{pmatrix} A^T A & A^T w \\ w^T A & w^T w \end{pmatrix} = \begin{pmatrix} A^T A & 0 \\ 0 & |w|^2 \end{pmatrix} = \det(A^T A)\,|w|^2.
\end{aligned}$$

By the induction hypothesis, we can take the square root on both sides to get

$$\sqrt{\det(B^T B)} = \sqrt{\det(A^T A)}\,|w| = V(u_1, \cdots, u_{k-1})\ |w| = V(u_1, \cdots, u_k).$$

2.3 Note: In the special case that $A = (u_1, u_2, \cdots, u_n) \in M_n(\mathbb{R})$, we have

$$V(u_1, \cdots, u_n) = \sqrt{\det(A^T A)} = \sqrt{\det(A)^2} = \big|\det(A)\big|.$$

2.4 Remark: There is a similar formula for the volume of an l-simplex in $\mathbb{R}^n$. For the l-simplex $S = [a_0, a_1, \cdots, a_l]$ (this means that S is the smallest convex set which contains each of the points a_k), if we let $A = (u_1, u_2, \cdots, u_l) \in M_{n\times l}(\mathbb{R})$ where $u_k = a_k - a_0$, then the volume of S is given by

$$V[a_0, a_1, \cdots, a_l] = \frac{1}{l\,!} V(u_1, \cdots, u_l) = \frac{1}{l\,!}\sqrt{\det(A^T A)}.$$

2.5 Definition: Let F be a field. For $n \geq 2$ we define the **cross product**

$$X : M_{n\times(n-1)}(F) = \prod_{k=1}^{n-1} F^n \to F^n$$

as follows. Given $A = (u_1, u_2, \cdots, u_{n-1}) \in M_{n\times(n-1)}(F)$, we define $X(A)$, also written as $X(u_1, u_2, \cdots, u_{n-1})$, to be the vector in F^n with entries

$$X(A)_j = X(u_1, u_2, \cdots, u_{n-1})_j = (-1)^{n+j} \det A^{(j)}$$

where $A^{(j)} \in M_{n-1}(F)$ is the matrix obtained from A by removing the j^{th} row. For $u \in F^2$ we write $X(u)$ as $u^\times$, and for $u, v \in F^3$ we write $X(u, v)$ as $u \times v$.

2.6 Example: Given $u \in F^2$ we have

$$u^\times = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^\times = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}.$$

Given $u, v \in F^3$ we have

$$u \times v = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \\ -\det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix} \\ \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

2.7 Note: Because the determinant is n-linear and alternating, it follows that the cross product is $(n-1)$-linear and alternating. Thus for $u_i, v, w \in F^n$ and $t \in F$ we have

(1) $X(u_1, \cdots, v + w, \cdots, u_{n-1}) = X(u_1, \cdots, v, \cdots, u_{n-1}) + X(u_1, \cdots, w, \cdots, u_{n-1})$,
(2) $X(u_1, \cdots, t\,u_k, \cdots, u_{n-1}) = t\,X(u_1, \cdots, u_k, \cdots, u_{n-1})$,
(3) $X(u_1, \cdots, u_k, \cdots, u_l, \cdots, u_{n-1}) = -X(u_1, \cdots, u_l, \cdots, u_k, \cdots, u_{n-1})$.

2.8 Definition: Recall that for $u_1, \cdots, u_n \in \mathbb{R}^n$, the set $\{u_1, \cdots, u_n\}$ is a basis for $\mathbb{R}^n$ if and only if $\det(u_1, \cdots, u_n) \neq 0$. For an ordered basis $\mathcal{A} = (u_1, \cdots, u_n)$, we say that $\mathcal{A}$ is **positively oriented** when $\det(u_1, \cdots, u_n) > 0$ and we say that $\mathcal{A}$ is **negatively oriented** when $\det(u_1, \cdots, u_n) < 0$.

2.9 Theorem: *(Properties of the Cross Product) For $u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}, w \in \mathbb{R}^n$,*

(1) $X(u_1, \cdots, u_{n-1}) \bullet w = \det(u_1, \cdots, u_{n-1}, w)$,
(2) $X(u_1, \cdots, u_{n-1}) \bullet u_k = 0$ for $1 \leq k < n$.
(3) $X(u_1, \cdots, u_{n-1}) = 0$ if and only if $\{u_1, \cdots, u_{n-1}\}$ is linearly dependent.
(4) When $w = X(u_1, \cdots, u_{n-1}) \neq 0$ we have $\det(u_1, \cdots, u_{n-1}, w) > 0$ so that the n-tuple $(u_1, \cdots, u_{n-1}, w)$ is a positively oriented basis for $\mathbb{R}^n$,
(5) $\big|X(u_1, \cdots, u_{n-1})\big|$ is equal to the volume of the parallelotope on $u_1, \cdots, u_{n-1}$,
(6) $X(u_1, \cdots, u_{n-1}) \bullet X(v_1, \cdots, v_{n-1}) = \det(B^TA)$ where $A = (u_1, \cdots, u_{n-1}) \in M_{n\times n-1}(\mathbb{R})$ and $B = (v_1, \cdots, v_{n-1}) \in M_{n\times n-1}(\mathbb{R})$, and
(7) $X\big(u_1, \cdots, u_{n-2}, X(v_1, \cdots, v_{n-1})\big) = \sum_{i=1}^{n-1} (-1)^{n+i} \det\big((B^TA)^{(i)}\big)\, v_i$ where $A = (u_1, \cdots, u_{n-2})$ and $B = (v_1, \cdots, v_{n-1})$, and $(B^TA)^{(i)}$ is obtained from B^TA by removing the i^{th} row.

