Lecture Notes on Complex Analysis

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Chapter 1. Complex Numbers

1.1 Definition: A **complex number** is a vector in $\mathbf{R}^2$. The **complex plane**, denoted by $\mathbf{C}$, is the set of complex numbers:

$$\mathbf{C} = \mathbf{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x \in \mathbf{R}, y \in \mathbf{R} \right\} .$$

In $\mathbf{C}$ we usually write $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $iy = yi = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix} .$$

If $z = x+iy$ with $x, y \in \mathbf{R}$ then x is called the **real** part of z and y is called the **imaginary** part of z, and we write

$$\operatorname{Re} z = x \text{ , and } \operatorname{Im} z = y \, .$$

1.2 Definition: We define the **sum** of two complex numbers to be the usual vector sum:

$$(a + ib) + (c + id) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix} = (a + c) + i(b + d) \, ,$$

where $a, b \in \mathbf{R}$. We define the **product** of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative and associative and distributive over the sum:

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc) \, .$$

1.3 Example: Let $z = 2 + i$ and $w = 1 + 3i$. Find $z + w$ and zw.

Solution: $z+w = (2+i)+(1+3i) = (2+1)+i(1+3) = 3+4i$, and $zw = (2+i)(1+3i) = 2 + 6i + i - 3 = -1 + 7i$.

1.4 Example: Show that every non-zero complex number has a unique inverse z^{-1} and find a formula for the inverse.

Solution: We let $z = a+ib$ where $a, b \in \mathbf{R}$ with $a^2+b^2 \neq 0$, and we solve $(a+ib)(x+iy) = 1$ to find $z^{-1} = x + iy$:

$$\begin{aligned} (a + ib)(x + iy) = 1 &\iff (ax - by) + i(ay + bx) = 1 \\ &\iff \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\ &\iff x + iy = \frac{1}{a^2 + b^2} (a - ib) \, . \end{aligned}$$

Thus $(a + ib)^{-1} = \dfrac{a}{a^2 + b^2} - i \dfrac{b}{a^2 + b^2}$.

1.5 Notation: For $z, w \in \mathbf{C}$ we use the following notation:

$$-z = -1z\ ,\quad w - z = w + (-z)\ ,\quad \frac{1}{z} = z^{-1}\ \text{ and }\ \frac{w}{z} = wz^{-1}.$$

1.6 Example: Find $\dfrac{(4-i)-(1-2i)}{1+2i}$.

Solution: $\dfrac{(4-i)-(1-2i)}{1+2i} = \dfrac{3+i}{1+2i} = (3+i)(1+2i)^{-1} = (3+i)(\frac{1}{5} - \frac{2}{5}i) = 1 - i.$

1.7 Note: The set of complex numbers is a **field** under the operations of addition and multiplication. This means that for all u, v and w in $\mathbf{C}$ we have

$$\begin{aligned}
&u + v = v + u \\
&(u+v)+w = u+(v+w) \\
&0 + u = u \\
&u + (-u) = 0 \\
&uv = vu \\
&(uv)w = u(vw) \\
&1u = u \\
&u\,u^{-1} = 1 \text{ if } u \neq 0 \\
&u(v+w) = uv + uw
\end{aligned}$$

1.8 Definition: If $z = x + iy$ with $x, y \in \mathbf{R}$ then we define the **conjugate** of z to be

$$\overline{z} = x - iy\,.$$

and we define the **length** (or **magnitude**) of z to be

$$|z| = \sqrt{x^2 + y^2}\,.$$

1.9 Note: For z and w in $\mathbf{C}$ the following identities are all easy to verify.

$$\begin{aligned}
&\overline{\overline{z}} = z \\
&z + \overline{z} = 2\mathrm{Re}\,z\ ,\quad z - \overline{z} = 2i\mathrm{Im}\,z \\
&z\overline{z} = |z|^2\ ,\quad |\overline{z}| = |z| \\
&\overline{z+w} = \overline{z} + \overline{w}\ ,\quad \overline{zw} = \overline{z}\,\overline{w}\ ,\quad |zw| = |z||w|
\end{aligned}$$

1.10 Note: We do *not* have inequalities between complex numbers. We can *only* write $a < b$ or $a \leq b$ in the case that a and b are both *real* numbers. But there are several inequalities between real numbers which concern complex numbers. For $z \in \mathbf{C}$ and $w \in C$,

$$\begin{aligned}
&|\mathrm{Re}\,(z)| \leq |z|\ ,\quad |\mathrm{Im}\,(z)| \leq |z| \\
&|z+w| \leq |z| + |w|\ ,\quad \text{this is called the } \textbf{triangle inequality} \\
&|z+w| \geq \big||z| - |w|\big|
\end{aligned}$$

The first two inequalities follow from the fact that $|z|^2 = |\mathrm{Re}\,(z)|^2 + |\mathrm{Im}\,(z)|^2$. We can then prove the triangle inequality as follows: $|z+w|^2 = (z+w)(\overline{z}+\overline{w}) = |z|^2 + |w|^2 + (w\overline{z} + z\overline{w}) = |z|^2 + |w|^2 + 2\mathrm{Re}\,(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$. The last inequality follows from the triangle inequality since $|z| = |z + w - w| \leq |z+w| + |w|$ and $|w| = |z + w - z| \leq |z+w| + |z|$. (Alternatively, the last two inequalities can be proven using the Law of Cosines).

1.11 Example: Given complex numbers a and b, describe the set $\left\{z \in \mathbf{C} \middle| |z-a| < |z-b|\right\}$.

Solution: Geometrically, this is the set of all z such that z is closer to a than to b, so it is the **half-plane** which contains a and lies on one side of the perpendicular bisector of the line segment ab.

1.12 Example: Given a complex number a, describe the set $\left\{z \in \mathbf{C} \middle| 1 < |z-a| < 2\right\}$.

Solution: $\left\{z \middle| |z-a| = 1\right\}$ is the circle centred at a of radius 1 and $\left\{z \middle| |z-a| = 2\right\}$ is the circle centred at a of radius 2, and $\left\{z \in \mathbf{C} \middle| 1 < |z-a| < 2\right\}$ is the region between these two circles. Such a region is called an **annulus**.

1.13 Example: Show that every non-zero complex number has exactly two complex square roots, and find a formula for the two square roots of $z = x + iy$.

Solution: Let $z = x + iy$ where $x, y \in \mathbf{R}$ with x and y not both zero. We need to solve $w^2 = z$ for $w \in \mathbf{C}$. Write $w = u + iv$ with $u, v \in \mathbf{R}$. We have

$$\begin{aligned} w^2 = z &\iff (u+iv)^2 = x + iy \iff (u^2 - v^2) + i(2uv) = x + iy \\ &\iff \left(u^2 - v^2 = x \text{ and } 2uv = y\right). \end{aligned}$$

To solve this pair of equations for u, square both sides of the second equation to get $4u^2v^2 = y^2$, then multiply the first equation by $4u^2$ to get $4u^4 - 4u^2v^2 = 4x\,u^2$, that is $4u^4 - 4x\,u^2 - y^2 = 0$. By the quadratic formula,

$$u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}.$$

In the case that $y \neq 0$, we must use the + sign so that the right side is non-negative, so we obtain

$$u = \pm\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}.$$

A similar calculation gives

$$v = \pm\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

All four choices of sign will satisfy the equation $u^2 - v^2 = x$, but to satisfy $2uv = y$ notice that when $y > 0$, u and v have the same sign, and when $y < 0$, u and v have the opposite sign. It remains only to consider the case that $y = 0$, and we leave this case as an exercise. The final result is that

$$w = \begin{cases} \pm\left(\sqrt{\dfrac{x + \sqrt{x^2 + y^2}}{2}} + i\sqrt{\dfrac{-x + \sqrt{x^2 + y^2}}{2}}\right) & \text{, if } y > 0, \\ \pm\left(\sqrt{\dfrac{x + \sqrt{x^2 + y^2}}{2}} - i\sqrt{\dfrac{-x + \sqrt{x^2 + y^2}}{2}}\right) & \text{, if } y < 0, \\ \pm\sqrt{x} & \text{, if } y = 0 \text{ and } x > 0, \\ \pm\, i\sqrt{|x|} & \text{, if } y = 0 \text{ and } x < 0. \end{cases}$$

1.14 Note: When working with real numbers, for $0 < x \in \mathbf{R}$ it is customary to write $\sqrt{x}$ or $x^{1/2}$ to denote the unique positive square root of x. When working with complex numbers, for $0 \neq z \in \mathbf{C}$ we sometimes write $\sqrt{z}$ or $z^{1/2}$ to denote one of the two square roots of z, and we sometimes write $\sqrt{z}$ or $z^{1/2}$ to denote both square roots of z.

1.15 Example: Find $\sqrt{3-4i}$.

Solution: Using the formula derived in the previous example, we have

$$\sqrt{3-4i} = \pm\left(\sqrt{\tfrac{3+\sqrt{3^2+4^2}}{2}} - i\sqrt{\tfrac{-3+\sqrt{3^2+4^2}}{2}}\right) = \pm\left(\sqrt{\tfrac{3+5}{2}} - i\sqrt{\tfrac{-3+5}{2}}\right) = \pm(2-i)\,.$$

1.16 Note: The Quadratic Formula can be used for complex numbers. Indeed for $a, b, c, z \in \mathbf{C}$ with $a \neq 0$ we have

$$\begin{aligned} az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}\,z + \frac{c}{a} = 0 \iff z^2 + \frac{b}{2a}\,z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\ &\iff \left(z + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2} \iff z + \frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a} \\ &\iff z = \frac{-b+\sqrt{b^2-4ac}}{2a}\,, \end{aligned}$$

where $\sqrt{b^2-4ac}$ is being used to denote both square roots in the case that $b^2 - 4ac \neq 0$.

1.17 Example: Solve $i\,z^2 - (2+3i)z + 5(1+i) = 0$.

Solution: By the Quadratic Formula, we have

$$\begin{aligned} z &= \frac{(2+3i) + \sqrt{(2+3i)^2 - 20i(1+i)}}{2i} = \frac{(2+3i) + \sqrt{-5+12i+20-20i}}{2i} \\ &= \frac{(2+3i) + \sqrt{15-8i}}{2i} \end{aligned}$$

and by the formula for square roots we have

$$\sqrt{15-8i} = \pm\left(\sqrt{\tfrac{15+\sqrt{15^2+8^2}}{2}} - i\sqrt{\tfrac{-15+\sqrt{15^2+8^2}}{2}}\right) = \pm\left(\sqrt{\tfrac{15+17}{2}} - i\sqrt{\tfrac{-15+17}{2}}\right) = \pm\,(4-i)$$

and so

$$z = \frac{(2+3i) \pm (4-i)}{2i} = \frac{6+2i}{2i} \text{ or } \frac{-2+4i}{2i} = 1-3i \text{ or } 2+i\,.$$

1.18 Definition: If $z \neq 0$, we define the **angle** (or **argument**) of z to be the angle $\theta(z)$ from the positive x-axis counterclockwise to z. In other words, $\theta(z)$ is the angle such that

$$z = |z|\big(\cos\theta(z) + i\sin\theta(z)\big)\,.$$

1.19 Note: We can think of the angle $\theta(z)$ in several different ways. We can require, for example, that $0 \leq \theta(z) < 2\pi$ so that the angle is uniquely determined. Or we can allow $\theta(z)$ to be any real number, in which case the angle will be unique up to a multiple of 2π. Then again, we can think of $\theta(z)$ as the infinite set of real numbers $\theta(z) = \{\theta_0 + 2\pi k | k \in z\}$, that is we can regard $\theta(z)$ as an element of $\mathbf{R}/2\pi$, the set of real numbers modulo 2π.

1.20 Notation: For $\theta \in \mathbf{R}$ (or for $\theta \in \mathbf{R}/2\pi$) we shall write

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

1.21 Note: If $z \neq 0$ and we have $x = \mathrm{Re}\,(z)$, $y = \mathrm{Im}\,(z)$, $r = |z|$ and $\theta = \theta(z)$ then

$$\begin{aligned} &x = r\cos\theta\ , \quad y = r\sin\theta \\ &r = \sqrt{x^2+y^2}\ , \quad \tan\theta = \frac{y}{x},\ \text{if } x \neq 0 \\ &z = re^{i\theta}\ , \quad \overline{z} = r\,e^{-i\,\theta}\ , \quad z^{-1} = \frac{1}{r}e^{-i\,\theta} \end{aligned}$$

We say that $x + i\,y$ is the **cartesian** form of z and $re^{i\,\theta}$ is the **polar** form.

1.22 Example: Let $z = -3 - 4i$. Express z in polar form.

Solution: We have $|z| = 5$ and $\tan\theta(z) = \frac{4}{3}$. Since $\theta(z)$ is in the third quadrant, we have $\theta(z) = \pi + \tan^{-1}\frac{4}{3}$. So $z = 5e^{i(\pi+\tan^{-1}(4/3))}$.

1.23 Example: Let $z = 10e^{i\,\tan^{-1}3}$. Express z in cartesian form.

Solution: $z = 10\big(\cos(\tan^{-1}3) + i\,\sin(\tan^{-1}3)\big) = 10\left(\frac{1}{\sqrt{10}} + i\frac{3}{\sqrt{10}}\right) = \sqrt{10} + 3\sqrt{10}\,i$.

1.24 Example: Find a formula for multiplication in polar coordinates.

Solution: For $z = re^{i\alpha}$ and $w = e^{i\beta}$ we have $zw = rs(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\big) = rs\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big)$ and so we obtain the formula

$$re^{i\alpha}se^{i\beta} = rs\,e^{i(\alpha+\beta)}\,.$$

1.25 Note: An immediate consequence of the above example is that

$$(r\,e^{i\,\theta})^n = r^n e^{i\,n\theta}$$

for $r, \theta \in \mathbf{R}$ and for $n \in \mathbf{Z}$. This result is known as **De Moivre's Law**.

1.26 Example: Find $(1+i)^{10}$.

Solution: This can be done in cartesian coordinates using the binomial theorem (which holds for complex numbers), but it is easier in polar coordinates. We have $1 + i = \sqrt{2}e^{i\,\pi/4}$ so $(1+i)^{10} = (\sqrt{2}e^{i\,\pi/4})^{10} = (\sqrt{2})^{10}e^{i\,10\pi/4} = 32e^{i\,\pi/2} = 32i$.

1.27 Example: Find a formula for the n^{th} roots of a complex number. In other words, given $z = re^{i\theta}$, solve $w^n = z$.

Solution: Let $w = se^{i\alpha}$. We have $w^n = z \iff (se^{i\alpha})^n = re^{i\theta} \iff s^n e^{i\,n\alpha} = re^{i\theta} \iff s^n = r$ and $n\alpha = \theta + 2\pi k$ for some $k \in \mathbf{Z} \iff s = \sqrt[n]{r}$ and $\alpha = \dfrac{\theta + 2\pi k}{n}$ for some $k \in \mathbf{Z}$. Notice that when $z \neq 0$ there are exactly n solutions obtained by taking $0 \leq k < n$. So we obtain the formula

$$(r\,e^{i\,\theta})^{1/n} = \sqrt[n]{r}\,e^{i\,(\theta+2\pi k)/n} \quad , \qquad k \in \{0, 1, \ldots, n-1\}\,.$$

In particular, $(r\,e^{i\,\theta})^{1/2} = \pm\sqrt{r}\,e^{i\,\theta/2}$. For $0 < a \in \mathbf{R}$ we have $z^2 = a \iff z = \pm\sqrt{a}$, and for $0 > a \in \mathbf{R}$ we have $z^2 = a \iff z = \pm\sqrt{|a|}\,i$.

1.28 Note: When working with complex numbers, for $0 \neq z \in \mathbf{C}$ and for $0 < n \in \mathbf{Z}$, we sometimes write $\sqrt[n]{z}$ or $w^{1/n}$ to denote one of the n solutions to $w^n = z$, and we sometimes write $\sqrt[n]{z}$ or $z^{1/n}$ to denote the set of all n^{th} roots.

1.29 Note: For $z, w \in \mathbf{C}$, the rule

$$(zw)^{1/n} = z^{1/n} w^{1/n}$$

does hold provided that $z^{1/n}$ is used to denote the set of all n^{th} roots, but it does not always hold when $z^{1/n}$ is used to denote one of the n^{th} roots. Consider the following amusing "proof" that $1 = -1$:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i^2 = -1\,.$$

1.30 Example: Find $\sqrt[3]{-2+2i}$.

Solution: Note that $-2+2i = 2\sqrt{2}\, e^{i\,3\pi/4}$, and so the formula for n^{th} roots gives

$$\begin{aligned}\sqrt[3]{-2+2i} &= \sqrt[3]{2\sqrt{2}\, e^{3\pi/4}} \\ &= \sqrt{2}\, e^{i\left(\pi/4 + \frac{2\pi}{3}\,k\right)}, k \in \{0,1,2\} \\ &= \sqrt{2}\, e^{i\,\pi/3}, \sqrt{2}\, e^{i\,11\pi/12}, \sqrt{2}\, e^{i\,19\pi/12}\,.\end{aligned}$$

1.31 Note: The remaining examples in this chapter illustrate situations in which we can use complex numbers as a tool to help solve certain problems which only involve real numbers.

1.32 Example: Let $x_0 = 1$ and $x_1 = 1$, and for $n \geq 2$ let $x_n = 2x_{n-1} - 5x_{n-2}$. Find a closed-form formula for x_n.

Solution: If a sequence x_n satisfies the recursion formula $a\,x_n + b\,x_{n-1} + c\,x_{n-2} = 0$ and if the associated quadric $az^2 + bz + c = 0$ has distinct roots α and β, then it can be shown that $x_n = A\alpha^n + B\beta^n$ for some constants A and B (if you have not seen this fact before, then try to prove it by induction). For the given sequence, the associated quadratic is $z^2 - 2z + 5 = 0$ which has roots $z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$, and so we have

$$x_n = A(1+2i)^n + B(1-2i)^n$$

for some constants A and B. To get $x_0 = 1$ and $x_1 = 1$, we need $A + B = 1$ and $A(1+2i) + B(1-2i) = 1$. Solving these two equations gives $A = B = \frac{1}{2}$, so we have

$$\begin{aligned}x_n &= \tfrac{1}{2}\left((1+2i)^n + (1-2i)^n\right) = \tfrac{1}{2}\left(\left(\sqrt{5}\, e^{i\,\theta}\right)^n + \left(\sqrt{5}\, e^{-i\,\theta}\right)^n\right) = \tfrac{(\sqrt{5})^n}{2}\left(e^{i\,n\theta} + e^{-i\,n\theta}\right) \\ &= \tfrac{(\sqrt{5})^n}{2}\,(2\cos n\theta) = \left(\sqrt{5}\right)^n \cos n\theta\end{aligned}$$

where $\theta = \theta(1+2i) = \tan^{-1} 2$. Thus we obtain

$$x_n = (\sqrt{5})^n \cos\left(n \tan^{-1} 2\right)\,.$$

1.33 Example: Find $\displaystyle\sum_{i=0}^{n}\binom{3n}{3i}$.

Solution: Let $\alpha = e^{i\,2\pi/3}$. Note that $1+\alpha+\alpha^2 = 1+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=0$. By the Binomial Theorem we have

$$\begin{aligned}(1+1)^{3n} &= \binom{3n}{0}+\binom{3n}{1} \quad\; +\binom{3n}{2} \quad\;\; +\binom{3n}{3}+\binom{3n}{4} \quad\; +\cdots+\binom{3n}{3n}\\ (1+\alpha)^{3n} &= \binom{3n}{0}+\binom{3n}{1}\alpha\; +\binom{3n}{2}\alpha^2+\binom{3n}{3}+\binom{3n}{4}\alpha\; +\cdots+\binom{3n}{3n}\\ (1+\alpha^2)^{3n} &= \binom{3n}{0}+\binom{3n}{1}\alpha^2+\binom{3n}{2}\alpha\;\; +\binom{3n}{3}+\binom{3n}{4}\alpha^2+\cdots+\binom{3n}{3n}\end{aligned}$$

Adding these three equations gives $(1+1)^{3n}+(1+\alpha)^{3n}+(1+\alpha^2)^{3n}=3\displaystyle\sum_{i=0}^{n}\binom{3n}{3i}$. Note that $1+\alpha = 1-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\frac{1}{2}+\frac{\sqrt{3}}{2}i=e^{i\,\pi/3}$ and similarly $1+\alpha^2=e^{-i\,\pi/3}$, and so

$$\begin{aligned}\sum_{i=0}^{n}\binom{3n}{3i} &= \tfrac{1}{3}\left((1+1)^{3n}+(1+\alpha)^{3n}+(1+\alpha^2)^{3n}\right)=\tfrac{1}{3}\left(2^{3n}+\left(e^{i\,\pi/3}\right)^{3n}+\left(e^{-i\,\pi/3}\right)^{3n}\right)\\ &= \tfrac{1}{3}\left(2^{3n}+e^{i\,n\pi}+e^{-i\,n\pi}\right)=\frac{2^{3n}+2(-1)^n}{3}\,.\end{aligned}$$

1.34 Note: The Fundamental Theorem of Algebra (which we shall prove later in this course) states that every non-constant polynomial over **C** has a root in **C**. It follows that every such polynomial factors into linear factors over **C**. If a polynomial $f(x)$ has real coefficients, and α is a complex root of f so that $f(\alpha)=0$, then we have $f(\overline{\alpha})=\overline{f(\alpha)}=0$ so that $\overline{\alpha}$ is also a root of f. Notice that in this case

$$(x-\alpha)(x-\overline{\alpha})=x^2-(\alpha+\overline{\alpha})x+\alpha\overline{\alpha}=x^2-2\mathrm{Re}\,(\alpha)+|\alpha|^2\,,$$

which has real coefficients. It follows that every non-constant polynomial over **R** factors into linear and quadratic factors over **R**.