Proof: Since $X(u_1, \cdots, u_{n-1}) = \sum_{i=1}^{n} (-1)^{n+i} \det A^{(i)}\, e_i$ we have

$$X(u_1, u_2, \cdots, u_{n-1}) \cdot w = \sum_{i=1}^{n} (-1)^{n+i} \det A^{(i)}\, w_i = \det(u_1, \cdots, u_{n-1}, w),$$

where the last equality follows by expanding the determinant along the last column. This proves Part (1), and Part (2) follows from Part (1) since $\det(u_1, \cdots, u_k, \cdots, u_{n-1}, u_k) = 0$.

To prove Part (3), let $A = (u_1, \cdots, u_{n-1})$. Then $\{u_1, \cdots, u_{n-1}\}$ is linearly independent if and only $\mathrm{rank}(A) = n-1$ if and only if some set of $n-1$ rows of A are linearly independent if and only if $A^{(i)}$ is invertible for some index i if and only if $X(u_1, \cdots, u_{n-1}) \neq 0$.

Part (4) holds because when $w = X(u_1, \cdots, u_{n-1}) \neq 0$ we have $|w|^2 > 0$ so that

$$0 < |w|^2 = w \cdot w = X(u_1, \cdots, u_{n-1}) \cdot w = \det(u_1, \cdots, u_{n-1}, w).$$

To prove Part (6), let $x = X(u_1, \cdots, u_{n-1})$, $y = X(v_1, \cdots, v_{n-1})$, $A = (u_1, \cdots, u_{n-1})$ and $B = (v_1, \cdots, v_{n-1})$. Using Part (1) we see that $x \cdot y = \det(u_1, \cdots, u_{n-1}, y) = \det(A, y)$ and also $x \cdot y = \det(v_1, \cdots, v_{n-1}, x) = \det(B, x)$, and so

$$(x \cdot y)^2 = \det(A, y) \det(B, x) = \det\big((B, x)^T (A, y)\big) = \det \begin{pmatrix} B^T A & B^T y \\ x^T A & x^T y \end{pmatrix}.$$

By Part (2), x is perpendicular to the columns of A and y is perpendicular to the columns of B and so we have $A^T x = 0 = B^T y$ and so

$$(x \cdot y)^2 = \det \begin{pmatrix} B^T A & 0 \\ 0 & x \cdot y \end{pmatrix} = (x \cdot y) \det(B^T A).$$

When $x \cdot y \neq 0$, we can divide both sides by $x \cdot y$ to get $x \cdot y = \det(B^T A)$, as required.

We shall now provide two proofs to deal with the case in which $x \cdot y = 0$. For the first proof, we consider both sides of the above equality, namely $(x \cdot y)^2$ and $(x \cdot y) \det(B^T A)$, to be polynomials in the entries of the vectors u_i and v_j. By unique factorization of polynomials (in many variables), we obtain $(x \cdot y) = \det(B^T A)$, as required.

Here is an alternate proof. Suppose that $x \cdot y = 0$. First we consider the case that $x = 0$ or $y = 0$. In this case, either $\mathrm{rank}(A) < n-1$ or $\mathrm{rank}(B) < n-1$, and in either case we have $\mathrm{rank}(B^T A) < n-1$ so that $B^T A$ is not invertible, hence $\det(B^T A) = 0 = x \cdot y$. Finally, we consider the case that $x \cdot y = 0$ with $x \neq 0$ and $y \neq 0$. In this case, since $x \cdot y = 0$ we have $y \in \mathrm{Span}\{x\}^\perp$. Since $x \neq 0$, the set $\{u_1, \cdots, u_{n-1}\}$ is linearly independent by Part (3) and so we have $y \in \mathrm{Span}\{x\}^\perp = \mathrm{Span}\{u_1, \cdots, u_{n-1}\} = \mathrm{Col}(A)$. But also, by Part (2), we have $y \in \mathrm{Span}\{v_1, \cdots, v_{n-1}\}^\perp = \mathrm{Col}(B)^\perp = \mathrm{Null}(B^T)$. Since $0 \neq y \in \mathrm{Col}(A)$ we can write $y = At$ for some $0 \neq t \in \mathbb{R}^{n-1}$, and since $y \in \mathrm{Null}(B^T)$ we have $0 = B^T y = B^T A t$. Since $t \neq 0$ and $B^T A t = 0$ it follows that $B^T A$ is not invertble, so again we find that $\det(B^T A) = 0 = x \cdot y$. This completes the proof of Part (6).

Note that Part (5) follows from Part (6). Indeed when $A = (u_1, \cdots, u_{n-1})$ we have

$$\big|X(u_1, \cdots, u_{n-1})\big|^2 = X(u_1, \cdots, u_{n-1}) \cdot X(u_1, \cdots, u_{n-1}) = \det(A^T A)$$

and so

$$\big|X(u_1, \cdots, u_{n-1})\big| = \sqrt{\det(A^T A)} = V(u_1, \cdots, u_{n-1}).$$

In order to prove Part (7), we shall obtain a change of variables formula for the cross product. Let $A = (u_1, \cdots, u_{n-1}) \in M_{n\times(n-1)}(\mathbb{R})$ and let $P = (v_1, \cdots, v_n) \in M_n(\mathbb{R})$. Note that the i^{th} entry of $P^T X(PA)$ is

$$\big(P^T X(PA)\big)_i = {v_i}^T X(PA) = X(PA)\cdot v_i = \det(PA, v_i).$$

Recall that $\operatorname{Cof}(P)\,P = P\operatorname{Cof}(P) = \det(P)\,I$, where $\operatorname{Cof}(P)$ is the cofactor matrix of P (or the transpose of the cofactor matrix of P, depending on convention), so we have

$$\begin{aligned}(\det P)^n\big(P^T X(PA)\big)_i &= \det\big(P\operatorname{Cof}(P)\big)\det(PA, v_i) = \det(P)\det\big(\operatorname{Cof}(P)PA, \operatorname{Cof}(P)v_i\big)\\ &= \det(P)\det\big((\det P)A, (\operatorname{Cof}(P)P)_i\big) = \det(P)\det\big((\det P)A, (\det P)e_i\big)\\ &= (\det P)^{n+1}\det(A, e_i) = (\det P)^{n+1}(-1)^{n+i}\det A^{(i)} = (\det P)^{n+1}X(A)_i.\end{aligned}$$

Thus $(\det P)^n P^T X(PA) = (\det P)^{n+1} X(A)$. When P is invertible, we can divide both sides by $(\det P)^n$ to get $P^T X(PA) = (\det P)\,X(A)$. Even when P is not invertible, we can regard both sides of the equality $(\det P)^n P^T X(PA) = (\det P)^{n+1} X(A)$ as polynomials in the entries of the vectors u_i and v_j, and then by unique factorization we obtain the change of variables formula

$$P^T X(PA) = (\det P)\,X(A).$$

Alternatively, replacing P by P^T, we obtain

$$P\,X(P^T A) = (\det P)\,X(A).$$

Finally, let us prove Part (7). Let $A = (u_1, \cdots, u_{n-2})$ and $B = (v_1, \cdots, v_{n-1})$, and let $y = X(B) = X(v_1, \cdots, v_{n-1})$, so that we have

$$X\big(u_1, \cdots, u_{n-2}, X(v_1, \cdots, v_{n-1})\big) = X(A, y).$$

Let $P = (B, y) = (v_1, \cdots, v_{n-1}, y)$. Note that

$$\det P = X(v_1, \cdots, v_{n-1})\cdot y = y\cdot y = |y|^2.$$

By the above change of variables formula, we have

$$\begin{aligned}|y|^2 X(A, y) &= (\det P)X(A, y) = P\,X\big(P^T(A, y)\big)\\ &= P\,X\left(\begin{pmatrix} B^T \\ y^T\end{pmatrix}(A, y)\right) = P\,X\begin{pmatrix} B^T A & B^T y \\ y^T A & y^T y\end{pmatrix} = P\,X\begin{pmatrix} B^T A & 0 \\ y^T A & |y|^2\end{pmatrix}\\ &= P\left(\left(\sum_{i=1}^{n-1}(-1)^{n+i}\det\begin{pmatrix}(B^T A)^{(i)} & 0 \\ y^T A & |y|^2\end{pmatrix} e_i\right) + 0\cdot e_n\right)\\ &= (v_1, \cdots, v_{n-1}, y)\left(\sum_{i=1}^{n-1}(-1)^{n+i}|y|^2\det(B^T A)^{(i)}\ e_i\ + 0\cdot e_n\right)\\ &= |y|^2\sum_{i=1}^{n-1}(-1)^{n+i}\det\big((B^T A)^{(i)}\big)v_i\end{aligned}$$

Regarding both sides of the equality $|y|^2 X(A, y) = |y|^2 \sum_{i=1}^{n-1}(-1)^{n+i}\det\big((B^T A)^{(i)}\,v_i$ as polynomials in the entries of the vectors u_i and v_j, we can divide both sides by $|y|^2$ to obtain $X(A, y) = \sum_{i=1}^{n-1}\det\big((B^T A)^{(i)}\,v_i$, as required.