1.35 Example: Let $f(x)=x^4+2x^2+4$. Solve $f(z)=0$ for $z\in\mathbf{C}$, factor $f(z)$ over the complex number, and then factor $f(x)$ over the real numbers.

Solution: By the quadratic formula, $f(z)=0$ when $z^2=-1\pm\sqrt{3}\,i$ or in polar coordinates $z=2e^{\pm i\,2\pi/3}$. Thus the roots of f are $z=\pm\sqrt{2}e^{\pm i\,\pi/3}$, and so f factors over **C** as

$$z^4+2z^2+4=\left(z-\sqrt{2}e^{i\,\pi/3}\right)\left(z-\sqrt{2}e^{-i\,\pi/3}\right)\left(z+\sqrt{2}e^{i\,\pi/3}\right)\left(z+\sqrt{2}e^{-i\,\pi/3}\right).$$

Since $\left(z-\sqrt{2}e^{i\,\pi/3}\right)\left(z-\sqrt{2}e^{-i\,\pi/3}\right)=z^2-\sqrt{2}\,z+2$ and $\left(z+\sqrt{2}e^{i\,\pi/3}\right)\left(z+\sqrt{2}e^{-i\,\pi/3}\right)=z^2+\sqrt{2}\,z+2$, we see that over **R**, f factors as

$$f(x)=(x^2-\sqrt{2}\,x+2)(x^2+\sqrt{2}\,x+2)\,.$$

1.36 Note: Historically, complex numbers first arose in the study of cubic equations. An equation of the form $ax^3+bx^2+cx+d=0$, where $a,b,c,d\in\mathbf{C}$ with $a\neq 0$ can be solved as follows. First, divide by a to obtain an equation of the form $x^3+Bx^2+Cx+D=0$. Next, make the substitution $y=x+\frac{B}{3}$ and rewrite the equation in the form $y^3+py+q=0$. Then make the substitution $y=z-\frac{p}{3z}$ to convert the equation to the form $z^3+q-\frac{p^3}{27}z^{-3}=0$. Finally, multiply by z^3 to obtain $z^6+qz^3-\frac{p^3}{27}$ and solve for z^3 using the Quadratic Formula.

1.37 Example: Let $f(x)=x^3+3x^2+4x+1$. Note that $f'(x)=3x^2+6x+4=3(x+1)^2+1>0$, so f is increasing and hence has exactly one real root. Find the real root of f.

Solution: Let $y=x+1$. Then $x^3+3x^2+4x+1=(y-1)^3+3(y-1)^2+4(y-1)+1=y^3+y-1$. Try $y=z+rz^{-1}$ with $r=-\frac{1}{3}$, so we have $y^3+y-1=(z-\frac{1}{3}z^{-1})^3+(z-\frac{1}{3}z^{-1})-1=z^3-1-\frac{1}{27}z^{-3}$. We solve $z^6-z^3-\frac{1}{27}=0$ using the quadratic formula, and obtain $z^3=\frac{1\pm\sqrt{\frac{31}{27}}}{2}$. If $z=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}$ then $r\,z^{-1}=-\frac{1}{3}\sqrt[3]{\frac{2}{1+\sqrt{\frac{31}{27}}}}=-\frac{1}{3}\sqrt[3]{\frac{2(1-\sqrt{\frac{31}{27}})}{1-\frac{31}{27}}}=\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$. Similarly, if $z=\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$ then $r\,z^{-1}=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}$. In either case we have $y=z+rz^{-1}=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}+\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$, and $x=y-1=\sqrt[3]{\frac{\sqrt{\frac{31}{27}}+1}{2}}-\sqrt[3]{\frac{\sqrt{\frac{31}{27}}-1}{2}}-1$. (We did not use complex numbers in this example).

1.38 Example: Find the three real roots of $f(x)=x^3-3x+1$.

Solution: Let $x=z+rz^{-1}$ with $r=1$ so that $f(x)=(z+z^{-1})^3-3(z+z^{-1})+1=z^3+1+z^{-3}$. Multiply by z^3 and solve $z^6+z^3+1=0$ to get $z^3=\frac{-1\pm\sqrt{3}\,i}{2}=e^{\pm i\,2\pi/3}$. If $z^3=e^{i\,2\pi/3}$ then $z=e^{i\,2\pi/9}$, $e^{i\,8\pi/9}$ or $e^{i\,14\pi/9}$ and so $x=z+z^{-1}=z+\overline{z}=2\text{Re}\,(z)=2\cos(\frac{2\pi}{9})$, $2\cos(\frac{8\pi}{9})$ or $2\cos(\frac{14\pi}{9})$. If $z^3=e^{-i\,2\pi/3}$ then we obtain the same values for x. Thus the three real roots are $2\cos(40°)$, $-2\cos(20°)$ and $2\cos(80°)$.

Chapter 2. Complex Functions

2.1 Definition: Let X and Y be sets. We say that f is a **function** (or a **map**) from X to Y, and we write $f : X \to Y$, when to each element $x \in X$ there is assigned a unique element $y = f(x) \in Y$. The set X is called the **domain** of f, and the **image** (or **range**) of f is the set

$$\text{Image}\,(f) = f(X) = \big\{ f(x) \big| x \in X \big\} .$$

More generally, for $U \subseteq X$, the **image** of U under f is the set $f(U) = \big\{ f(x) \big| x \in U \big\}$. For $V \subseteq Y$, the **inverse image** of V under f is the set

$$f^{-1}(V) = \big\{ x \in X \big| f(x) \in U \big\} .$$

The **graph** of f is the set

$$\text{Graph}\,(f) = \big\{ (x, y) \in X \times Y \big| x \in X, y = f(x) \big\} .$$

We say that f is a **multi-function** from X to Y, and we use the same notation $f : X \to Y$, when f is a function from X to the set of all subsets of Y.

2.2 Note: A map $f : U \subseteq \mathbf{R} \to \mathbf{R}$ can be visualized by drawing a picture of its graph, which is a curve in $\mathbf{R}^2$.

2.3 Note: A map $f : U \subseteq \mathbf{R} \to \mathbf{C}$ can be visualized by drawing its image, which is typically a curve in $\mathbf{C}$.

2.4 Example: The **line segment** from $a \in \mathbf{C}$ to $b \in \mathbf{C}$ is the image of the map

$$z(t) = a + t(b - a) \ , \ 0 \le t \le 1 .$$

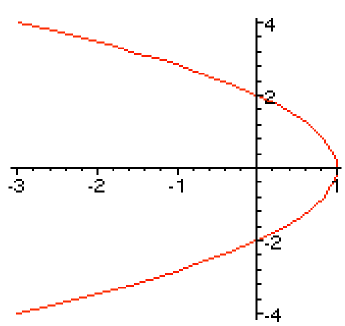
2.5 Example: The **circle** centred at $a \in \mathbf{C}$ with radius $r > 0$ is the image of the map

$$z(t) = a + r\,e^{i\,t} \ , \ 0 \le t \le 2\pi .$$

2.6 Example: Describe and sketch the image of the map $z(t) = (1 + i\,t)^2$.

Solution: We can sketch the image of *any* map $z(t)$ simply by plotting points. Try plotting the points $z(t)$ for $t = -2, -1, 0, 1, 2$. For this particular map, we can eliminate the parameter t to describe the image: $z(t) = (1 + i\,t)^2 = (1 - t^2) + i\,(2t)$ so we have $x = 1 - t^2$ and $y = 2t$, and so $x = 1 - \frac{1}{4}y^2$. This shows that the image is the parabola $x = 1 - \frac{1}{4}y^2$.

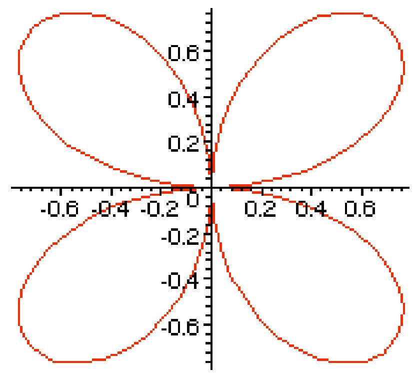
t	$z(t)$
-2	$-3 - 4i$
-1	$-2i$
0	1
1	$2i$
2	$3 + 4i$



2.7 Example: Describe and sketch the image of the map $z(t) = \sin(2t)e^{i\,t}$.

Solution: We have $z(t) = r(t)e^{i\,\theta(t)}$ where $r(t) = \sin(2t)$ and $\theta(t) = t$. Plot the points $r(t)e^{i\,\theta(t)}$ for $t = \frac{\pi}{12}\,k$, $k = 0, 1, 2, \cdots, 24$ on a polar grid (the cartesian grid consists of vertical lines $x =$ const. and horizontal lines $y =$ const., while the polar grid consists of cirles $r =$ const. and rays $\theta =$ const.). You will see that the curve is a four-leafed rose: it consists of one loop in each of the four quadrants.

$\theta = t$	$r = \sin(2t)$
0	0
$\pi/12$	$1/2$
$\pi/6$	$\sqrt{3}/2$
$\pi/4$	1
$\pi/3$	$\sqrt{3}/2$
$5\pi/6$	$1/2$
$\pi/2$	0



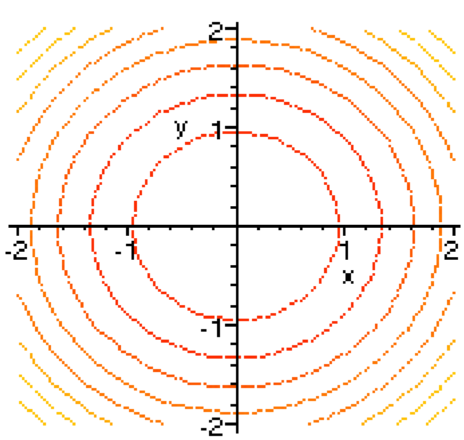
2.8 Note: To visualize a map $f : U \subseteq \mathbf{C} \to \mathbf{R}$ we can draw the **level curves** (also called **contour lines**). These are the inverse images $f^{-1}(u) = \{z \in \mathbf{C} | f(z) = u\}$ of constant values $u \in \mathbf{R}$, and they are typically curves in $U \subseteq \mathbf{C}$. We can also use the level curves of f to help draw its graph, which is a surface in $\mathbf{R}^3$.

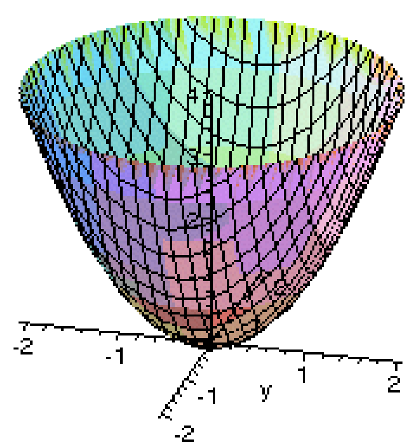
2.9 Example: Describe the level curves and the graph of the map $u = f(z) = \mathrm{Re}\,(z)$.

Solution: For $u \in \mathbf{R}$ we have $f^{-1}(u) = \{u + i\,y \,|\, y \in \mathbf{R}\}$, which is the line $x = u$. Also, $\mathrm{Graph}\,(f) = \big\{(x, y, z) \in \mathbf{R}^3 \big| u = x\big\}$, which is the plane through the origin perpendicular to the vector $(1, 0, -1)$.

2.10 Example: Sketch some level curves and sketch the graph of $u = f(z) = |z|^2$.

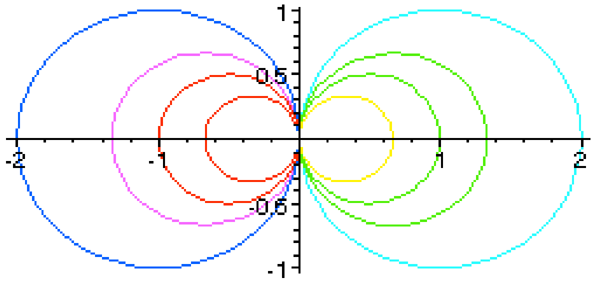
Solution: For $u \in \mathbf{R}$ we have $f^{-1}(u) = \{x + i\,y \,|\, x^2 + y^2 = u\}$. When $u < 0$, this is empty, when $u = 0$ it is the origin, and when $u > 0$ it is the circle about the origin of radius $\sqrt{u}$. Also, we have $\mathrm{Graph}\,(f) = \big\{(x, y, z) \big| u = x^2 + y^2\big\}$, which is a paraboloid.





2.11 Example: Sketch some level curves of $u = f(z) = \mathrm{Re}\,(1/z)$.

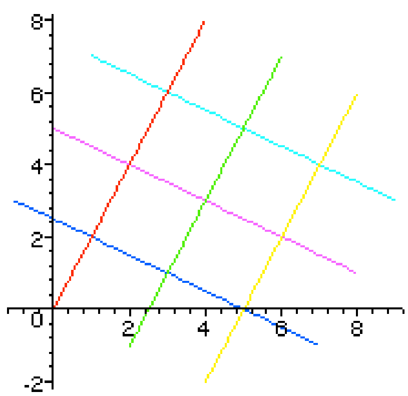
Solution: We have $u(x+i\,y) = \dfrac{x}{x^2+y^2}$. When $u=0$ we have $x=0$, and when $u \neq 0$ we have $\dfrac{x}{x^2+y^2} = u \iff x = u\,x^2 + u\,y^2 \iff x^2 - \frac{x}{u} + y^2 = 0 \iff (x - \frac{1}{2u})^2 + y^2 = \frac{1}{4u^2}$ so the level curve u =constant is the circle centred at $(\frac{1}{2u}, 0)$ with radius $\frac{1}{2u}$. These circles all go through the origin. If you sketch several of them you will see that they form the pattern which is made by the electric field of a dipole (a small bar magnet).



2.12 Note: To visualize a map $f : U \subseteq \mathbf{C} \to \mathbf{C}$ we can sketch the images of various curves in the domain (if $z = x + i\,y$ then we usually draw the images of the lines $x=$ const. and $y=$ const. while if $z = r\,e^{i\,\theta}$ then we draw the images of the circles $r=$ const. and the rays $\theta=$ const.). Alternatively, we can draw the inverse images of various curves in the range (if $w = f(z)$ with $w = u + i\,v$ then we might draw the inverse images of the lines $u=$ const. and $v=$ const.)

2.13 Example: Give a geometric description of the map $w(z) = a\,z + b$ where $a \in \mathbf{C}$ and $b \in \mathbf{C}$. Sketch the images of the lines $x = -1, 0, 1$ and $y = -1, 0, 1$ when $z = x + i\,y$ and $a = 1 + 2i$ and $b = 4 + 3i$.

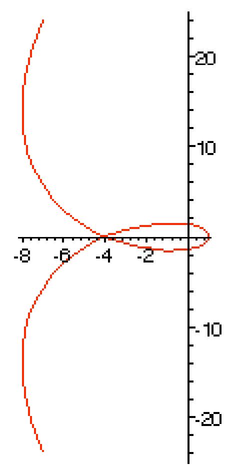
Solution: If $a = re^{i\,\alpha}$ and $z = s\,e^{i\,\beta}$ then $a\,z = (r\,s)e^{i(\alpha+\beta)}$, so multiplying z by a has the effect of scaling z by a factor of $r = |a|$ and rotating the result about the origin by the angle $\alpha = \theta(a)$. Adding b is the same as translating by b. This geometric description shows that the three vertical lines $x = -1, 0, 1$ will be sent to the three lines which are parallel to $a\,i = -2 + i$ and which pass through the points $w(-1) = 3 + i$, $w(0) = 4 + 3i$ and $w(1) = 5 + 5i$, respectively, and the three horizontal lines $y = -1, 0, 1$ are sent to the three lines parallel to $a = 1 + 2i$ through $w(-i) = 6 + 2i$, $w(0) = 4 + 3i$ and $w(i) = 2 + 4i$, respectively. This can also be shown algebraically. For example, the vertical line $x = c$ is given parametrically by $z(t) = c + i\,t$, $t \in \mathbf{R}$, and it is sent to $w(z(t)) = a(c + i\,t) + b = ac + b + i\,at = w(c) + at$, which is the line through $w(c)$ parallel to $i\,a$.



2.14 Example: Let $w(z) = z^4$. Describe the images of the circles $r =$ const. and the rays $\theta =$ const. where $z = r\,e^{i\,\theta}$. Also, sketch the image of the line $x = 1$, where $z = x + i\,y$.

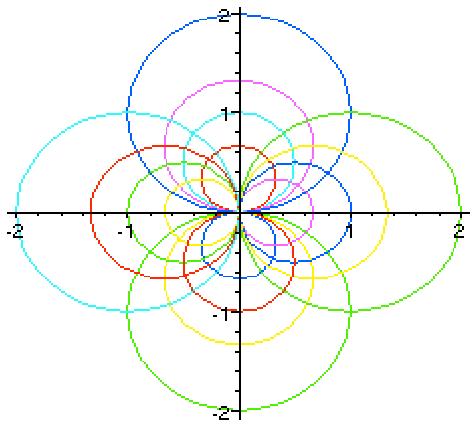
Solution: We have $w = (r\,e^{i\,\theta})^4 = r^4 e^{i\,4\theta}$, so if $w = s\,e^{i\,\phi}$ then we have $s = r^4$ and $\phi = 4\theta$. Thus the circle $r = c$ is mapped to the circle $s = c^4$ and the ray $\theta = \alpha$ is mapped to the ray $\phi = 4\alpha$. The line $x = 1$ is given parametrically by $z = 1 + i\,t$ and it is mapped to the curve $w(t) = (1 + i\,t)^4 = 1 + 4t\,i - 6t^2 - 4t^3\,i + t^4 = (1 - 6t^2 + t^4) + i\,(4t - 4t^3)$, so its image is the curve given parametrically by $u(t) = 1 - 6t^2 + t^4$ and $v(t) = 4t - 4t^3$. The u-intercepts occur when $v = 0$, that is when $t = 0, \pm 1$ and the v-intercepts occur when $u = 0$, that is when $t^2 = 3 \pm 2\sqrt{2}$. Also, We have $u'(t) = -12t + 4t^3 = 4t(t^2 - 3)$ and $v'(t) = 4 - 12t^2 = 4(1 - 3t^2)$, and so the curve is vertical when $u'(t) = 0$, that is when $t = 0, \pm\sqrt{3}$ and it is horizontal when $v'(t) = 0$, that is when $t = \pm 1/\sqrt{3}$. To sketch the curve, plot the points when $t = 0, \pm 1/\sqrt{3}, \pm 1, \pm\sqrt{3}, \pm 2$, and perhaps also when $t = \pm\sqrt{3 \pm 2\sqrt{2}}$.

t	u	v
-2	7	25
$-\sqrt{3}$	-8	$8\sqrt{3}$
-1	-4	0
$-1/\sqrt{3}$	$-8/9$	$-8\sqrt{3}/9$
0	1	0
$1/\sqrt{3}$	$-8/9$	$8\sqrt{3}/9$
1	-4	0
$\sqrt{3}$	-8	$-8\sqrt{3}$
2	7	-25



2.15 Example: Let $w(z) = \dfrac{1}{\bar{z}}$. Describe the images of the circles $r =$ const. and the rays $\theta =$ const., and then describe the images of the lines $x =$ const. and $y =$ const.

Solution: If $z = r\,e^{i\,\theta}$ and $w = s\,e^{i\,\phi}$ then we have $w = \dfrac{1}{r\,e^{-i\,\theta}} = \frac{1}{r}e^{i\,\theta}$ so that $s = \frac{1}{r}$ and $\phi = \theta$. This map is known as the **inversion** in the unit circle: the circle $r = c$ is mapped to the circle $s = 1/c$ while the ray $\theta = \alpha$ is mapped to itself. If $z = x + i\,y$ and $w = u + i\,v$ then the vertical line $x = c$ is given parametrically by $z(t) = c + i\,t$ and it is sent to $w(z(t)) = \dfrac{c + i\,t}{c^2 + t^2}$, so its image is the curve given by $u(t) = \dfrac{c}{c^2 + t^2}$ and $v(t) = \dfrac{t}{c^2 + t^2}$. When $c = 0$ we have $u = 0$ and $v = t/t^2 = 1/t$, so the line $x = 0$ (excluding the origin) is mapped to the line $u = 0$ (excludind the origin). When $c \neq 0$, we can use the expression for $u(t)$ to solve for t to get $t^2 = (c - u\,c^2)/u$ and then we can substitute this into the expression $v^2(t) = t^2/(c^2 + t^2)^2$ and simplify to get $v^2 = \frac{1}{c}u - u^2$ or equivalently $(u - \frac{1}{2c})^2 + v^2 = (\frac{1}{2c})^2$. Thus the image of the line $x = c$, $c \neq 0$ is the circle centred at $\frac{1}{2c}$ with radius $\frac{1}{2|c|}$, excluding the origin. Similarly, the image of the horizontal line $y = c$ is the circle centred at $\frac{1}{2c}\,i$ with radius $\frac{1}{2|c|}$, excluding the origin.



2.16 Definition: We define the **exponential** function by

$$e^{x+i\,y} = e^x e^{i\,y} = e^x \cos y + i\, e^x \sin y\,.$$

We also write $\exp(z) = e^z$.

2.17 Note: It is not hard to check that the exponential function has the following properties for all complex numbers z and w:

$$\begin{aligned}
&e^0 = 1 \\
&e^{-z} = 1/e^z\ , \quad e^{nz} = (e^z)^n, n \in \mathbf{Z} \\
&e^{z+w} = e^z e^w\ , \quad e^{z-w} = e^z/e^w \\
&e^z = e^w \iff w = z + i\,2\pi k \text{ for some } k \in \mathbf{Z}
\end{aligned}$$

2.18 Example: Let $w(z) = e^z$. Describe the images of the lines $x =$ const. and $y =$ const. where $z = x + i\,y$.

Solution: We have $w = e^x e^i\, y$, so if $w = r\, e^{i\,\theta}$ then we have $r = e^x$ and $\theta = y$. So the vertical line $x = c$ is mapped to the circle $r = e^c$, and the horizontal line $y = c$ is mapped to the ray $\theta = c$. Notice that the domain of e^z is all of $\mathbf{C}$ while the range is $\mathbf{C} \setminus \{0\}$.

2.19 Definition: We define the **trigonometric** functions by

$$\sin z = \frac{e^{i\,z} - e^{-i\,z}}{2i}\ , \quad \cos z = \frac{e^{i\,z} + e^{-i\,z}}{2}\ , \quad \tan z = \frac{\sin z}{\cos z}$$

and $\sec z = 1/\cos z$, $\csc z = 1/\sin z$ and $\cot z = \cos z/\sin z$. We define the **hyperbolic** functions by

$$\sinh z = \frac{e^z - e^{-z}}{2}\ , \quad \cosh z = \frac{e^z + e^{-z}}{2}\ , \quad \tanh z = \frac{\sinh z}{\cosh z}$$

and $\coth z = \cosh z/\sinh z$.

2.20 Note: It is not hard to verify the following properties, where $z,w\in\mathbf{C}$:

$$\begin{aligned}
&\sin(z+2\pi)=\sin z\ ,\quad \cos(z+2\pi)=\cos z\\
&\sin(-z)=-\sin z\ ,\quad \cos(-z)=\cos z\\
&\sin^2 z+\cos^2 z=1\\
&\sin(z+w)=\sin z\cos w+\cos z\sin w\ ,\quad \sin(2z)=2\sin z\cos z\\
&\cos(z+w)=\cos z\cos w-\sin z\sin w\ ,\quad \cos(2z)=\cos^2 z-\sin^2 z\\
&\sinh(-z)=-\sinh z\ ,\quad \cosh(-z)=\cosh z\\
&\cosh^2 z-\sinh^2 z=1\\
&\sinh(z+w)=\sinh z\cosh w+\cosh z\sinh w\ ,\quad \sinh(2z)=2\sinh z\cosh z\\
&\cosh(z+w)=\cosh z\cosh w+\sinh z\sinh w\ ,\quad \cosh(2z)=\cosh^2 z+\sinh^2 z
\end{aligned}$$

In fact *all* of the trigonometric identities and hyperbolic identies which hold for real numbers also hold for complex numbers. Here are some more properties:

$$\begin{aligned}
&\sinh(z+i\,2\pi)=\sinh z\ ,\quad \cosh(z+i\,2\pi)=\cosh z\\
&\sinh(i\,z)=i\,\sin z\ ,\quad \cosh(i\,z)=\cos z\\
&\sin(i\,z)=i\,\sinh z\ ,\quad \cos(i\,z)=\cosh z\\
&\sin(x+i\,y)=\sin x\cosh y+i\,\cos x\sinh y\ ,\quad |\sin(x+i\,y)|^2=\sin^2 x+\sinh^2 y\\
&\cos(x+i\,y)=\cos x\cosh y-i\,\sin x\sinh y\ ,\quad |\cos(x+i\,y)|^2=\cos^2 x+\sinh^2 y\\
&\sinh(x+i\,y)=\sinh x\cos y+i\,\cosh x\sin y\ ,\quad |\sinh(x+i\,y)|^2=\sinh^2 x+\sin^2 y\\
&\cosh(x+i\,y)=\cosh x\cos y+i\,\sinh x\sin y\ ,\quad |\cosh(x+i\,y)|^2=\sinh^2 x+\cos^2 y
\end{aligned}$$

2.21 Example: Find $\sin(\frac{\pi}{6}+i\,\ln 2)$.

Solution: We have

$$\sin(\tfrac{\pi}{6}+i\,\ln 2)=\sin(\tfrac{\pi}{6})\cosh(\ln 2)+i\,\cos(\tfrac{\pi}{6})\sinh(\ln 2)=\tfrac{1}{2}\cdot\tfrac{5}{4}+i\,\tfrac{\sqrt{3}}{2}\cdot\tfrac{3}{4}=\tfrac{5+3\sqrt{3}\,i}{8}\,.$$

2.22 Example: Solve $\cosh z=-2$.

Solution: If $z=x+i\,y$ then we have $\cosh z=\cosh x\cos y+i\,\sinh x\sin y$, so we have $\cosh z=-2$ when $\cosh x\cos y=-2$ and $\sinh x\sin y=0$. We cannot have $\sinh x=0$, since if $\sinh x=0$ then $x=0$ so $\cosh x\cos y=\cos y\neq -2$. So we must have $\sin y=0$ and so $y=k\pi$ for some $k\in\mathbf{Z}$ and we have $\cos y=\pm 1$. To have $\cosh x\cos y=-2$, we must have $\cos y=-1$ and $\cosh x=2$ (since $\cosh x$ is always positive). We can solve $\cosh x=2$ as follows: $\cosh x=2\iff e^x+e^{-x}=4\iff (e^x)^2-4e^x+1=0\iff e^x=2\pm\sqrt{3}$ so we have $x=\ln(2\pm\sqrt{3})$ or equivalently $x=\pm\ln(2+\sqrt{3})$. Thus $z=\pm\ln(2+\sqrt{3})+i\,(\pi+2\pi k)$ for some $k\in\mathbf{Z}$.

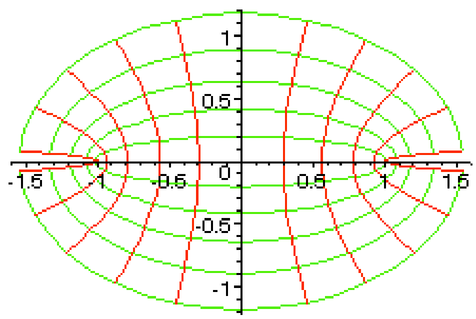
2.23 Example: Let $w(z)=\sin z$. Describe the images of the lines $x=$ const. and $y=$ const. where $z=x+i\,y$.

Solution: The vertical line $x=c$ is given parametrically by $z(t)=c+i\,t$ and it is mapped to the curve $w(t)=\sin(c+i\,t)=\sin c\cosh t+i\,\cos c\sinh t$. If $w=u+i\,v$ then we have $u(t)=\sin c\cosh t$ and $v(t)=\cos c\sinh t$. Using the identity $\cosh^2 t-\sinh^2 t=1$ we obtain

$\dfrac{u^2}{\sin^2 c} - \dfrac{v^2}{\cos^2 c} = 1$, provided that $t \neq \frac{\pi}{2}k, k \in \mathbf{Z}$. This is the equation of a hyperbola. The image of the line $x = c$ will be one of the two branches of this hyperbola; when $\sin c$ is positive $u(t)$ is also positive and the image is the branch on the right; when $\sin c$ is negative, the image is the branch on the left. When $\sin c = 0$ (so that $c = \pi k$), the image is the line $u = 0$, that is, the v-axis. When $\cos c = 0$, the image lies on the line $v = 0$ (the u-axis) and it is either the interval $[1, \infty)$ (when $\sin c = 1$) or else the interval $(-\infty, -1]$ (when $\sin c = -1$).

The horizontal line $y = c$ is given parametrically by $z(t) = t + i\,c$ and it is mapped to $w(t) = \sin t \cosh c + i\, \cos t \sinh c$ so we have $u(t) = \sin t \cosh c$ and $v(t) = \cos t \sinh t$. Since $\sin^2 t + \cos^2 t = 1$ we have $\dfrac{u^2}{\cosh^2 c} + \dfrac{v^2}{\sinh^2 c} = 1$. The line $y = c$ is mapped to this ellipse, unless $c = 2\pi k\, i$ in which case the image can be seen to be the line segment $[-1, 1]$ on the u-axis.

If you sketch a few of these hyperbolas and ellipses, you will get a nice picture showing two orthogonal families of curves. You will see that the domain and the range of $\sin z$ are both equal to $\mathbf{C}$.



2.24 Definition: Let X and Y be sets and let $f : X \to Y$. We say that f is **one-to-one**, written as f is 1:1, (or that f is **injective**) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. We say that f is **onto** (or **surjective**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. We say that f is **invertible** (or **bijective**) when f is both one-to-one and onto, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$.

When a map f from X to Y is invertible, it has a unique **inverse function** from Y to X, denoted by f^{-1}, which is defined by

$$f(x) = y \iff f^{-1}(y) = x$$

or equivalently by

$$f(f^{-1}(y)) = y \ , \quad f^{-1}(f(x)) = x$$

for all $x \in X$ and $y \in Y$. When f from X to Y is not invertible, we define its **inverse multi-function** from Y to X given by

$$f^{-1}(y) = \big\{ x \in X \big| f(x) = y \big\} \, .$$

2.25 Note: When a map $f : X \to Y$ is 1:1 (but perhaps not onto), the map $f : X \to f(X)$ is both 1:1 and onto, and hence invertible. When a map $f : X \to Y$ is not 1:1, then sometimes we can find a subset $U \subset X$ such that the restriction $f : U \subset X \to Y$ is 1:1, and then the map $f : U \to f(U)$ is invertible. In this case, the inverse function of the map $f : U \to f(U)$ is called a **branch** of the inverse multi-function. When working with complex-valued functions of a complex variable, we shall sometimes use the notation f^{-1} to denote the inverse multi-function of f, and we shall sometimes use the notation f^{-1} to denote some branch of the inverse multi-function.

2.26 Example: In real variable calculus, to define $\sin^{-1} x$ it is customary to restrict the domain of $\sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ so that it becomes 1:1. If we thought instead of $\sin^{-1} x$ as a multi-function then for example we would have $\sin^{-1}(\frac{1}{2}) = \left\{ \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k \middle| k \in \mathbf{Z} \right\}$.

2.27 Example: The polar coordinates map $f : \left\{ (r, \theta) \middle| r > 0, \theta \in \mathbf{R} \right\} \to \mathbf{R}^2$ given by $f(r,\theta) = (r\cos\theta, r\sin\theta)$ is not 1:1. We can make it 1:1 by restricting the domain in various ways. For example for any $\theta_0 \in \mathbf{R}$, if we restrict the domain to

$$\{(r,\theta)|r > 0, \theta_0 < \theta < \theta_0 + 2\pi\}$$

then f becomes 1:1 and its inverse is given by $f^{-1}(x,y) = (r,\theta)$ where $r = |x + i\,y|$ and $\theta = \theta(x + i\,y)$ with $\theta_0 < \theta(x + i\,y) < \theta_0 + 2\pi$. Alternatively, if we think of f^{-1} as a multi-function, then we can still write $f^{-1}(x,y) = (r,\theta)$ where $r = |x + i\,y|$ and $\theta = \theta(x + i\,y)$, but this time $\theta(x + i\,y)$ denotes an infinite set of the form $\theta(x + i\,y) = \left\{ \theta_0 + 2\pi\,k \middle| k \in \mathbf{Z} \right\}$ with say $0 \le \theta_0 < 2\pi$.

2.28 Definition: The inverse of the exponential function e^z is the **logarithmic** function (or the logarithmic multi-function), denoted by $\log z$.

2.29 Example: Find a formula for $\log z$.

Solution: Let $z = r\,e^{i\,\theta}$ and $w = u + i\,v$. Then $w = \log z \iff e^w = z \iff e^u e^{i\,v} = r\,e^{i\,\theta}$, which happens when $e^u = r$ and $v = \theta + 2\pi k$ for some $k \in \mathbf{Z}$. Thus

$$\log\left(r\,e^{i\,\theta}\right) = \ln r + i(\theta + 2\pi k),\ k \in \mathbf{Z}\,.$$

This is the formula for the multi-valued logarithm. We can obtain a branch of the logarithm by restricting the domain of the exponential function in various ways. For example, for any $\theta_0 \in \mathbf{R}$, if we restrict its domain to the set $\left\{ re^{i\,\theta} \middle| r > 0, \theta_0 < \theta < \theta_0 + 2\pi \right\}$, then it becomes invertible with inverse function

$$\log\left(re^{i\,\theta}\right) = \ln r + i\,\theta\ ,\ \text{where } \theta_0 < \theta < \theta_0 + 2\pi\,.$$

2.30 Example: Find $\log(1 - i)$

Solution: $\log(1 - i) = \log(\sqrt{2}e^{-i\,\pi/4}) = \ln\sqrt{2} + i\,(-\frac{\pi}{4} + 2\pi k),\ k \in \mathbf{Z}$.

2.31 Note: For the multi-valued logarithm, you should convince yourself that the following formulas make sense and they all hold:

$$\begin{aligned} &e^{\log z} = z \\ &\log(z\,w) = \log z + \log w \\ &\log(z/w) = \log z - \log w \end{aligned}$$

2.32 Definition: We can use the logarithm to define **complex exponents**: given $a \in \mathbf{C}$ we define

$$z^a = \exp(a \log z)\,.$$

2.33 Example: Find i^{-2i}.

Solution: $i^{-2i} = \exp(-2i\,\log i) = \exp(-2i\,(i\,(\frac{\pi}{2} + 2\pi k)) = \exp(\pi + 4\pi k),\ k \in \mathbf{Z}$.

2.34 Example: Find a formula for $z^{2/3}$.

Solution: Write $z = r\,e^{i\,\theta}$. Then we have

$$z^{2/3} = \exp(\tfrac{2}{3}\log z) = \exp\left(\tfrac{2}{3}\big(\ln r + i(\theta + 2\pi\,k)\big)\right) = r^{2/3} e^{i(2\theta+4\pi k)/3}, k \in \mathbf{Z}\,.$$

Notice that there are three distinct values of $z^{2/3}$ obtained by taking $k \in \{0, 1, 2\}$.

2.35 Note: Check that

$$\begin{aligned} z^n &= \exp(n \log z) \quad \text{is single valued for } 0 < n \in \mathbf{Z} \\ z^{1/n} &= \exp\left(\tfrac{1}{n}\log z\right) \ \text{ takes } n \text{ values for } 0 < n \in \mathbf{Z} \\ z^{-a} &= (z^a)^{-1} \end{aligned}$$

2.36 Definition: The **inverse trigonometric functions** are denoted by $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$ and so on. The **inverse hyperbolic** functions are denoted by $\sinh^{-1} z$, $\cosh^{-1} z$, $\tanh^{-1} z$ and so on.

2.37 Note: Since the trigonometric and the hyperbolic functions are defined using the exponential function, their inverses can be expressed in terms of the logarithmic function:

$$\begin{aligned} \sin^{-1} z &= -i\,\log\left(i\,z + \sqrt{1 - z^2}\right) \\ \cos^{-1} z &= -i\,\log\left(z + \sqrt{z^2 - 1}\right) \\ \tan^{-1} z &= \frac{i}{2}\log\frac{i+z}{i-z} \\ \sinh^{-1} z &= \log\left(z + \sqrt{z^2 + 1}\right) \\ \cosh^{-1} z &= \log\left(z + \sqrt{z^2 - 1}\right) \\ \tanh^{-1} z &= \frac{1}{2}\log\frac{1+z}{1-z} \end{aligned}$$

where the square roots are double valued. Let us derive the formula for $\sin^{-1} z$. We have $w = \sin^{-1} z \iff z = \sin w \iff z = (e^{iw} - e^{-iw})/2i \iff (e^{iw})^2 - 2iz(e^{iw}) - 1 = 0 \iff e^{iw} = iz + \sqrt{1 - z^2}$ so we obtain $iw = \log\left(iz + \sqrt{1 - z^2}\right)$, as required.

2.38 Example: Find $\cosh^{-1}(-2)$.

Solution: We already did this in example 2.22, but let us do it again using the above logarithmic formula. We have

$$\cosh^{-1}(-2) = \log(-2 \pm \sqrt{3}) = \log((2 \pm \sqrt{3})e^{i\,\pi}) = \ln(2 \pm \sqrt{3}) + i\,(\pi + 2\pi k), k \in \mathbf{Z}\,.$$

Chapter 3. Topology in Euclidean Space

3.1 Definition: For vectors $x, y \in \mathbf{R}^n$ we define the **dot product** of x and y to be

$$x \cdot y = y^T x = \sum_{i=1}^{n} x_i y_i \,.$$

When $z, w \in \mathbf{C} = \mathbf{R}^2$ we have $z \cdot w = \mathrm{Re}\,(z\overline{w})$.

3.2 Theorem: *(Properties of the Dot Product) For all $x, y, z \in \mathbf{R}^n$ and all $t \in \mathbf{R}$ we have*
(1) (Bilinearity) $(x+y)\cdot z = x\cdot z + y\cdot z\ ,\ (tx)\cdot y = t(x\cdot y)$
$x\cdot(y+z) = x\cdot y + x\cdot z\ ,\ x\cdot(ty) = t(x\cdot y)$,
(2) (Symmetry) $x\cdot y = y\cdot x$, *and*
(3) (Positive Definiteness) $x\cdot x \geq 0$ *with* $x\cdot x = 0$ *if and only if* $x = 0$.

Proof: The proof is left as an exercise.

3.3 Definition: For a vector $x \in \mathbf{R}^n$, we define the **norm** (or **length**) of x to be

$$|x| = \sqrt{x\cdot x} = \sqrt{\sum_{i=1}^{n} {x_i}^2}.$$

We say that x is a **unit vector** when $|x| = 1$.

3.4 Theorem: *(Properties of Length) Let $x, y \in \mathbf{R}^n$ and let $t \in \mathbf{R}$. Then*
(1) (Positive Definiteness) $|x| \geq 0$ *with* $|x| = 0$ *if and only if* $x = 0$,
(2) (Scaling) $|tx| = |t||x|$,
(3) $|x \pm y|^2 = |x|^2 \pm 2(x\cdot y) + |y|^2$.
(4) (The Polarization Identities) $x\cdot y = \frac{1}{2}\big(|x+y|^2 - |x|^2 - |y|^2\big) = \frac{1}{4}\big(|x+y|^2 - |x-y|^2\big)$,
(5) (The Cachy-Schwarz Inequality) $|x\cdot y| \leq |x|\,|y|$ *with* $|x\cdot y| = |x|\,|y|$ *if and only if the set* $\{x, y\}$ *is linearly dependent, and*
(6) (The Triangle Inequality) $|x+y| \leq |x| + |y|$.

Proof: We leave the proofs of Parts (1), (2) and (3) as an exercise, and we note that (4) follows immediately from (3). To prove part (5), suppose first that $\{x, y\}$ is linearly dependent. Then one of x and y is a multiple of the other, say $y = tx$ with $t \in \mathbf{R}$. Then

$$|x\cdot y| = |x\cdot(tx)| = |t(x\cdot x)| = |t|\,|x|^2 = |x|\,|tx| = |x|\,|y|.$$

Suppose next that $\{x, y\}$ is linearly independent. Then for all $t \in \mathbf{R}$ we have $x + ty \neq 0$ and so

$$0 \neq |x+ty|^2 = (x+ty)\cdot(x+ty) = |x|^2 + 2t(x\cdot y) + t^2|y|^2.$$

Since the quadratic on the right is non-zero for all $t \in \mathbf{R}$, it follows that the discriminant of the quadratic must be negative, that is

$$4(x\cdot y)^2 - 4|x|^2|y|^2 < 0.$$

Thus $(x\cdot y)^2 < |x|^2|y|^2$ and hence $|x\cdot y| < |x|\,|y|$. This proves part (5).

Using part (5) note that

$$|x+y|^2 = |x|^2 + 2(x\cdot y) + |y|^2 \leq |x+y|^2 + 2|x\cdot y| + |y|^2 \leq |x|^2 + 2|x|\,|y| + |y|^2 = \big(|x| + |y|\big)^2$$

and so $|x+y| \leq |x| + |y|$, which proves part (6).

3.5 Definition: For points $a, b \in \mathbf{R}^n$, we define the **distance** between a and b to be

$$\text{dist}(a, b) = |b - a|.$$

3.6 Theorem: *(Properties of Distance) Let $a, b, c \in \mathbf{R}^n$. Then*
(1) (Positive Definiteness) $\text{dist}(a, b) \geq 0$ *with* $\text{dist}(a, b) = 0$ *if and only if* $a = b$,
(2) (Symmetry) $\text{dist}(a, b) = \text{dist}(b, a)$, *and*
(3) (The Triangle Inequality) $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$.

Proof: The proof is left as an exercise.

3.7 Definition: For $a \in \mathbf{R}^n$ and $0 < r \in \mathbf{R}$, the **sphere**, the **open ball**, the **closed ball**, and the (open) **punctured ball** in $\mathbf{R}^n$ centered at a of radius r are defined to be the sets

$$\begin{aligned} S(a, r) &= \big\{x \in \mathbf{R}^n \big| \text{dist}(x, a) = r\big\} = \big\{x \in \mathbf{R}^n \big| |a - x| = r\big\}, \\ B(a, r) &= \big\{x \in \mathbf{R}^n \big| \text{dist}(x, a) < r\big\} = \big\{x \in \mathbf{R}^n \big| |a - x| < r\big\}, \\ \overline{B}(a, r) &= \big\{x \in \mathbf{R}^n \big| \text{dist}(x, a) \leq r\big\} = \big\{x \in \mathbf{R}^n \big| |a - x| \leq r\big\}, \\ B^*(a, r) &= \big\{x \in \mathbf{R}^n \big| 0 < \text{dist}(x, a) < r\big\} = \big\{x \in \mathbf{R}^n \big| 0 < |a - x| < r\big\}. \end{aligned}$$

When $n = 2$, a sphere is also called a **circle** and a ball is also called a **disc**. For $a \in \mathbf{R}^2 = \mathbf{C}$ and $0 < r \in \mathbf{R}$ we also write $D(a, r) = B(a, r)$, $\overline{D}(a, r) = \overline{B}(a, r)$ and $D^*(a, r) = B^*(a, r)$.

3.8 Definition: For a set $A \subseteq \mathbf{R}^n$, we say that A is **open** (in $\mathbf{R}^n$) when for every $a \in A$ there exists $r > 0$ such that $B(a, r) \subseteq A$, and we say that A is **closed** (in $\mathbf{R}^n$) when its complement $A^c = \mathbf{R}^n \setminus A$ is open in $\mathbf{R}^n$.

3.9 Example: Show that for $a \in \mathbf{R}^n$ and $0 < r \in \mathbf{R}$, the set $B(a, r)$ is open and the set $\overline{B}(a, r)$ is closed.

Solution: Let $a \in \mathbf{R}^n$ and let $r > 0$. We claim that $B(a, r)$ is open. We need to show that for all $b \in B(a, r)$ there exists $s > 0$ such that $B(b, s) \subseteq B(a, r)$. Let $b \in B(a, r)$ and note that $|b - a| < r$. Let $s = r - |b - a|$ and note that $s > 0$. Let $x \in B(b, s)$, so we have $|x - b| < s$. Then, by the Triangle Inequality, we have

$$|x - a| = |x - b + b - a| \leq |x - b| + |b - a| < s + |b - a| = r$$

and so $x \in B(a, r)$. This shows that $B(b, s) \subseteq B(a, r)$ and hence $B(a, r)$ is open.

Next we claim that $\overline{B}(a, r)$ is closed, that is $\overline{B}(a, r)^c$ is open. Let $b \in \overline{B}(a, r)^c$, that is let $b \in \mathbf{R}^n$ with $b \notin \overline{B}(a, r)$. Since $b \notin \overline{B}(a, r)$ we have $|b - a| > r$. Let $s = |b - a| - r > 0$. Let $x \in B(b, s)$ and note that $|x - b| < s$. Then we have

$$|b - a| = |b - x + x - a| \leq |b - x| + |x - a| < s + |x - a|$$

and so $|x - a| > |b - a| - s = r$. Since $|x - a| > r$ we have $x \notin \overline{B}(a, r)$ and so $x \in \overline{B}(a, r)^c$. This shows that $B(b, s) \subseteq \overline{B}(a, r)^c$ and it follows that $\overline{B}(a, r)^c$ is open and hence that $\overline{B}(a, r)$ is closed.

3.10 Theorem: *(Basic Properties of Open Sets)*

(1) The sets $\emptyset$ *and* $\mathbf{R}^n$ *are open in* $\mathbf{R}^n$.
(2) If S *is a set of open sets then the union* $\bigcup S = \bigcup_{U \in S} U$ *is open.*
(3) If S *is a finite set of open sets then the intersection* $\bigcap S = \bigcap_{U \in S} U$ *is open.*

Proof: The empty set is open because any statement of the form "for all $x \in \emptyset$ F" (where F is any statement) is considered to be true (by convention). The set $\mathbf{R}^n$ is open because given $a \in \mathbf{R}^n$ we can choose any value of $r > 0$ and then we have $B(a, r) \subseteq \mathbf{R}^n$ by the definition of $B(a, r)$. This proves Part (1).

To prove Part (2), let S be any set of open sets. Let $a \in \bigcup S = \bigcup_{U \in S} U$. Choose an open set $U \in S$ such that $a \in U$. Since U is open we can choose $r > 0$ such that $B(a, r) \subseteq U$. Since $U \in S$ we have $U \subseteq \bigcup S$. Since $B(a, r) \subseteq U$ and $U \subseteq \bigcup S$ we have $B(a, r) \subseteq \bigcup S$. Thus $\bigcup S$ is open, as required.

To prove Part (3), let S be a finite set of open sets. If $S = \emptyset$ then we use the convention that $\bigcap S = \mathbf{R}^n$, which is open. Suppose that $S \neq \emptyset$, say $S = \{U_1, U_2, \cdots, U_m\}$ where each U_k is an open set. Let $a \in \bigcap S = \bigcap_{k=1}^m U_k$. For each index k, since $a \in U_k$ we can choose $r_k > 0$ so that $B(a, r_k) \subseteq U_k$. Let $r = \min\{r_1, r_2, \cdots, r_m\}$. Then for each index k we have $B(a, r) \subseteq B(a, r_k) \subseteq U_k$. Since $B(a, r) \subseteq U_k$ for every index k, it follows that $B(a, r) \subseteq \bigcap_{k=1}^m U_k = \bigcap S$. Thus $\bigcap S$ is open, as required.

3.11 Theorem: *(Basic Properties of Closed Sets)*

(1) The sets $\emptyset$ *and* $\mathbf{R}^n$ *are closed in* $\mathbf{R}^n$.
(2) If S *is a set of closed sets then the intersection* $\bigcap S = \bigcap_{K \in S} K$ *is closed.*
(3) If S *is a finite set of closed sets then the union* $\bigcup S = \bigcup_{K \in S} K$ *is closed.*

Proof: The proof is left as an exercise

3.12 Definition: Let $A \subseteq \mathbf{R}^n$. The **interior** and the **closure** of A (in $\mathbf{R}^n$) are the sets

$$A^0 = \bigcup \left\{ U \subseteq \mathbf{R}^n \middle| U \text{ is open, and } U \subseteq A \right\},$$

$$\overline{A} = \bigcap \left\{ K \subseteq \mathbf{R}^n \middle| K \text{ is closed and } A \subseteq K \right\}.$$

3.13 Theorem: *Let* $A \subseteq \mathbf{R}^n$.

(1) The interior of A *is the largest open set which is contained in* A*. In other words,* $A^0 \subseteq A$ *and* A^0 *is open, and for every open set* U *with* $U \subseteq A$ *we have* $U \subseteq A^0$.

(2) The closure of A *is the smallest closed set which contains* A*. In other words,* $A \subseteq \overline{A}$ *and* $\overline{A}$ *is closed, and for every closed set* K *with* $A \subseteq K$ *we have* $\overline{A} \subseteq K$.

Proof: Note that A^0 is open by Part (2) of Theorem 8.10, because A^0 is equal to the union of a set of open sets. Also note that $A^0 \subseteq A$ because A^0 is equal to the union of a set of subsets of A. Finally note that for any open set U with $U \subseteq A$ we have $U \in S$ so that $U \subseteq \bigcup S = A^0$. This completes the proof of Part (1), and the proof of Part (2) is similar.

3.14 Corollary: *Let* $A \subseteq \mathbf{R}^n$.
(1) $(A^0)^0 = A^0$ *and* $\overline{\overline{A}} = \overline{A}$.
(2) A *is open if and only if* $A = A^0$
(3) A *is closed if and only if* $A = \overline{A}$.

Proof: The proof is left as an exercise.

3.15 Definition: Let $A \subseteq \mathbf{R}^n$. An **interior point** of A is a point $a \in A$ such that for some $r > 0$ we have $B(a,r) \subseteq A$. A **limit point** of A is a point $a \in \mathbf{R}^n$ such that for every $r > 0$ we have $B^*(a,r) \cap A \neq \emptyset$. A **boundary point** of A is a point $a \in \mathbf{R}^n$ such that for every $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$. The set of all limit points of A is denoted by A'. The **boundary** of A, is the set of all boundary points of A.

3.16 Theorem: *(Equivalent Topological Definitions) Let $A \subseteq \mathbf{R}^n$.*
(1) A is closed if and only if $A' \subseteq A$.
(2) $\overline{A} = A \cup A'$.
(3) A^0 is equal to the set of all interior points of A.
(4) $\partial A = \overline{A} \setminus A^0$.

Proof: To prove Part (1) note that when $a \notin A$ we have $B(a,r) \cap A = B^*(a,r) \cap A$ and so

$$\begin{aligned} A \text{ is closed } &\iff A^c \text{ is open} \\ &\iff \forall a \in A^c\ \exists r > 0\ B(a,r) \subseteq A^c \\ &\iff \forall a \in \mathbf{R}^n\ \big(a \notin A \Longrightarrow \exists r > 0\ B(a,r) \subseteq A^c \\ &\iff \forall a \in \mathbf{R}^n\ \big(a \notin A \Longrightarrow \exists r > 0\ B(a,r) \cap A = \emptyset\big) \\ &\iff \forall a \in \mathbf{R}^n\ \big(a \notin A \Longrightarrow \exists r > 0\ B^*(a,r) \cap A = \emptyset\big) \\ &\iff \forall a \in \mathbf{R}^n\ \big(\forall r > 0\ B^*(a,r) \cap A \neq \emptyset \Longrightarrow a \in A\big) \\ &\iff \forall a \in \mathbf{R}^n\ \big(a \in A' \Longrightarrow a \in A\big) \\ &\iff A' \subseteq A. \end{aligned}$$

To prove Part (2) we shall prove that $A \cup A'$ is the smallest closed set which contains A. It is clear that $A \cup A'$ contains A. We claim that $A \cup A'$ is closed, that is $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$, that is let $a \in \mathbf{R}^n$ with $a \notin A$ and $a \notin A'$. Since $a \notin A'$ we can choose $r > 0$ so that $B(a,r) \cap A = \emptyset$. We claim that because $B(a,r) \cap A = \emptyset$ it follows that $B(a,r) \cap A' = \emptyset$. Suppose, for a contradiction, that $B(a,r) \cap A' \neq emptyset$. Choose $b \in B(a,r) \cap A'$. Since $b \in B(a,r)$ and $B(a,r)$ is open, we can choose $s > 0$ so that $B(b,s) \subseteq B(a,r)$. Since $b \in A'$ it follows that $B(b,s) \cap A \neq \emptyset$. Choose $x \in B(b,s) \cap A$. Then we have $x \in B(b,s) \subseteq B(a,r)$ and $x \in A$ and so $x \in B(a,r) \cap A$, which contradicts the fact that $B(a,r) \cap A = \emptyset$. Thus $B(a,r) \cap A' = \emptyset$, as claimed. Since $B(a,r) \cap A = \emptyset$ and $B(a,r) \cap A' = \emptyset$ it follows that $B(a,r) \cap (A \cup A') = \emptyset$ hence $B(a,r) \subseteq (A \cup A')^c$. Thus proves that $(A \cup A')^c$ is open, and hence $A \cup A'$ is closed.

It remains to show that for every closed set K with $A \subseteq K$ we have $A \cup A' \subseteq K$. Let K be a closed set in $\mathbf{R}^n$ with $A \subseteq K$. Note that since $A \subseteq K$ it follows that $A' \subseteq K'$ because if $a \in A'$ then for all $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ hence $B(a,r) \cap K \neq \emptyset$ and so $a \in K'$. Since K is closed we have $K' \subseteq K$ by Part (1). Since $A' \subseteq K'$ and $K' \subseteq K$ we have $A' \subseteq K$. Since $A \subseteq K$ and $A' \subseteq K$ we have $A \cup A' \subseteq K$, as required. This completes the proof of Part (2). We leave the proofs of Parts (3) and (4) as an exercise.

3.17 Definition: Let $A \subseteq \mathbf{R}^n$. We say that A is **disconnected** when there exist open sets U and V in $\mathbf{R}^n$ such that

$$U \cap A \neq \emptyset\,,\ V \cap A \neq \emptyset\,,\ U \cap V = \emptyset \text{ and } A \subseteq U \cup V.$$

When A is disconnected, such open sets U and V are said to **separate** A. We say that A is **connected** when it is not disconnected.

3.18 Theorem: *The connected sets in* $\mathbf{R}$ *are the intervals, that is the sets of one of the forms*

$$(a,b)\,,\,[\,a,b)\,,\,(a,b\,]\,,\,[\,a,b\,]\,,\,(a,\infty)\,,\,[\,a,\infty)\,,\,(-\infty,b)\,,\,(-\infty,b\,]\,,\,(-\infty,\infty)$$

for some $a,b\in\mathbf{R}$ *with* $a\le b$. *We include the case that* $a=b$ *in order to include the degenerate intervals* $\emptyset=(a,a)$ *and* $\{a\}=[a,a]$.

Proof: I may include a proof later.

3.19 Definition: Let $A\subseteq\mathbf{R}^n$. We say that A is **bounded** when there exists $R>0$ such that $A\subseteq B(0,R)$.

3.20 Exercise: Show that A is bounded if and only if there exists $a\in\mathbf{R}^n$ and $r>0$ such that $A\subseteq B(a,r)$.

3.21 Definition: Let $A\subseteq\mathbf{R}^n$. An **open cover** of A is a set S of open sets such that $A\subseteq\bigcup S$. A **subcover** of an open cover S of A is a subset $T\subseteq S$ such that $A\subseteq\bigcup T$. We say that A is **compact** when every open caver of A has a finite subcover.

3.22 Exercise: Show that the set $A=\left\{\frac{1}{n}\middle|n\in\mathbf{Z}^+\right\}$ is not compact, but that the set $B=A\cup\{0\}$ is compact.

3.23 Definition: A **closed rectangle** in $\mathbf{R}^n$ is a set of the form

$$\begin{aligned}R&=[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]\\&=\left\{(x_1,x_2,\cdots,x_n)\in\mathbf{R}^n\middle|a_i\le x_i\le b_i\text{ for all }i\right\}.\end{aligned}$$

3.24 Theorem: *(Nested Rectangles) Let* $R_1,R_2,R_3,\cdots$ *be closed rectangles in* $\mathbf{R}^n$ *with* $R_1\supseteq R_2\supseteq R_3\supseteq\cdots$. *Then*

$$\bigcap_{k=1}^{\infty}R_k\neq\emptyset.$$

Proof: I may include a proof later.

3.25 Theorem: *(Compactness of Rectangles) Every closed rectangle in* $\mathbf{R}^n$ *is compact.*

Proof: I may include a proof later.

3.26 Theorem: *(The Heine-Borel Theorem) Let $A \subseteq \mathbf{R}^n$. Then A is compact if and only if A is closed and bounded.*

Proof: Suppose that A is compact. Suppose, for a contradiction, that A is not bounded. For each $k \in \mathbf{Z}^+$ let $U_k = B(0,k)$ and let $S = \{U_k \big| k \in \mathbf{Z}^+\}$. Then $\bigcup S = \mathbf{R}^n$ so S is an open cover of A. Let T be any finite subset of S. If $T = \emptyset$ then $\bigcup T = \emptyset$ and $A \not\subseteq \bigcup T$. Suppose that $T \neq \emptyset$, say $T = \{U_{k_1}, U_{k_2}, \cdots, U_{k_m}\}$ with $k_1 < k_2 < \cdots < k_m$. Since $U_{k_1} \subseteq U_{k_2} \subseteq \cdots \subseteq U_{k_m}$ we have $\bigcup T = \bigcup_{i=1}^m U_{k_i} = U_{k_m} = B(0,k_m)$. Since A is not bounded we have $A \not\subseteq B(0,k_m)$ and so $A \not\subseteq \bigcup T$. This shows that the open cover S has no finite subcover T, which contradicts the fact that A is compact.

Next suppose, for a contradiction, that A is not closed. By Part (1) of Theorem 8.16, it follows that $A' \not\subseteq A$. Choose $a \in A'$ with $a \notin A$. For each $k \in \mathbf{Z}^+$ let U_k be the open set $U_k = \overline{B}\left(a, \frac{1}{k}\right)^c = \left\{x \in \mathbf{R}^n \big| |x-a| > \frac{1}{k}\right\}$ and let $S = \{U_k \big| k \in \mathbf{Z}^+\}$. Note that $\bigcup S = \mathbf{R}^n \setminus \{a\}$ so S is an open cover of A. Let T be any finite subset of S. If $T = \emptyset$ then $\bigcup T = \emptyset$ so $A \not\subseteq \bigcup T$ (since A is not closed so $A \neq \emptyset$). Suppose that $T \neq \emptyset$, say $T = \{U_{k_1}, U_{k_2}, \cdots, U_{k_m}\}$ with $k_1 < k_2 < \cdots < k_m$. Since $U_{k_1} \subseteq U_{k_2} \subseteq \cdots \subseteq U_{k_m}$ we have $\bigcup T = \bigcup_{i=1}^m U_{k_i} = U_{k_m} = \overline{B}\left(a, \frac{1}{k_m}\right)^c$. Since a is a limit point of A we have $B\left(a, \frac{1}{k_m}\right) \neq \emptyset$ hence $\overline{B}\left(a, \frac{1}{k_m}\right) \cap A \neq \emptyset$ and so $A \not\subseteq \overline{B}\left(a, \frac{1}{k_m}\right)^c$, hence $A \not\subseteq \bigcup T$. This shows that the open cover S has no finite subcover T, which again contradicts the fact that A is compact.

Suppose, conversely, that A is closed and bounded. Since A is bounded we can choose $r > 0$ so that $A \subseteq B(0,r)$. Let R be the closed rectangle $R = \left\{x \in \mathbf{R}^n \big| |x_k| \leq r \text{ for all } k\right\}$. Note that $B(0,r) \subseteq R$ since when $x = (x_1, \cdots, x_n) \in B(0,r)$, for each index k we have

$$|x_k| = \left({x_k}^2\right)^{1/2} \leq \left(\sum_{i=1}^n {x_i}^2\right)^{1/2} = |x| < r.$$

We claim that since A is closed and $A \subseteq R$ and R is compact, it follows that A is compact. Let S be an open cover of A. Since A is closed, the set A^c is open, and since $A \subseteq \bigcup S$ we have $\bigcup \left(S \cup \{A^c\}\right) = \mathbf{R}^n$ and so the set $S \cup \{A^c\}$ is an open cover of the rectangle R. Since R is compact, we can choose a finite subset $T \subseteq \left(S \cup \{A^c\}\right)$ such that $R \subseteq \bigcup T$. Then we have $A \subseteq R \subseteq \bigcup T$ and so $A \subseteq \bigcup \left(T \setminus \{A^c\}\right)$. Thus the open cover S of A does have a finite subcover, namely the set $T \setminus \{A^c\}$. This proves that A is compact, as required.

3.27 Definition: Let $A \subseteq S \subseteq \mathbf{R}^n$. We say that A is **open in** S when there exists an open set U in $\mathbf{R}^n$ such that $A = S \cap U$. We say that A is **closed in** S when its complement $A^c = S \setminus A$ is open in S.

3.28 Theorem: *Let $A \subseteq S \subseteq \mathbf{R}^n$.*

(1) A is open in S if and only if for all $a \in A$ there exists $r > 0$ such that $B(a,r) \cap S \subseteq A$.
(2) A is closed in S if and only if there exists a closed set K in $\mathbf{R}^n$ such that $A = S \cap K$.
(3) S is disconnected if and only if S has a nonempty proper subset which is both open and closed in S.

Proof: I may include a proof later.

3.29 Definition: Let $\langle a_n\rangle_{n\geq p}$ be a sequence in $\mathbf{R}^m$. We say the sequence $\langle a_n\rangle_{n\geq p}$ is **bounded** when

$$\exists R>0\ \forall n\in\mathbf{Z}_{\geq p}\ |a_n|\leq R.$$

For $b\in\mathbf{R}^m$, we say that the sequence $\langle a_n\rangle_{n\geq p}$ **converges to** b and write $\lim\limits_{n\to\infty}a_n=b$ (or $a_n\to b$) when

$$\forall\epsilon>0\ \exists N\in\mathbf{Z}_{\geq p}\ \forall n\in\mathbf{Z}_{\geq p}\big(n\geq N\Longrightarrow|a_n-b|<\epsilon\big).$$

We say the sequence $\langle a_n\rangle_{n\geq p}$ **diverges to** ∞ and write $\lim\limits_{n\to\infty}a_n=\infty$ (or $a_n\to\infty$) when

$$\forall R>0\ \exists N\in\mathbf{Z}_{\geq p}\ \forall n\in Z_{\geq p}\big(n\geq N\Longrightarrow|a_n|\geq R\big).$$

We say that the sequence $\langle a_n\rangle_{n\geq p}$ **converges** when it converges to some point $b\in\mathbf{R}^m$ and otherwise we say that it **diverges**.

3.30 Theorem: *Let $\langle a_n\rangle_{n\geq p}$ be a sequence in $\mathbf{R}^m$, say $a_n=\big(a_{n,1},a_{n,2},\cdots,a_{n,m}\big)\in\mathbf{R}^m$.*
(1) $\langle a_n\rangle_{n\geq p}$ is bounded if and only if $\langle a_{n,i}\rangle_{n\geq p}$ is bounded for all indices i.
(2) For $b=(b_1,\cdots,b_m)\in\mathbf{R}^m$ we have $\lim\limits_{n\to\infty}a_n=b$ if and only if $\lim\limits_{n\to\infty}a_{n,i}=b_i$ for all i.
In particular, if $u,v,x_n,y_n\in\mathbf{R}$ and $a_n=x_n+i\,y_n$, then

$$\lim_{n\to\infty}a_n=u+iv\iff\Big(\lim_{n\to\infty}x_n=u\text{ and }\lim_{n\to\infty}y_n=v\Big).$$

Proof: The proof is left as an exercise.

3.31 Theorem: *Let $\langle a_n\rangle_{n\geq p}$ be a sequence in $\mathbf{R}^m$ and let $u,v\in\mathbf{R}^m\cup\{\infty\}$.*
(1) If $\lim\limits_{n\to\infty}a_n=u$ and $\lim\limits_{n\to\infty}a_n=v$ then $u=v$.
(2) If $\lim\limits_{n\to\infty}a_n=u$ and $\langle a_{n_j}\rangle_{j\geq q}$ is a subsequence of $\langle a_n\rangle$ then $\lim\limits_{j\to\infty}a_{n_j}=u$.
(3) If $\langle a_n\rangle_{n\geq p}$ converges then it is bounded.

Proof: The proof is left as an exercise.

3.32 Theorem: *Let $c\in\mathbf{R}$, let $u,v\in\mathbf{R}^m$, and let $\langle a_n\rangle$ and $\langle b_n\rangle$ be sequences in $\mathbf{R}^m$ with $\lim\limits_{n\to\infty}a_n=u$ and $\lim\limits_{n\to\infty}b_n=v$. Then*
(1) $\lim\limits_{n\to\infty}(ca_n)=cu$,
(2) $\lim\limits_{n\to\infty}(a_n\pm b_n)=u\pm v$,
(3) $\lim\limits_{n\to\infty}(u_n\bullet v_n)=u\bullet v$,
(4) if $m=2$ so that $u,v\in\mathbf{C}$ then $\lim\limits_{n\to\infty}(a_nb_n)=uv$, and
(5) if $m=2$ so that $u,v\in\mathbf{C}$ and if $v\neq 0$ then $\lim\limits_{n\to\infty}a_n/b_n=u/v$.

Proof: The proof is left as an exercise.

3.33 Theorem: *(Bolzano-Weierstrass) Every bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.*

Proof: I may include a proof later.

3.34 Definition: Let $\langle a_n\rangle_{n\geq p}$ be a sequence in $\mathbf{R}^n$. We say that $\langle a_n\rangle$ is **Cauchy** when

$$\forall\epsilon>0\ \exists N\in\mathbf{Z}_{\geq p}\ \forall k,\ell\in\mathbf{Z}_{\geq p}\Big(k,l\geq N\Longrightarrow|a_k-a_\ell|<\epsilon\Big).$$

3.35 Theorem: *(The Completeness of $\mathbf{R}^n$) For every sequence in $\mathbf{R}^n$, the sequence converges if and only if it is Cauchy.*

Proof: I may include a proof later.

3.36 Definition: Let $A \subseteq \mathbf{R}^n$ and let $f : A \to \mathbf{R}^m$. When a is a limit point of A and $b \in \mathbf{R}^m$, we say that $f(x)$ **converges to** b as x tends to a, and we write $\lim_{x \to a} f(x) = b$ when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies |f(x) - b| < \epsilon\right).$$

When a is a limit point of A, we say that $f(x)$ **diverges to** ∞ and we write $\lim_{x \to a} f(x) = \infty$ when

$$\forall R > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies |f(x)| \geq R\right).$$

3.37 Theorem: *(Sequential Characterization of Limits) Let $A \subseteq \mathbf{R}^n$, let $f : A \to \mathbf{R}^m$, let a be a limit point of A and let $u \in \mathbf{R}^m \cup \{\infty\}$. Then $\lim_{x \to a} f(x) = u$ if and only if $\lim_{n \to \infty} f(x_n) = u$ for every sequence $\langle x_n \rangle$ in $A \setminus \{a\}$ with $\lim_{n \to \infty} x_n = a$.*

Proof: The proof is left as an exercise.

3.38 Corollary: *Let $A \subseteq \mathbf{R}^n$ and let $f : A \to \mathbf{R}^m$, let a be a limit point of A, let $b = (b_1, b_2, \cdots, b_m) \in \mathbf{R}^n$ and say $f(x) = \big(f_1(x), \cdots, f_m(x)\big) \in \mathbf{R}^m$ for each $x \in A$ so that $f_i : A \to \mathbf{R}$ for each i. Then $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f_i(x) = b_i$ for all i.*

Proof: The proof is left as an exercise.

3.39 Corollary: *Let $A \subseteq \mathbf{R}^m$, let $f : A \to \mathbf{R}^m$, let a be a limit point of A, and let $u, v \in \mathbf{R}^m \cup \{\infty\}$. If $\lim_{x \to a} = u$ and $\lim_{x \to a} = v$ then $u = v$.*

Proof: The proof is left as an exercise.

3.40 Corollary: *Let $A \subseteq \mathbf{R}^n$, let $f, g : A \to \mathbf{R}^m$, let a be a limit point of A, let $u, v \in \mathbf{R}^m$ and suppose that $\lim_{x \to a} f(x) = u$ and $\lim_{x \to a} g(x) = v$. Then*

(1) $\lim_{x \to a} c\, f(x) = cu$,
(2) $\lim_{x \to a} f(x) + g(x) = uv$,
(3) $\lim_{x \to a} f(x) \bullet g(x) = u \bullet v$,
(4) if $m = 2$ so that $u, v \in \mathbf{C}$ then $\lim_{x \to a} f(x)g(x) = uv$, and
(5) if $m = 2$ so that $u, v \in \mathbf{C}$ and if $v \neq 0$ then $\lim_{x \to a} f(x)/g(x) = u/v$.

Proof: The proof is left as an exercise.

3.41 Definition: Let $A \subseteq \mathbf{R}^n$, let $B \subseteq \mathbf{R}^m$, and let $f : A \to B$. For $a \in A$, we say that f is **continuous at** a when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A\ \big(|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon\big).$$

We say that f is **continuous** (in A) when f is continuous at a for every $a \in A$. We say that f is **uniformly continuous** in A when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall a \in A\ \forall x \in A\ \big(|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon\big).$$

3.42 Theorem: *Let $A \subseteq \mathbf{R}^n$, let $f : A \to \mathbf{R}^n$, and let $a \in A$. Then*
(1) if a is not a limit point of A then f is continuous at a, and
(2) if a is a limit point of A then f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

Proof: The proof is left as an exercise.

3.43 Theorem: *(Sequential Characterization of Continuity) Let $A \subseteq \mathbf{R}^n$, let $f : A \to \mathbf{R}^n$, and let $a \in A$. Then f is continuous at a if and only if $\lim\limits_{n \to \infty} f(x_n) = f(a)$ for every sequence $\langle x_n \rangle_{n \geq p}$ in A with $\lim\limits_{n \to \infty} x_n = a$.*

Proof: The proof is left as an exercise.

3.44 Corollary: *Let $A \subseteq \mathbf{R}^n$, let $f : A \to \mathbf{R}^m$ be given by $f(x) = \big(f_1(x), \cdots, f_m(x)\big)$ where $f_k : A \to \mathbf{R}$ for each k, and let $a \in A$. Then f is continuous at a if and only if each function f_k is continuous at a.*

Proof: The proof is left as an exercise.

3.45 Corollary: *Let $c \in \mathbf{R}$, let $A \subseteq \mathbf{R}^n$, let $f, g : A \to \mathbf{R}^m$, let $a \in A$, and suppose that f and g are both continuous at a. Then the functions cf, $f \pm g$, and $f \bullet g$ are all continuous at a, and in the case that $m = 2$ so that $f, g : A \to \mathbf{C}$ the function fg is continuous at a, and in the case that $m = 2$ and $g(a) \neq 0$ the function f/g is continuous at a.*

Proof: The proof is left as an exercise.

3.46 Example: Let $U = \{r\,e^{i\,\theta} | r > 0, 0 < \theta < 2\pi\}$. Let $\theta : U \to (0, 2\pi)$ be the angle function. Show that θ is continuous in U.

Solution: Write $z = x + i\,y$ with $x, y \in \mathbf{R}$. For $\mathrm{Im}\,(z) > 0$, the angle function is given by the formula $\theta(x + i\,y) = \cos^{-1}\big(x/\sqrt{x^2 + y^2}\big)$. This formula expresses $\theta(x + i\,y)$ using sums, products, quotients and composites of known continuous functions, and so it must be continuous, by parts b) and c) of the above theorem. Thus $\theta(z)$ is continuous at all points z with $\mathrm{Im}\,(z) > 0$.

Similarly, for $\mathrm{Re}\,(z) < 0$, $\theta(z)$ is given by the formula $\theta(x + i\,y) = \pi + \tan^{-1}\big(y/x\big)$, and for $\mathrm{Im}\,(z) < 0$ we have $\theta(x + i\,y) = 2\pi - \cos^{-1}\big(x/\sqrt{x^2 + y^2}\big)$. These are both continuous and so $\theta(z)$ is continuous for all $z \in U$.

3.47 Example: As an excercise, show that for the angle function $\theta : \mathbf{C}^* \to [0, 2\pi)$ and for $a > 0$, the limit $\lim\limits_{z \to a} \theta(z)$ does not exist, so $\theta : \mathbf{C}^* \to [0, 2\pi)$ is not continuous in $\mathbf{C}^*$. In fact it is impossible to choose $\theta(z) \in \mathbf{R}$ so that $\theta : \mathbf{C}^* \to \mathbf{R}$ is continuous in $\mathbf{C}^*$. As in the previous example, we must restrict the domain to make the angle function continuous. Indeed, for any $\alpha \in \mathbf{R}$, if we restrict the domain to $U_\alpha = \{r\,e^{i\,\theta} | r > 0, \alpha < \theta < \alpha + 2\pi\}$ and choose $\theta(z)$ with $\alpha < \theta(z) < \alpha + 2\pi$ then $\theta : U_\alpha \to (\alpha, \alpha + 2\pi)$ will be continuous.

3.48 Note: We have found formulas for the real and imaginary parts of the identity $f(z) = z$, the exponential $f(z) = e^z$, the trigonometric functions, and the hyperbolic functions. These formulas reveal that they are all continuous in their domains. Also, any branch of the logarithm $\log z = \ln|z| + i\,\theta(z)$ is continuous provided that $\theta(z)$ is chosen to be continuous. The inverse trigonometric and inverse hyperbolic functions can all be expressed in terms of the logarithm, and so they are also continuous provided that $\theta(z)$ is chosen to be continuous. Any complex function which can be expressed using sums, products, quotients and composites of the above functions will be continuous in its domain.

3.49 Theorem: *(Topological Characterization of Continuity) Let $A \subseteq \mathbf{R}^n$, let $B \subseteq \mathbf{R}^m$, and let $f : A \to B$.*

(1) f is continuous if and only if $f^{-1}(U)$ is open in A for every open set U in B.
(2) f is continuous if and only if $f^{-1}(K)$ is closed in A for every closed set K in B.

3.50 Exercise: Show that the set $U = \left\{(x, y) \in \mathbf{R}^2 \middle| y > x^2\right\}$ is open in $\mathbf{R}^2$.

3.51 Theorem: *(Properties of Continuous Functions) Let $A \subseteq \mathbf{R}^n$, let $B \subseteq \mathbf{R}^m$, and let $f : A \to B$ be continuous.*

(1) If A is bounded then $f(A)$ is bounded.
(2) If A is connected then $f(A)$ is connected.
(3) If A is compact then $f(A)$ is compact.
(4) If A is compact the f is uniformly continuous on A.
(5) If A is compact and $m = 1$ then $f(x)$ attains its maximum and minimum values on A.

3.52 Definition: Let $A \subseteq \mathbf{R}^n$ and let $a, b \in A$. A (continuous) **path** from a to b in A is a continuous function $f : [0, 1] \to A$ with $f(0) = a$ and $f(1) = b$. We say that A is **path-connected** when for every $a, b \in A$ there exists a continuous path from a to b in A.

3.53 Theorem: *Let $A \subseteq \mathbf{R}^n$. If A is path-connected then A is connected.*

Proof: I may include a proof later.

3.54 Exercise: Show that for $a \in \mathbf{R}^n$ and $r > 0$, the set $B(a, r)$ is connected.

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Chapter 4. Derivatives

4.1 Note: From now on, we shall always use the letter U to denote an open set.

4.2 Definition: Recall that for a function $f : U \subseteq \mathbf{R} \to \mathbf{R}$ we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists, and then we say that f is **differentiable** at $x = a$ and $f'(a)$ is called the (real) **derivative** of f at a. Equivalently, we see that f is differentiable at $x = a$ if there exists a real number $f'(a)$ such that $\lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| = 0$. This last condition can be rewritten as $\lim_{x \to a} \frac{|R(x)|}{|x - a|} = 0$, where $R(x) = f(x) - \big(f(a) + f'(a)(x - a)\big)$. In this way we obtain a definition which applies to functions $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$.

A function $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is **differentiable** at $x = a$ if there exists an $m \times n$ matrix $Df(a)$ such that $\lim_{x \to a} \frac{|R(x)|}{|x - a|} = 0$, where $R(x) = f(x) - \big(f(a) + Df(a)(x - a)\big)$. The matrix $Df(a)$ is called the (real) **derivative matrix** of f at $x = a$. We say that $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is **differentiable** in U if it is differentiable at every point $a \in U$.

For a map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}$, the j^{th} **partial derivative** of f is given by

$$f_{x_j}(a) = \frac{\partial f}{\partial x_j}(a) = g'(0)\,,$$

if it exists, where $g(t) = f(a + t\, e_j)$ with e_j denoting the j^{th} standard basis vector in $\mathbf{R}^n$.

We now recall (without proof) some theorems from vector calculus.

4.3 Theorem: *Let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$, and let f_i be the components of f so that $f(x) = \big(f_1(x), \cdots f_m(x)\big)$. Then if f is differentiable at $x = a$ then the partial derivatives $\frac{\partial f_i}{\partial x_j}$ all exist and*

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

4.4 Theorem: *If $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is $\mathcal{C}^1$ in U, which means that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ all exist and are continuous in U, then f is differentiable in U.*

4.5 Theorem: *If $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a then f is continuous at a.*

4.6 Theorem: *If $f, g : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ are both differentiable at $x = a$, then*
(a) $D(c\,f)(a) = c\,Df(a)$ where $c \in \mathbf{R}$.
(b) $D(f \pm g)(a) = Df(a) \pm Dg(a)$.
(c) (The Product Rule) If $m = 1$ then $D(fg)(a) = Df(a)g(a) + f(a)Dg(a)$.
(d) (The Quotient Rule) If $m = 1$ then $D(f/g)(a) = \big(Df(a)g(a) - f(a)D(a)\big)/g^2(a)$.

4.7 Theorem: *(The Chain Rule) If $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a, and $g : V \subseteq \mathbf{R}^m \to \mathbf{R}^l$ is differentiable at $f(a)$ then $h(x) = g(f(x))$ is differentiable at a and $h'(a) = g'(f(a))f'(a)$.*

4.8 Theorem: *(The Inverse Function Theorem) If $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is $\mathcal{C}^1$ in U and $Df(a)$ is invertible, then we can restrict the domain of f to some open set $V \subseteq U$ with $a \in V$ so that f is invertible, $g = f^{-1}$ is $\mathcal{C}^1$, and $Dg\big(f(x)\big) = Df(x)^{-1}$.*

4.9 Definition: For a differentiable map $f : U \subseteq \mathbf{R} \to \mathbf{R}^n$ given by $f(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, we have

$$Df(t) = f'(t) = \begin{pmatrix} {x_1}'(t) \\ \vdots \\ {x_n}'(t) \end{pmatrix}.$$

The vector $Df(a) = f'(a)$ is the **tangent vector** to the curve $x = f(t)$ at the point $f(a)$. In particular, for a differentiable map $f : U \subseteq \mathbf{C} \to \mathbf{C}$ given by $f(t) = z(t) = x(t) + i\, y(t)$, we have

$$Df(t) = z'(t) = x'(t) + i\, y'(t).$$

4.10 Definition: For a differentiable map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}$ we have

$$Df(x) = \Big(\tfrac{\partial f}{\partial x_1}(x) \quad \tfrac{\partial f}{\partial x_2}(x) \quad \ldots \quad \tfrac{\partial f}{\partial x_n}(x) \Big).$$

We define the **gradient** of f at a to be $\nabla f = Df(x)^T$. Given a point $a \in U$ and a vector $v \in \mathbf{R}^n$, we define the **directional derivative** $D_v f(a)$ of f at a with respect to v as follows. Choose any curve $\alpha : \mathbf{R} \to U$ with $\alpha(0) = a$ and $\alpha'(0) = v$, and set $\beta(t) = f(\alpha(t))$. By the chain rule, we have $\beta'(t) = Df(\alpha(t))\alpha'(t)$ and so $\beta'(0) = Df(a)v = \nabla f(a) \cdot v$. We define

$$D_v f(a) = \beta'(0) = Df(a)v = \nabla f(a) \cdot v.$$

Notice that the gradient $\nabla f(a)$ is perpendicular to the level set $f(x) = f(a)$. To see this, choose any curve $x(t)$ with $x(0) = a$ and with $f(x(t)) = f(a)$ (so that $x(t)$ lies in the level set). Then by the chain rule we have $Df(x(t))x'(t) = 0$, and setting $t = 0$ gives $Df(a)x'(0) = 0$ or equivalently $\nabla f(a) \cdot x'(a) = 0$. Thus $\nabla f(a)$ is perpendicular to $x'(0)$.

4.11 Example: Given a differentiable map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$, notice that the i^{th} row of the matrix $Df(a)$ is equal to $Df_i(a) = \nabla f_i(a)^T$, where f_i is the i^{th} component of f. So the i^{th} row is perpendicular to the level set $f_i(x) = f_i(a)$.

4.12 Example: For a differentiable map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$, we denote the j^{th} column of the matrix $Df(a)$ by $f_{x_j}(a)$ $\big($or by $\frac{\partial f}{\partial x_j}\big)$, so we have

$$f_{x_j}(a) = \frac{\partial f}{\partial x_j}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{pmatrix}.$$

Notice that this is equal to the tangent vector to the curve $\beta(t) = f(a + t\, e_j)$, where e_j is the j^{th} standard basis vector; indeed if $\alpha(t) = a + t\, e_j$ so $\alpha(0) = a$ and $\alpha'(0) = e_j$, and if $\beta(t) = f\big(\alpha(t)\big)$, then by the Chain Rule we have $\beta'(t) = Df\big(\alpha(t)\big)\alpha'(t)$ so $\beta'(0) = Df(a)\, e_j$, which is the j^{th} column of $Df(a)$.

4.13 Example: For a differentiable map $f : U \subseteq \mathbf{C} \to \mathbf{C}$ given by $w(z) = u(z) + i\,v(z)$ with $z = x + i\,y$, we have

$$Df(a) = \begin{pmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{pmatrix}.$$

The columns $f_x = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$ and $f_y = \begin{pmatrix} u_y \\ v_y \end{pmatrix}$ are the tangent vectors to the curves $f(a+t)$ and $f(a+i\,t)$ respectively, and the rows $Du = \begin{pmatrix} u_x & u_y \end{pmatrix}$ and $Dv = \begin{pmatrix} v_x & v_y \end{pmatrix}$ are perpendicular to the level curves $u = u(a)$ and $v = v(b)$ respectively.

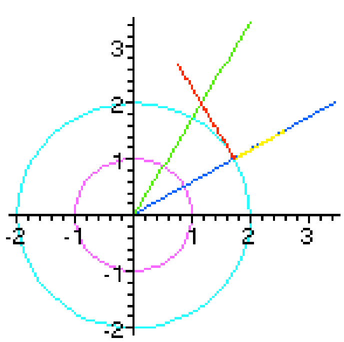
4.14 Example: Let f be the polar coordinates map $(x, y) = f(r, \theta) = (r\cos\theta, r\sin\theta)$. Then

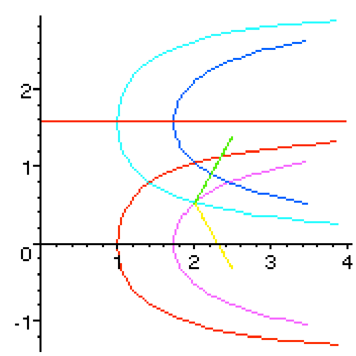
$$Df(r,\theta) = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

At $(r,\theta) = (2, \frac{\pi}{6})$, we have $(x, y) = f(2, \frac{\pi}{6}) = (\sqrt{3}, 1)$ and $Df(2, \frac{\pi}{6}) = \begin{pmatrix} \sqrt{3}/2 & -1 \\ 1/2 & \sqrt{3} \end{pmatrix}$. Below on the left, is a picture showing the images of the lines $r = 0, 1, 2$ and $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ (the images are circles and rays), and the tangent vectors $f_r = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$ and $f_\theta = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$ are shown at the point $(x, y) = (\sqrt{3}, 1)$. On the right, there is a picture showing the level curves $x = 0, 1, \sqrt{3}$ and $y = 0, 1, \sqrt{3}$ (they are multiples of $r = \sec\theta$ and $r = \csc\theta$), and the gradient vectors $Dx = \begin{pmatrix} \frac{\sqrt{3}}{2} & -1 \end{pmatrix}$ and $Dy = \begin{pmatrix} \frac{1}{2} & \sqrt{3} \end{pmatrix}$ are shown at the point $(r,\theta) = (2, \frac{\pi}{6})$ draw This map f is not $1:1$ so it does not have an inverse, but since the matrix $Df(2, \frac{\pi}{6})$ is invertible, we know that we can make f invertible by restricting its domain. If $g = f^{-1}$ near the point $(r,\theta) = (2, \frac{\pi}{3})$, then we have

$$Dg(\sqrt{3}, 1) = \begin{pmatrix} \sqrt{3}/2 & -1 \\ 1/2 & \sqrt{3} \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}.$$

This can also be verified by finding an explicit formula for g, for example if we restrict the domain of f to $r > 0$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ then $(r,\theta) = g(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}(y/x)\right)$.





4.15 Note: We now wish to interpret the real derivative matrix Df of a map of the form $f : U \subseteq \mathbf{C} \to \mathbf{C}$ in terms of complex numbers.

4.16 Note: A real 2×2 matrix A corresponds to two complex numbers in the following two ways. Let $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, and write $z=x+i\,y$ with $x,y\in\mathbf{R}$. Then we have

$$\begin{aligned} Az &= A\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} a\\ c\end{pmatrix}x+\begin{pmatrix} b\\ d\end{pmatrix}y \\ &=(a+i\,c)x+(b+i\,d)y \\ &=(a+ic)\frac{z+\overline{z}}{2}+(b+i\,d)\frac{z-\overline{z}}{2i} \\ &=\tfrac{1}{2}\big((a+d)+i\,(c-b)\big)\,z=\tfrac{1}{2}\big((a-d)+i\,(c+b)\big)\,\overline{z} \end{aligned} .$$

Thus we have $A\begin{pmatrix} x\\ y\end{pmatrix}=px+qy=uz+v\overline{z}$ where p and q are the columns of A, that is $p=a+i\,c$ and $q=b+i\,d$, and u and v are given by $u=\frac{1}{2}\big((a+d)+i\,(c-b)\big)=\frac{1}{2}(p-i\,q)$ and $v=\frac{1}{2}\big((a-d)+i\,(c+b)\big)=\frac{1}{2}(p+i\,q)$. Note also that $p=u+v$ and $q=i\,(u-v)$.

Conversely, given $u=\alpha+i\,\beta$ and $v=\gamma+i\,\delta$, with $\alpha,\beta,\gamma,\delta\in\mathbf{R}$, we have

$$\begin{aligned} u\,z&=(\alpha+i\,\beta)(x+i\,y)=(\alpha x-\beta y)+i\,(\beta x+\alpha y)=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} \\ v\overline{z}&=(\gamma+i\,\delta)(x-i\,y)=(\gamma x+\delta y)+i\,(\delta x-\gamma y)=\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}\end{aligned}$$

and so $u\,z+v\,\overline{z}=Az$ where $A=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}+\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}$.

4.17 Definition: For a map $f:U\subseteq\mathbf{C}\to\mathbf{C}$ given by $f(z)=u(z)+i\,v(z)$ with $z=x+i\,y$, wich is differentiable at $a\in U$ so that $Df(a)=\begin{pmatrix} u_x(a) & u_y(a)\\ v_x(a) & v_y(a)\end{pmatrix}$, we define

$$\begin{aligned} f_x(a)&=u_x(a)+i\,v_x(a)=f_z(a)+f_{\overline{z}}(a) \\ f_y(a)&=u_y(a)+i\,v_y(a)=i\,(f_z(a)-f_{\overline{z}}(a)) \\ f_z(a)&=\tfrac{1}{2}(f_x(a)-i\,f_y(a))=\tfrac{1}{2}\big((u_x(a)+v_y(a))+i\,(v_x(a)-u_y(a))\big) \\ f_{\overline{z}}(a)&=\tfrac{1}{2}(f_x(a)+i\,f_y(a))=\tfrac{1}{2}\big((u_x(a)-v_y(a))+i\,(u_y(a)-v_x(a))\big)\,. \end{aligned}$$

Note that if $f_z(a)=\alpha+i\,\beta$ and $f_{\overline{z}}(a)=\gamma+i\,\delta$ with $\alpha,\beta,\gamma,\delta\in\mathbf{R}$ then

$$Df(a)=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}+\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}.$$

When $w=f(z)$, other notations for these include

$$\begin{aligned} f_x&=\tfrac{\partial f}{\partial x}=w_x=\tfrac{\partial w}{\partial x} \\ f_y&=\tfrac{\partial f}{\partial y}=w_y=\tfrac{\partial w}{\partial y} \\ f_z&=\tfrac{\partial f}{\partial z}=\partial f=w_z=\tfrac{\partial w}{\partial z}=\partial w \\ f_{\overline{z}}&=\tfrac{\partial f}{\partial \overline{z}}=\overline{\partial} f=w_{\overline{z}}=\tfrac{\partial w}{\partial z}=\overline{\partial} w\,. \end{aligned}$$

4.18 Note: For $f : U \subseteq \mathbf{C} \to \mathbf{C}$ with $a \in U$, then from the above note and definition, we see that

$$\begin{aligned}
&f \text{ is differentiable at } a \\
&\iff \text{there exists a real } 2\times 2 \text{ matrix } Df(a) \text{ such that} \\
&\qquad \lim_{z\to a} \frac{|R(z)|}{|z-a|} = 0, \text{ where } R(z) = f(z) - \big(f(a) + Df(a)(z-a)\big) \\
&\iff \text{there exist two complex numbers } f_x(a) \text{ and } f_y(a) \text{ such that} \\
&\qquad \lim_{z\to a} \frac{|R(z)|}{|z-a|} = 0, \text{ where } R(z) = f(z) - \big(f(a) + f_x(a)\mathrm{Re}\,(z-a) + f_y(a)\mathrm{Im}\,(z-a)\big) \\
&\iff \text{there exist two complex numbers } f_z(a) \text{ and } f_{\overline{z}}(a) \text{ such that} \\
&\qquad \lim_{z\to a} \frac{|R(z)|}{|z-a|} = 0, \text{ where } R(z) = f(z) - \big(f(a) + f_z(a)(z-a) + f_{\overline{z}}(\overline{z}-\overline{a})\big).
\end{aligned}$$

4.19 Example: Show that $\frac{\partial z}{\partial z} = \frac{\partial \overline{z}}{\partial \overline{z}} = 1$, $\frac{\partial z}{\partial \overline{z}} = \frac{\partial \overline{z}}{\partial z} = 0$, and $\frac{\partial a}{\partial z} = \frac{\partial a}{\partial \overline{z}} = 0$, where $a \in \mathbf{C}$.

Solution: If $f(z) = z$, then we have $f(x+i\,y) = u(x,y) + i\,v(x,y)$, where $u(x,y) = x$ and $v(x,y) = y$. So $Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f_x = u_x + i\,v_x = 1$, $f_y = u_y + i\,v_y = i$, $f_z = \frac{1}{2}(f + x - i\,f_y) = 1$ and $f_{\overline{z}} = \frac{1}{2}(f_x - i\,f_y) = 0$.

If $f(z) = \overline{z}$, then we have $u(x,y) = x$ and $v(x,y) = -y$. So $Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f_x = 1$, $f_y = -i$, $f_z = 0$ and $f_{\overline{z}} = 1$.

If $f(z) = a \in \mathbf{C}$ then $u(x,y) = \mathrm{Re}\,(a)$ and $v(x,y) = \mathrm{Im}\,(a)$. So $Df = 0$ and hence $f_x = f_y = f_z = f_{\overline{z}} = 0$.

4.20 Theorem: *Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$ be differentiable in U. For $\alpha = x, y, z$ or $\overline{z}$ we have*
(a) $(cf)_\alpha = c\,f_\alpha$
(b) $(f \pm g)_\alpha = f_\alpha \pm f_\alpha$
(c) (The Product Rule) $(fg)_\alpha = f_\alpha g + f g_\alpha$
(d) (The Quotient Rule) $(f/g)_\alpha = (f_\alpha g - f g_\alpha)/g^2$, when $g \neq 0$

Proof: We prove the product rule, and leave the other parts as an exercise. We write $f = u + i\,v$ and $g = s + i\,t$ where u, v, s and t are real-valued. Then $fg = (us - vt) + i\,(ut + vs)$. The Product Rule in Theorem 4.6 applies to the functions u, v, s and t, so we have

$$\begin{aligned}
(fg)_x &= (us - vt)_x + i\,(ut + vs)_x \\
&= (u_x s + u s_x - v_x t - v t_x) + i\,(u_x t + u t_x + v_x s + v s_x) \\
&= (u_x + i\,v_x)(s + i\,t) + (u + i\,v)(s_x + i\,t_x) \\
&= f_x g + f g_x
\end{aligned}$$

Similarly, $(fg)_y = f_y g + f g_y$. Then, using this result, we have

$$\begin{aligned}
(fg)_z &= \tfrac{1}{2}\big((fg)_x - i\,(fg)_y\big) \\
&= \tfrac{1}{2}\big((f_x g + f g_x) - i\,(f_y g + f g_y)\big) \\
&= \tfrac{1}{2}(f_x - i\,f_y)\,g + f\,\tfrac{1}{2}(g_x - i\,g_y) \\
&= f_z g + f g_z\,,
\end{aligned}$$

and similarly, $(fg)_{\overline{z}} = f_{\overline{z}} g + f\,g_{\overline{z}}$.

4.21 Theorem: *For $f : U \subseteq \mathbf{C} \to \mathbf{C}$, define $\overline{f} : U \to \mathbf{C}$ by $\overline{f}(z) = \overline{f(z)}$. Then we have $\overline{f}_z = \overline{f_{\overline{z}}}$ and $\overline{f}_{\overline{z}} = \overline{f_z}$.*

Proof: Write $f = u + i\,v$ with u and v real-valued. Then $\overline{f} = u - i\,v$ so $D\overline{f} = \begin{pmatrix} u_x & u_y \\ -v_x & -v_y \end{pmatrix}$ and hence $\overline{f}_x = u_x - i\,v_x = \overline{f_x}$ and $\overline{f}_y = u_y - i\,v_y = \overline{f_y}$. So we have

$$\overline{f}_z = \tfrac{1}{2}(\overline{f}_x - i\,\overline{f}_y) = \tfrac{1}{2}(\overline{f_x} - i\,\overline{f_y}) = \overline{\tfrac{1}{2}(f_x + i\,f_y)} = \overline{f_{\overline{z}}}\,,$$

and similarly, $\overline{f}_{\overline{z}} = \overline{f_z}$.

4.22 Theorem: *(The Chain Rule) Suppose $f : U \to V \subseteq \mathbf{C}$ and $g : V \to \mathbf{C}$ are both differentiable, and let $h(z) = g(f(z))$. Then h is differentiable, and if we write $w = f(z)$ and $q = g(w)$, then*

$$\begin{pmatrix} q_z & q_{\overline{z}} \\ \overline{q}_z & \overline{q}_{\overline{z}} \end{pmatrix} = \begin{pmatrix} q_w & q_{\overline{w}} \\ \overline{q}_w & \overline{q}_{\overline{w}} \end{pmatrix} \begin{pmatrix} w_z & w_{\overline{z}} \\ \overline{w}_z & \overline{w}_{\overline{z}} \end{pmatrix}.$$

Equivalently, we have $\dfrac{\partial q}{\partial z} = \dfrac{\partial q}{\partial w}\dfrac{\partial w}{\partial z} + \dfrac{\partial q}{\partial \overline{w}}\dfrac{\partial \overline{w}}{\partial z}$ *and* $\dfrac{\partial q}{\partial \overline{z}} = \dfrac{\partial q}{\partial w}\dfrac{\partial w}{\partial \overline{z}} + \dfrac{\partial q}{\partial \overline{w}}\dfrac{\partial \overline{w}}{\partial \overline{z}}$.

Proof: Write $z = x + i\,y$, $f(z) = w = u + i\,v$ and $g(w) = q = s + i\,t$. Then

$$\begin{aligned} q_z &= \tfrac{1}{2}\big((s_x + t_y) + i\,(t_x - s_y)\big) \\ &= \tfrac{1}{2}\big((s_u u_x + s_v v_x + t_u u_y + t_v v_y) + i\,(t_u u_x + t_v v_x - s_u u_y - s_v v_y)\big)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} q_w w_z + q_{\overline{w}}\overline{w}_z &= q_w w_z + q_{\overline{w}}\overline{w_{\overline{z}}} \\ &= \tfrac{1}{2}\big((s_u + t_v) + i\,(t_u - s_v)\big)\,\tfrac{1}{2}\big((u_x + v_y) + i\,(v_x - u_y)\big) \\ &+ \tfrac{1}{2}\big((s_u - t_v) + i\,(t_u + s_v)\big)\,\tfrac{1}{2}\big((u_x - v_y) - i\,(v_x + u_y)\big)\,. \end{aligned}$$

Expanding and simplifying this last expression shows that $q_z = q_w w_z + q_{\overline{w}}\overline{w}_z$. Similarly, we can show that $q_{\overline{z}} = q_w w_{\overline{z}} + q_{\overline{w}}\overline{w}_{\overline{z}}$.

4.23 Example: Let $f(z) = z^2 + 3z\overline{z}$. Find f_z and $f_{\overline{z}}$.

Solution: We solve this using two methods. First, by Example 4.19 and Theorem 4.20, we can calculate $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ using all the same rules that we use to find partial derivatives of real functions of two real variables. We have $f_z = 2z + 3\overline{z}$ and $f_{\overline{z}} = 3z$.

Our second solution is to express f in terms of real variables, and then use Definition 4.17. We have $f(z) = f(x + i\,y) = (x + i\,y)^2 + 3(x + i\,y)(x - i\,y) = (4x^2 + 2y^2) + i\,(2xy)$, and so $Df = \begin{pmatrix} 8x & 4y \\ 2y & 2x \end{pmatrix}$. Thus we have $f_x = 8x + i\,2y$ and $f_y = 4y + i\,2x$, and so

$$\begin{aligned} f_z &= \tfrac{1}{2}(f_x - i\,f_y) = \tfrac{1}{2}(8x + i\,2y - i\,4y + 2x) = 5x - i\,y = 5\tfrac{z+\overline{z}}{2} - i\,\tfrac{z-\overline{z}}{2i} = 2z + 3\overline{z} \\ f_{\overline{z}} &= \tfrac{1}{2}(f_x + i\,f_y) = \tfrac{1}{2}(8x + i\,2y + i\,4y - 2x) = 3x + i\,3y = 3z\,. \end{aligned}$$

4.24 Example: Let $f(z) = \dfrac{(z+\overline{z})z}{2+z\overline{z}}$. Find $f_z(1+i)$ and $f_{\overline{z}}(1+i)$.

Solution: By the Product and Quotient rules, we have

$$\frac{\partial f}{\partial z} = \frac{(z+(z+\overline{z}))(2+z\overline{z}) - (z+\overline{z})z\overline{z}}{(2+z\overline{z})^2}\ , \text{ so } \frac{\partial f}{\partial z}(1+i) = \frac{(3+i)(4)-(2)(2)}{(4)^2} = \frac{2+i}{4}$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{(z)(2+z\overline{z}) - (z+\overline{z})z^2}{(2+z\overline{z})^2}\ , \text{ so } \frac{\partial f}{\partial \overline{z}}(1+i) = \frac{(1+i)(4)-(2)(2i)}{(4)^2} = \frac{1}{4}\,.$$

4.25 Example: Let $w = f(z) = iz + \overline{z}$, let $q = g(w) = w^2 - \overline{w}$, and let $h(z) = g(f(z))$. Find $h_z(1+2i)$ and $h_{\overline{z}}(1+2i)$.

Solution: We provide three solutions to this problem. Our first solution uses the Chain Rule in theorem 4.22. We have

$$\begin{pmatrix} h_z & h_{\overline{z}} \end{pmatrix} = \begin{pmatrix} g_w & g_{\overline{w}} \end{pmatrix}\begin{pmatrix} f_z & f_{\overline{z}} \\ \overline{f}_z & \overline{f}_{\overline{z}} \end{pmatrix} = \begin{pmatrix} 2w & -1 \end{pmatrix}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 2wi-1 & 2w+i \end{pmatrix}.$$

When $z = 1+2i$ we have $w = f(z) = i(1+2i) + (1-2i) = -1-i$ and so we obtain $h_z = 2wi - 1 = 2(-1-i)\,i - 1 = 1-2i$ and $h_{\overline{z}} = 2w + i = 2(-1-i) + i = -2-i$.

Our second solution is to expand the composite $g(f(z))$ so that we can avoid using the Chain Rule. We have $h(z) = g(f(z)) = (iz+\overline{z})^2 - (-i\overline{z}+z) = -z^2 + 2iz\overline{z} + \overline{z}^2 + i\,\overline{z} - z$. Thus we have $h_z = -2z + 2i\overline{z} - 1$ so $h_z(1+2i) = -2(1+2i) + 2i(1-2i) - 1 = 1-2i$ and we have $h_{\overline{z}} = 2iz + 2\overline{z} + i$ so $h_{\overline{z}} = 2i(1+2i) + 2(1-2i) + i = -2-i$.

The third solution is to express f, g and h in terms of real variables. Write $z = x + i\,y$, $w = f(z) = u + i\,v$ and $q = h(z) = s + i\,t$. Then $f(x+i\,y) = i(x+i\,y) + (x - i\,y) = (x-y) + i\,(x-y)$ so $u = x-y$ and $v = x-y$, and $g(u+i\,v) = (u+i\,v)^2 - (u-i\,v) = (u^2-v^2-u) + i\,(2uv-v)$ so $s = u^2 - v^2 - u$ and $t = 2uv + v$. By the Chain Rule for real variables,

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 2u-1 & -2v \\ 2v & 2u+1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2u-2v-1 & -2u+2v+1 \\ 2u+2v+1 & -2u-2v-1 \end{pmatrix}$$

so $h_z = \frac{1}{2}\big((2u-2v-1)+(-2u-2v-1)\big) + \frac{i}{2}\big((2u+2v+1)-(-2u+2v+1)\big) = (-2v-1)+i\,(2u)$ and $h_{\overline{z}} = \frac{1}{2}\big((2u-2v-1)-(-2u-2v-1)\big) + \frac{i}{2}\big((2u+2v+1)+(-2u+2v+1)\big) = 2u+i\,(2v+1)$. When $z = 1+2i$, we have $w = f(z) = i(1+2i) + (1-2i) = -1-i$, so $u = v = -1$ and hence $h_z(1+2i) = (-2v-1) + i\,(2u) = 1-2i$ and $h_{\overline{z}}(1+2i) = 2u + i\,(2v+1) = -2-i$.

4.26 Definition: Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$. We define

$$f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z-a}$$

provided that the limit exists, and in this case we say that f is **holomorphic** at $z = a$ and thet $f'(a)$ is the **derivative** of f at a. Equivalently, we say that f is holomorphic at $z = a$ if there exists a complex number $f'(a)$ such that

$$\lim_{z\to a} \frac{|S(z)|}{|z-a|} = 0\,,$$

where $S(z) = f(z) - \big(f(a) + f'(a)(z-a)\big)$. We say that f is **holomorphic** in U if it is holomorphic at every point in U. When $w = f(z)$ we also write $f' = \frac{df}{dz} = w' = \frac{dw}{dz}$.

4.27 Definition: For $f : U \subseteq \mathbf{C} \to \mathbf{C}$ we define

$$f^\times(a) = \lim_{z \to a} \frac{f(z) - f(a)}{\overline{z} - \overline{a}}$$

provided the limit exists, and if so we say that f is **conjugate-holomorphic** at $z = a$. Equivalently, f is conjugate-holomorphic at a if there exists a complex number $f^\times(a)$ such that $\lim_{z \to a} \frac{|T(z)|}{|z - a|} = 0$ where $T(z) = f(z) - \big(f(a) + f^\times(a)(\overline{z} - \overline{a})\big)$.

4.28 Theorem: *Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$ and let $a \in U$.*
(a) f is holomorphic at a
$\iff$ *f is differentiable at a and $f_{\overline{z}}(a) = 0$.*
$\iff$ *f is differentiable at a and $u_x(a) = v_y(a)$ and $u_y(a) = -v_x(a)$*
$\iff$ *f is differentiable at a and $Df(a)$ is of the form* $Df(a) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.
In this case we have $f'(a) = f_z(a) = u_x + i\,v_x = \alpha + i\,\beta$.
(b) f is conjugate-holomorphic at a
$\iff$ *f is differentiable at a and $f_z(a) = 0$.*
$\iff$ *f is differentiable at a and $u_x(a) = -v_y(a)$ and $u_y(a) = v_x(a)$*
$\iff$ *f is differentiable at a and $Df(a)$ is of the form* $Df(a) = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}$.
In this case we have $f^\times(a) = f_{\overline{z}}(a) = u_x + i\,v_x = \gamma + i\,\delta$.

Proof: This follows immediately from Definition 4.17, Note 4.18 and Definitions 4.26, 4.27.

4.29 Definition: The two differential equations $u_x = v_y$ and $u_y = -v_x$ are called the **Cauchy-Riemann** equations. Note that if $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is $\mathcal{C}^1$ in U, then it is differentiable in U, and if f also satisfies the Cauchy-Riemann equations in U, then it is holomorphic in U.

4.30 Example: Let $f(z) = z^2 + 2|z|^2$. Determine where f is holomorphic and where it is conjugate-holomorphic.

Solution: We have $f(z) = z^2 + 2z\overline{z}$, so $f_z = 2z + 2\overline{z} = 4\mathrm{Re}\,(z)$, and $f_{\overline{z}} = 2z$. Thus f is conjugate-holomorphic when $f_z = 4\mathrm{Re}\,(z) = 0$, that is along the y-axis, and f is holomorphic when $f_{\overline{z}} = z = 0$, that is at the origin.

4.31 Theorem: *If $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is holomorphic (or conjugate-holomorphic) at a then f is continuous at a.*

Proof: We have $\lim_{z \to a} \big(f(z) - f(a)\big) = \lim_{z \to a} \left(\frac{f(z) - f(a)}{z - a} (z - a) \right) = f'(a) \cdot 0 = 0$.

4.32 Theorem: *If $f, g : U \subseteq \mathbf{C} \to \mathbf{C}$ are both be holomorphic at a, then*
(a) $(cf)'(a) = c\,f'(a)$
(b) $(f \pm g)'(a) = f'(a) \pm g'(a)$
(c) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
(d) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$, provided $g(a) \neq 0$.
Similar results hold when f and g are both conjugate-holomorphic.

Proof: This follows immediately from Theorem 4.20.

4.33 Theorem: *(The Chain Rule) Let* $f, h : U \subseteq \mathbf{C} \to V \subseteq \mathbf{C}$ *and let* $g, k : V \to \mathbf{C}$ *with* f *and* g *holomorphic and* h *and* k *conjugate-holomorphic. Then*
(a) $g \circ f$ *is holomorphic with* $(g \circ f)'(z) = g'(f(z))f'(z)$.
(b) $k \circ f$ *is conjugate-holomorphic with* $(h \circ f)^{\times}(z) = h^{\times}(f(z))\overline{f'(z)}$.
(c) $g \circ h$ *is conjugate-homorphic with* $(f \circ h)^{\times}(z) = \underline{f'(h(z))}h^{\times}(z)$.
(d) $k \circ h$ *is holomorphic with* $(k \circ h)'(z) = k^{\times}(h(z))\overline{h^{\times}(z)}$.

Proof: These all follows from the Chain Rule in Theorem 4.22. We prove part (a). Write $w = f(z)$ and $q = g(w)$. Since f and g are holomorphic, we have $\frac{\partial w}{\partial \overline{z}} = 0$ and $\frac{\partial q}{\partial \overline{w}} = 0$. So by the Chain Rule in Theorem 4.22 , we have $\frac{\partial q}{\partial \overline{z}} = \frac{\partial q}{\partial w}\frac{\partial w}{\partial \overline{z}} + \frac{\partial q}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial \overline{z}} = 0$, which shows that $g \circ f$ is holomorphic, and $\frac{\partial q}{\partial z} = \frac{\partial q}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial q}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial z} = \frac{\partial q}{\partial w}\frac{\partial w}{\partial z}$.

4.34 Theorem: *(The Inverse Function Theorem) If* $f : U \subseteq \mathbf{C} \to \mathbf{C}$ *is holomorphic in* U *and* $f'(a) \neq 0$ *then we can make* f *invertible by restricting its domain, and then the inverse function* $g = f^{-1}$ *will be holomorphic near* $f(a)$ *with* $g'(f(z)) = 1/f'(z)$. *A similar result holds when* f *is conjugate-holomorphic.*

Proof: We give a proof which uses the Inverse Function for real functions, under the additional assumption that $f'(z)$ is continuous in U (we shall prove later that when f is holomorphic in U, its derivative is also holomorphic, and hence continuous). Suppose that f is holomorphic in U with $f'(z) = \alpha(z) + i\,\beta(z)$, where α and β are continuous, and that $f'(a) \neq 0$. Then we have $Df = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, and $u_x = v_y = \alpha$ and $u_y = -v_x = \beta$. Since α and β are continuous in U, f is $\mathcal{C}^1$ in U. Also, since $f'(a) = \alpha(a) + i\,\beta(a) \neq 0$ we have $\big|Df(a)\big| = \alpha(a)^2 + \beta(a)^2 \neq 0$, so $Df(a)$ is invertible. By the Inverse Function Theorem 4.8, we can restrict the domain of f so that it becomes invertible and has a $\mathcal{C}^1$ inverse g with $Dg(f(z)) = Df(z)^{-1}$. Note that

$$Dg\big(f(z)\big) = \begin{pmatrix} \alpha(z) & -\beta(z) \\ \beta(z) & \alpha(z) \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\alpha(z)}{\alpha^2(z)+\beta^2(z)} & \frac{\beta(z)}{\alpha^2(z)+\beta^2(z)} \\ \frac{-\beta(z)}{\alpha^2(z)+\beta^2(z)} & \frac{\alpha(z)}{\alpha^2(z)+\beta^2(z)} \end{pmatrix}.$$

Since g is $\mathcal{C}^1$ in U and satisfies the Cauchy-Riemann Equations in U, it is holomorphic in U, and we have

$$g'\big(f(z)\big) = \frac{\alpha(z)}{\alpha^2(z) + \beta^2(z)} + i\,\frac{-\beta(z)}{\alpha^2(z) + \beta^2(z)} = \frac{1}{f'(z)}\,.$$

4.35 Theorem: *The maps* z^n, $n \in \mathbf{Z}$, *the exponential map* e^z, *the trigonometric fuctions and the hyperbolic functions are all holomorphic in their domains. Also, any continuous branch of the logarithm* $\log z$, *(with an open domain) is holomorphic. We have*
(a) $(z^n)' = n\,z^{n-1}$, *where* $n \in \mathbf{Z}$.
(b) $(e^z)' = e^z$.
(c) $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$, $(\tan z)' = \sec^2 z$.
(d) $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$, $(\tanh z)' = \operatorname{sech}^2 z$.
(e) $(\log z)' = \dfrac{1}{z}$ *for any branch of* $\log z$.

Proof: For Part (a), let $f(z) = z^n$, $0 < n \in \mathbf{Z}$. Then we have

$$f'(z) = \lim_{w \to z} \frac{w^n - z^n}{w - z} = \lim_{w \to z} \left(w^{n-1} + w^{n-2} z + \cdots + w\, z^{n-2} + z^{n-1} \right) = n\, z^{n-1} .$$

For $f(z) = z^n$ with $n < 0$, say $n = -m$, we have $f(z) = \frac{1}{z^m}$, so by the Quotient Rule

$$f'(z) = \frac{-m\, z^{m-1}}{z^{2m}} = -m\, z^{-m-1} = n\, z^{n-1} .$$

For Part (b), let $f(z) = e^z$ and write $z = x + i\,y$ and $f(z) = u(z) + i\,v(z)$. Then

$$f(z) = e^{x+i\,y} = e^x \cos y + i\, e^x \sin y \ , \ Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} ,$$

and we see that f is holomorphic in $\mathbf{C}$ with $f'(z) = e^x \cos y + i\, e^x \sin y = e^z$.

For part (c), we only derive the formula for the derivative of $\sin z$, but we do this in two ways. One way is to let $f(z) = \sin z$ and write $z = x + i\,y$ and $f(z) = u(z) + i\,v(z)$. Then we have

$$f(z) = \sin x \cosh y + i\, \cos x \sinh y \ , \ Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \cos x \cosh y & \sin x \sinh y \\ -\sin x \sinh y & \cos x \cosh y \end{pmatrix}$$

and so f is holomorphic in $\mathbf{C}$ and $f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos(z)$.

Another way is to apply Part (b) and the differentiation rules in Theorem 4.32 b) to the definition of $\sin z$. Indeed

$$(\sin z)' = \frac{1}{2i}(e^{iz} - e^{-iz})' = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z .$$

We leave the other formulas in Parts (c) and (d) as an exercise.

For Part (e), let $f(z) = \log z$ and write $z = x + i\,y$ and $f(z) = u(z) + i\,v(z)$. Then since $\log(z) = \ln|z| + i\,\theta(z)$, we have $u(x + i\,y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$ and

$$v(x + i\,y) = \theta(x + i\,y) = \begin{cases} \tan^{-1} \dfrac{y}{x} + 2\pi\, k & , \text{if } x > 0 \\ \cos^{-1} \dfrac{x}{\sqrt{x^2 + y^2}} + 2\pi\, k & , \text{if } y > 0 \\ \tan^{-1} \dfrac{y}{x} + \pi + 2\pi\, k & , \text{if } x < 0 \\ -\cos_{-1} \dfrac{x}{\sqrt{x^2 + y^2}} + 2\pi\, k & , \text{if } y < 0 . \end{cases}$$

Verify that using any one of the four formulas for $v(x + i\,y) = \theta(x + i\,y)$ gives

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

and so f is holomorphic with $f'(z) = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} = \frac{1}{z}$.

4.36 Example: The two above theorems show that elementary complex functions can be differentiated much like the real elementary functions. For example, let $f(z) = (z^3 e^{\sin z})^5$, then $f'(z) = 5(z^3 e^{\sin z})^4 (3z^2 e^{\sin z} + z^3 e^{\sin z} \cos z)$.

4.37 Example: Let $f(z) = z^2 - 2z + 3$. Then $f'(z) = 2z - 2$ and we have $f(2) = 3$ and $f'(2) = 2$. Since $f'(2) \neq 0$, we can restrict the domain of f so that it is invertible. Let g be the inverse function. Find $g'(3)$.

Solution: By the Inverse Function Theorem, we have $g'(3) = \dfrac{1}{f'(2)} = \dfrac{1}{2}$.

4.38 Example: Find a formula for the derivative of one branch of z^w, where $w \in \mathbf{C}$.

Solution: Let $z^w = \exp(w \log z)$, where $\log z$ is a branch of the logarithm. Then

$$(z^w)' = \exp(w \log z) \frac{w}{z} = \frac{w\, z^w}{z}\,.$$

Notice that this is similar to the familiar formula $(z^w) = w\, z^{w-1}$; the familiar formula has the disadvantage that it does not specify which branch of z^{w-1} we should use.

4.39 Example: Let $f(z) = \overline{\sin\left(z^2 + (1+i)\overline{z}\right)}$. Find f_z and $f_{\overline{z}}$.

Solution: We have $f(z) = w(v(u(z)))$, where $u(z) = z^2 + (1+i)\overline{z}$, $v(u) = \sin u$ and $w(v) = \overline{v}$. Note that $u_z = 2z$, $u_{\overline{z}} = (1+i)$, $v_u = \cos u$, $v_{\overline{u}} = 0$, $w_v = 0$ and $w_{\overline{v}} = 1$. By the Chain Rule, $v_z = v_u u_z + v_{\overline{u}} \overline{u}_z = 2z \cos u$ and also $v_{\overline{z}} = v_u u_{\overline{z}} + v_{\overline{u}} \overline{u}_{\overline{z}} = (1+i)\cos u$. Using the Chain Rule again, we have $w_z = w_v v_z + w_{\overline{v}} \overline{v}_z = \overline{v}_z = \overline{v_{\overline{z}}} = (1-i)\overline{\cos u}$ and also $w_{\overline{z}} = w_v v_{\overline{z}} + w_{\overline{v}} \overline{v}_{\overline{z}} = \overline{v}_{\overline{z}} = \overline{v_z} = \overline{2z \cos u}$. Thus $f_z = (1-i)\overline{\cos u} = (1-i)\overline{\cos\left(z^2 + (1+i)\overline{z}\right)}$ and $f_{\overline{z}} = \overline{2z \cos u} = \overline{2z \cos\left(z^2 + (1+i)\overline{z}\right)}$.

An alternate solution is to note that for $z = x + i\,y$ we have

$$e^{\overline{z}} = e^{x - i\,y} = e^x(\cos y - i \sin y) = \overline{e^z}\,,$$

and so from the definition of $\sin z$ we also have $\sin(\overline{z}) = \overline{\sin z}$. Thus $f(z) = \sin\left(\overline{z}^2 + (1-i)z\right)$ and so $f_z(z) = (1-i)\cos\left(\overline{z}^2 + (1-i)z\right)$ and $f_{\overline{z}}(z) = 2\overline{z}\cos\left(\overline{z}^2 + (1-i)z\right)$.

Chapter 5. Conformal Maps

5.1 Note: Later on we shall see that every holomorphic function is $\mathcal{C}^\infty$, which means that all partial derivatives of all orders exist (and are continuous). For this chapter we shall assume that all functions are $\mathcal{C}^2$, which means that all the second order partial derivatives of f (namely u_{xx}, u_{xy}, u_{yx}, u_{yy}, v_{xx}, v_{xy}, v_{yx} and v_{yy}) exist and are continuous. We shall also use the fact that for $\mathcal{C}^2$ functions, we always have $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

5.2 Definition: A map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^n$ is said to **preserve orientation** at $x = a$ when $|Df(a)| > 0$, and it is said to **reverse orientation** at a if $|Df(a)| < 0$.

5.3 Note: Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$. If f is holomorphic at $z = a$ and $f'(a) \neq 0$ then f preserves orientation at a, since $|Df(a)| = \det \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha^2 + \beta^2 > 0$. On the other hand, if f is conjugate-holomorphic at a with $f^\times(a) \neq 0$ then f reverses orientation at a since $|Df(a)| = \det \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix} = -(\gamma^2 + \delta^2) < 0$.

5.4 Definition: A map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^n$ is called an **isometry** when it preserves distance, that is if $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbf{R}^n$. Using some linear algebra, one can show that f is an isometry if and only if f is of the form $f(x) = Ax + b$ for some vector $b \in \mathbf{R}^n$ and some **orthogonal** $n \times n$ matrix A (A is orthogonal means that $A^T A = I$).

5.5 Note: Since the 2×2 orthogonal matrices are the matrices either of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, we see that the isometries in $\mathbf{R}^2$ are the maps f which are either of the form $f(z) = az + b$ or of the form $f(z) = a\overline{z} + b$ for some $a, b \in \mathbf{C}$ with $|a| = 1$.

5.6 Definition: A map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^n$ is called a **similarity** of **scaling factor** $k > 0$ when it scales distances by a factor of k, that is if $|f(x) - f(y)| = k|x - y|$ for all $x, y \in \mathbf{R}^n$. It is not hard to see that f is a similarity of scaling factor k if and only if $\frac{1}{k} f$ is an isometry.

5.7 Note: A map $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is a similarity of scaling factor $k > 0$ if and only if f is either of the form $f(z) = az + b$ or of the form $f(z) = a\overline{z} + b$ for some $a, b \in \mathbf{C}$ with $|a| = k$.

5.8 Note: Let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ be differentiable at a. Given a vector $v \in \mathbf{R}^n$, choose a curve $\alpha(t)$ with $\alpha(0) = a$ and $\alpha'(0) = v$. The image of α under f is the curve $\beta(t) = f\big(\alpha(t)\big)$. By the Chain Rule, we have $\beta'(0) = Df(\alpha(0))\alpha'(0) = Df(a)v$, so we say that f sends the vector v at a to the vector $w = Df(a)v$ at $f(a)$.

5.9 Definition: A map $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is called **conformal** at a when it preserves angles between curves at a, or to be precise, f is conformal at a when

$$\frac{(Df\,v) \cdot (Df\,w)}{|Df\,v|\,|Df\,v|} = \frac{v \cdot w}{|v|\,|w|}$$

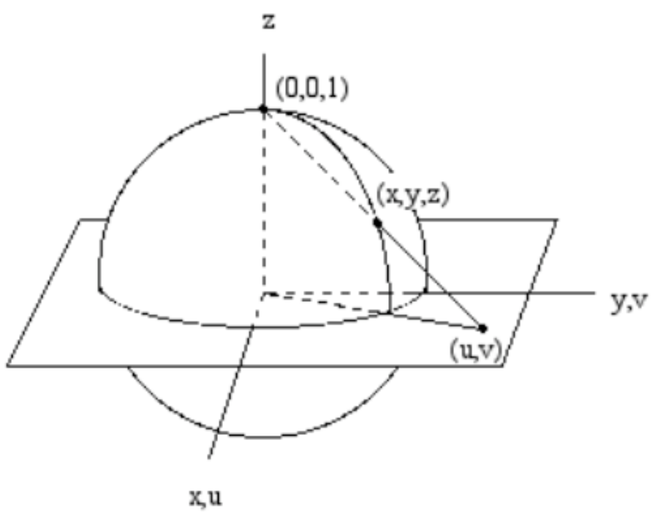
for all vectors $v, w \in \mathbf{R}^n$. We say f is conformal in U when it is conformal at every $a \in U$.

5.10 Note: Using linear algebra, one can show that f is conformal at a if and only if $Df(a)^T Df(a) = k\,I$ for some $k > 0$. We shall show only that the latter implies the former; suppose that $Df^T Df = k\,I$ with $k > 0$. Then

$$(Df\,v)\cdot(Df\,w) = (Df\,v)^T(Df\,w) = v^T Df^T Df\,w = v^T k\,I\,w = k\,v^T w = k\,v\cdot w$$

and in particular $|Df\,v| = \sqrt{(Df\,v)\cdot(Df\,v)} = \sqrt{k}\,|v|$, and similarly $|Df\,w| = \sqrt{k}\,|w|$. It follows that f is conformal; f behaves locally like a similarity of scaling factor $\sqrt{k}$.

5.11 Example: The **steriographic projection** from the unit sphere, with the north pole removed, to the complex plane is the map ϕ which sends the point (x,y,z) on the sphere to the point of intersection (u,v) of the line through (x,y,z) and the plane $z=0$. Find a formula for ϕ and ϕ^{-1}, and show that stereographic projection is conformal.



Solution: The line through $(0,0,1)$ and (x,y,z) is given by $(0,0,1)+t(x,y,z-1)$, $t\in\mathbf{R}$. The point of intersection of this line with the plane $z=0$ occurs when $1+t(z-1)=0$, that is when $t = 1/(1-z)$. The point of intersection is $(0,0,1)+\frac{1}{1-z}(x,y,z-1) = (\frac{x}{1-z},\frac{y}{1-z},0)$, so we have

$$(u,v) = \phi(x,y,z) = \left(\frac{x}{1-z},\frac{y}{1-z}\right).$$

Given (u,v) on the other hand, the line through $(0,0,1)$ and (u,v) is given by $\alpha(t) = (0,0,1)+t(u,v,-1) = (tu,tv,1-t)$. The point of intersection with the unit sphere occurs when $|\alpha(t)|=1$, so we need $(tu)^2+(tv)^2+(1-t)^2 = 1$, that is $t^2u^2+t^2v^2-2t+t^2=0$, or $t\,(tu^2+tv^2+t-2)=0$. The point of intersection occurs when $t=\frac{2}{u^2+v^2+1}$, so we obtain the formula

$$(x,y,z) = \phi^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Now let us show that ϕ^{-1} is conformal. We have

$$D\phi^{-1} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} = \frac{2}{(u^2+v^2+1)^2}\begin{pmatrix} -u^2+v^2+1 & -2uv \\ -2uv & u^2-v^2+1 \\ 2u & 2v \end{pmatrix}$$

and a quick calculation yields

$$(D\phi^{-1})^T(D\phi^{-1}) = \frac{4}{(u^2+v^2+1)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that near the point (u,v), ϕ^{-1} behaves like a similarity of scaling factor $2/(u^2+v^2+1)$.

5.12 Theorem: *Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$.*
(a) f is conformal at a if and only if either f is holomorphic at a with $f'(a) \neq 0$, in which case f preserves orientation, or f is conjugate-holomorhic at a with $f^\times(a) \neq 0$, in which case f reverses orientation.
(b) If U is connected, then f is conformal in U if and only if either f is homorphic in U with $f'(z) \neq 0$ for all $z \in U$, in which case f preserves orientation, or f is conjugate-holomorphic in U with $f^\times(z) \neq 0$ for all $z \in U$, in which case f reverses orientation.

Proof: To prove part (a), note that f is conformal at a if and ony if $Df(a)$ is a positive scalar multiple of an orthogonal matrix. Since the 2×2 orthogonal matrices are the matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, we see that f is conformal if and only if $Df = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ or $Df = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}$ for some α, β or $\gamma, \delta \in \mathbf{R}$ not both equal to zero.

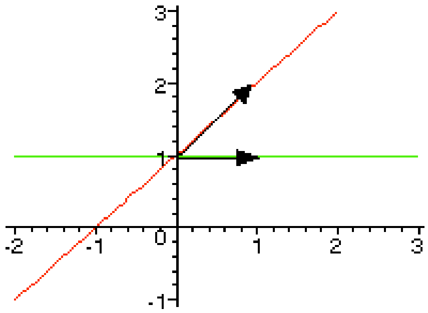
Part (b) involves a subtle point: if f is conformal in U then how do we know that f cannot be holomorphic at some points $a \in U$ and conjugate-holomorphic at other points? It is for this reason that we must assume that U is connected. Since we have assumed that all functions in this chapter are $\mathcal{C}^2$ we know that u_x, u_y, v_x and v_y are all continuous and so $|Df| = u_x v_y - u_y v_x$ is also continuous. At each point $a \in U$ we have $|Df(a)| \neq 0$, so $|Df|$ is a continuous map from U to $\mathbf{R}^*$. Since U is connected, we know that $|Df|(U)$ is also connected and lies in $\mathbf{R}^*$. This implies that either $|Df(a)| > 0$ for all a or $|Df(a)| < 0$ for all a.

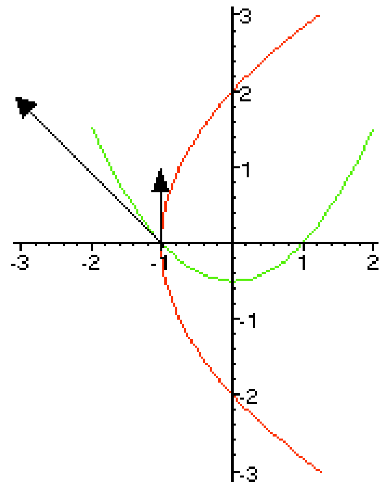
5.13 Note: If $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is holomorphic at a with $f(a) = b$ and $f'(a) = r\,e^{i\,\theta}$, where $r > 0$, then by the definition of the (complex) derivative, for z near a we have $f(z) \cong f(a) + f'(a)(z-a) = b + re^{i\theta}(z-a)$. This shows that locally, f behaves like the following similarity: translate by $-a$, rotate by θ, scale by r, then translate by b.

5.14 Example: Let $f(z) = z^2$. Then f is holomorphic in $\mathbf{C}$ and $f'(z) = 2z$ so $f'(z) \neq 0$ in $\mathbf{C}^*$. Hence $f(z) = z^2$ is conformal in $\mathbf{C}^*$ and preserves orientation. Verify directly that f preserves the oriented angle from $\alpha(t) = i + t$ to $\beta(t) = i + (1+i)\,t$.

Solution: We have $\alpha(0) = \beta(0) = i$, $\alpha'(0) = 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta'(0) = 1 + i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the angle from $\alpha'(0)$ to $\beta'(0)$ is $\frac{\pi}{4}$. The images are $\gamma(t) = f(\alpha(t)) = (i+t)^2 = (t^2-1) + i\,2t$ (this is the parabola $u = \frac{1}{4}v^2 - 1$) and $\delta(t) = f(\beta(t)) = (i + (1+i)t)^2 = -(1+2t) + i\,(2t+2t^2)$ (check that this is the parabola $v = \frac{1}{2}u^2 - \frac{1}{2}$). Note that $\gamma(0) = \delta(0) = -1$, so the two parabolas intersect at -1. We have $\gamma'(t) = 2t + 2i$ so $\gamma'(0) = 2i = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and we have $\delta'(t) = -2 + i\,(4t+2)$ so $\delta'(0) = -2 + 2i = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. So the angle from $\gamma'(0)$ to $\delta'(0)$ is $\frac{\pi}{4}$.

Notice also that α and β meet at i, and we have $f(i) = -1$ and $f'(i) = 2i = 2e^{i\,\pi/2}$. So near $z = i$, f can be approximated as follows: translate by $-i$, rotate by $\frac{\pi}{2}$, scale by 2, then translate by -1. Indeed, this is precisely what happens to the tangent vectors.





5.15 Definition: Let $u : U \subseteq \mathbf{C} \to \mathbf{C}$. The (2 dimensional) **Laplacian** is the differential operator ∇^2 given by

$$\nabla^2 u = u_{xx} + u_{yy} \,.$$

The map u is called **harmonic** in U when it is $\mathcal{C}^2$ and satisfies **Laplace's equation**

$$\nabla^2 u = 0 \,.$$

5.16 Note: There are several functions from physics which satisfy Laplace's equation. Steady state temperature (in a homogeneous medium), electrostatic potential (in a vaccume) and the velocity potential for a steady flow of fluid (irrotational and indecompressible) al satisfy Laplace's equation.

5.17 Example: As an exercise, you should check that the map $u(x, y, z) = \dfrac{-1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the 3 dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$, but that the map $u(x, y) = -\dfrac{1}{\sqrt{x^2 + y^2}}$ does not satisfy the 2 dimensional Laplace equation. The first map u represents the electic potential surrounding a point charge in $\mathbf{R}^3$, but the second map u does not represent the potential which surrounds a long straight wire. On the other hand, you can check that the map $u(x, y) = \ln \sqrt{x^2 + y^2}$ does satisfy the 2 dimensional Laplace equation, and this map u does represent the potential surrounding a wire.

5.18 Theorem: *If $f(z) = u(z) + i\,v(z)$ is holomorphic (or conjugate-holomorphic) in U then u and v are both harmonic functions. When $f = u + i\,v$ is holomorphic, we say that v is the* **harmonic conjugate** *of u.*

Proof: The Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ imply that

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}$$

and likewise $v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$.

5.19 Example: Let $f(z) = e^z$. Verify that u is harmonic, where $u = \mathrm{Re}\,(f)$.

Solution: Since $e^{x+i\,y} = e^x(\cos y + i\,siny)$, we have $u(x + i\,y) = e^x \cos y$. So $u_x = e^x \cos y$ and $u_{xx} = e^x \cos y$, while $u_y = -e^x \sin y$ and $u_{yy} = -e^x \cos y = -u_{xx}$.

5.20 Example: Let $f(z) = z^3$. Verify that u is harmonic, where $u = \text{Re}\,(f)$.

Solution: We have $f(x+i\,y) = (x+i\,y)^3 = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$ so $u = x^3 - 3xy^2$. We have $u_x = 3x^2 - 3y^2$ so $u_{xx} = 6x$ and $u_y = 3x^2 - 6xy$ and so $v_{yy} = -6x = -u_{xx}$.

5.21 Note: There is a partial converse to the above note which says that for certain sets U, for example when U is convex, if u is harmonic in U then there exists a harmonic function v such that the map $f = u + i\,v$ is holomorphic in U. The following example shows how to find v.

5.22 Example: Let $u = 2x^2 + 3xy - 2y^2$. Check that u is harmonic in $\mathbf{C}$, and find a harmonic conjugate v.

Solution: We have $u_x = 4x + 3y$, $u_{xx} = 4$, $u_y = 3x - 4y$ and $u_{yy} = -4 = -u_{xx}$, so u is harmonic. To find v such that $u + i\,v$ is holomorphic, we need to find v such that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. To get $v_y = u_x = 4x+3y$ we must take $v = \int 4x + 3y\;dy = 4xy + \frac{3}{2}y^2 + c(x)$. Then we have $v_x = 4y + c'(x)$. To get $v_x = -u_y = 4y - 3x$ we need to have $c'(x) = -3x$, so we choose $c(x) = -\frac{3}{2}x^2$. In this way we obtain $v = 4xy + \frac{3}{2}(y^2 - x^2)$. The function $f = u + i\,v$ should be holomorphic, and indeed you can check that $f(z) = \left(2 - \frac{3}{2}\,i\right) z^2$.

5.23 Example: A long strip of heat conducting material is modelled by the set

$$S = \{x + i\,y | 0 < y < 1\}\,.$$

Find the steady state temperature $u(x+i\,y)$ at each point in the strip given that the bottom edge is held at a constant temperature of a° and the top edge is held at b°. Describe the **isotherms**, that is the curves of constant temperature.

Solution: We must find a map $u : \overline{S} \to \mathbf{R}$ which is continuous on $\overline{S}$ and harmonic in S such that $u(x,0) = a$ and $u(x,1) = b$ for all x. We can take

$$u(x + i\,y) = a + (b-a)y\,.$$

It is easy to see that u is harmonic, indeed $u_{xx} = u_{yy} = 0$. Also notice that u is the imaginary part of the holomorphic map $f(z) = a\,i + (b-a)z$. The isotherm $u = c$ is the horizontal line $c = a + (b-a)y$, or $y = \frac{c-a}{b-a}$.

5.24 Theorem: *If $u : U \subseteq \mathbf{C} \to \mathbf{R}$ is harmonic and if $f : V \subseteq \mathbf{C} \to U \subseteq \mathbf{C}$ is holomorphic then $u \circ f$ is harmonic.*

Proof: Write $x + i\,y = f(s + i\,t)$, $u = u(x + i\,y)$, and $v = u \circ f$. The chain rule gives

$$v_s = u_x x_s + u_y y_s \qquad v_t = u_x x_t + u_y y_t\,.$$

Using the chain rule and the product rule, we obtain

$$\begin{aligned} v_{ss} &= (u_{xx}x_s + u_{xy}y_s)x_s + u_x x_{ss} + (u_{yx}x_s + u_{yy}y_s)y_s + u_y y_{ss} \\ v_{tt} &= (u_{xx}x_t + u_{xy}y_t)x_t + u_x x_{tt} + (u_{yx}x_t + u_{yy}y_t)y_t + u_y y_{tt} \end{aligned}$$

Adding these, using the fact that $u_{xy} = u_{yx}$ we obtain

$$v_{ss} + v_{tt} = u_{xx}({x_s}^2 + {x_y}^2) + u_{yy}({y_s}^2 + {y_t}^2) + u_{xy}(2x_s y_s + 2x_t y_t) + u_x(x_{ss} + y_{tt}) + u_y(y_{ss} + y_{tt})\,.$$

Since f is holomorphic, the Cauchy-Riemann equations $x_s = y_t$ and $x_t = -y_s$ imply that $({y_s}^2 + {y_t}^2) = ({x_s}^2 + {x_t}^2)$ and that $(2x_s y_s + 2x_t y_t) = 0$ and that x and y are each harmonic so that $(x_{ss} + x_{tt}) = 0$ and $(y_{ss} + y_{tt}) = 0$. So we are left with

$$v_{ss} + v_{tt} = (u_{xx} + u_{yy})({x_s}^2 + {x_y}^2)\,.$$

Finally, since u is harmonic, we have $(u_{xx} + u_{tt}) = 0$ and hence $v_{ss} + v_{tt} = 0$.

5.25 Note: We shall now consider problems of the following kind: given an open set $U \subseteq \mathbf{C}$, find a harmonic function $u : U \to \mathbf{R}$ which satisfies some given condition on the boundary ∂U; this kind of problem is called a **boundary value problem**. We solved an easy boundary value problem in example 5.23, in which the open set was the strip $S = \{x + i\,y | 0 < y < 1\}$. The above theorem allows us to use a solution to one boundary value problem on a set U to obtain a solution to another problem on a set V by mapping the set U to the set V using a holomorphic map.

5.26 Example: The upper half-plane $H = \{x + i\,y | y > 0\}$ is a model for a large heat conducting plate. Find the steady state temperature $v(z)$ at each point in the plate if the temperature along the bottom edge is held at a° for $x > 0$ and b° for $x < 0$. Also, describe the isotherms.

Solution: Notice that we can map the strip $S = \{x + i\,y | 0 < y < 1\}$ (which appeared in the example 5.23) to the half-plane $H = \{x + i\,y | y > 0\}$ using the map $f(z) = e^{\pi z}$. The bottom edge of S is mapped to the positive x-axis, and the top edge of S is mapped to the negative x-axis. To map H back to S we use the inverse map $g(z) = \frac{1}{\pi} \log z$, where $\log z$ is the branch of the logarithm given by $\log z = \ln |z| + i\theta(z)$ where $0 \le \theta(z) \le \pi$.

From example 5.23, the map $u(z) = \operatorname{Im}(a\,i + (b-a)z)$ is harmonic in the strip S with $u = a$ when $y = 0$ and $u = b$ when $y = 1$. To solve our problem in H, we take $v = u \circ g$. To be explicit, we take

$$v(z) = \operatorname{Im}\left(a\,i + \frac{b-a}{\pi}\log z\right) = a + \frac{b-a}{\pi}\theta(z)\,,$$

where $0 \le \theta(z) \le \pi$. The isotherm $u = c$ is the ray $c = a + \frac{b-a}{\pi}\,\theta(z)$ or $\theta(z) = \frac{c-a}{b-a}\,\pi$.

5.27 Example: Find the steady state temperature $u(z)$ inside a circular plate modelled by the disc $U = D(0,1)$, given that the top half of the boundary is held at $a^\circ = 1^\circ$ and the bottom half is held at $b^\circ = 5^\circ$. In particular, find the temperature at the point $\frac{1}{2}i$. Also describe the isotherm $u = 2$.

Solution: The map $f_1(z) = \dfrac{z+1}{2}$ maps the disc $D(0,1)$ to the disc $D(\frac{1}{2}, \frac{1}{2})$, and it sends the top half of the boundary of the first disc to the top half of the boundary of the second. The map $f_2(z) = \dfrac{1}{z}$ maps the disc $D(\frac{1}{2}, \frac{1}{2})$ to the half-plane $H_1 = \{x + i\,y | x > 1\}$, and it maps the top half of the boundary of the disc to the bottom half $\{1 + i\,y | y < 0\}$ of the boundary of H_1. The map $f_3(z) = z - 1$ translates the half-plane H_1 to $H_0 = \{x + i\,y | x > 0\}$. Finally the map $f_4(z) = i\,z$ rotates H_0 to the half-plane $H = \{x + i\,y | y > 0\}$ sending the bottom half of the boundary of H_0 to the right half $\{x > 0\}$ of the boundary of H. So we can use our solution $v(z)$ from the previous example to obtain the solution $u = v \circ f_4 \circ f_3 \circ f_2 \circ f_1$. To be explicit, we have $f_2(f_1(z)) = \dfrac{2}{1+z}$ and $f_3(f_2(f_1(z))) = \left(\dfrac{2}{1+z} - 1\right) = \left(\dfrac{1-z}{1+z}\right)$ and $f_4(f_3(f_2(f_1(z)))) = i\left(\dfrac{1-z}{1+z}\right)$, so our solution is

$$u(z) = a + \frac{b-a}{\pi}\,\theta\left(i\,\frac{1-z}{1+z}\right) = 1 + \frac{4}{\pi}\,\theta\left(i\,\frac{1-z}{1+z}\right)\,,$$

where $0 \le \theta\big(i\,\frac{1-z}{1+z}\big) \le \pi$. Since $\theta(i\,\frac{1-z}{1+z}) = \theta(\frac{1-z}{1+z}) + \frac{\pi}{2}$, we have

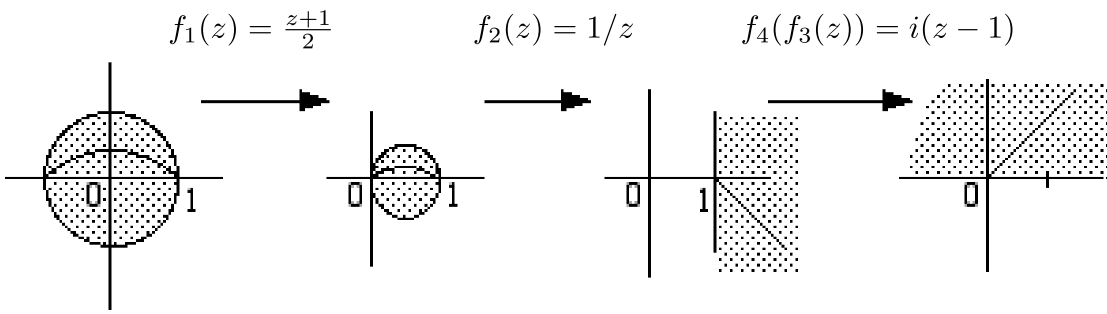
$$u(z) = 3 + \frac{4}{\pi}\,\theta\left(\frac{1-z}{1+z}\right)\,,$$

where $-\frac{\pi}{2} \le \theta\left(\frac{1-z}{1+z}\right) \le \frac{\pi}{2}$. In particular, the temperature at $\frac{1}{2}\,i$ is

$$u(\tfrac{1}{2}\,i) = 3 + \tfrac{4}{\pi}\,\theta\left(\tfrac{1-\frac{1}{2}i}{1+\frac{1}{2}i}\right) = 3 + \tfrac{4}{\pi}\,\theta\left(\tfrac{3-4i}{5}\right) = 3 + \tfrac{4}{\pi}\tan^{-1}\left(-\tfrac{4}{3}\right) \cong 1.82^\circ\,.$$

To find the isotherm $u = 2$, we recall that the corresponding isotherm $v = c = 2$ in example 5.26 was the ray $\theta(z) = \frac{c-a}{b-a}\,\pi = \frac{2-1}{5-1}\,\pi = \frac{\pi}{4}$. This ray is rotated by ${f_4}^{-1}(z) = -iz$ to the ray $\theta(z) = -\frac{\pi}{4}$, then translated by ${f_3}^{-1} = z+1$ to the ray $\theta(z-1) = -\frac{\pi}{4}$, this ray is the portion below the x-axis of the line whose nearest point to the origin is $\frac{1}{2}(1+i)$ and so it is mapped by ${f_2}^{-1}(z) = 1/z$ to the portion above the x-axis of the circle with diameter 0, $\frac{2}{1+i} = 1-i$, that is the circle $|z - \frac{1-i}{2}| = \frac{\sqrt{2}}{2}$, and finally this arc is translated and scaled by the map ${f_1}^{-1}(z) = 2z - 1$ to the portion above the x-axis of the circle $|z+i| = \sqrt{2}$. Thus the isotherm $u = 2$ is the arc $|z+i| = \sqrt{2}$, $z \in D(0,1)$.

We also remark that $\theta(\frac{1-z}{1+z}) = \mathrm{Im}\left(\log(\frac{1-z}{1+z})\right) = -2\,\mathrm{Im}\,(\tanh^{-1} z)$.



5.28 Example: Find the steady state temperature $u(z)$ in the plate shaped like the semi-infinite strip $U = \{x + i\,y | -1 < x < 1, y > 0\}$ given that the temperature along the bottom edge and the right edge is held at $a^\circ = 10^\circ$ and the temperature along the left edge of the boundary is held at $b^\circ = 40^\circ$. Also, find the temperature at $z = i$.

Solution: The map $f_1(z) = \frac{\pi}{2}z$ widens the strip U by a factor of $\frac{\pi}{2}$, and then the map $f_2(z) = \sin z$ sends the strip to the half plane $H = \{x + i\,y | y > 0\}$. The left edge of the boundary of U is mapped to the portion of the real axis with $x < -1$. Lastly, the map $f_3(z) = z + 1$ sends H to itself and it sends the portion of the real axis with $x < -1$ to the portion with $x < 0$. We can again use our solution $v(z)$ from example 5.26 to obtain the solution to this problem. We take $u = u \circ f_3 \circ f_2 \circ f_1$. To be explicit, we have $f_3(f_2(f_1(z))) = 1 + \sin\frac{\pi}{2}z$ and so

$$u(z) = a + \frac{b-a}{\pi}\,\theta\left(1 + \sin(\tfrac{\pi}{2}z)\right) = 10 + \frac{30}{\pi}\,\theta\left(1 + \sin(\tfrac{\pi}{2}z)\right),$$

where $0 \le \theta\left(1 + \sin(\frac{\pi}{2}\,z)\right) \le \pi$. In particular, we have

$$u(i) = 10 + \tfrac{30}{\pi}\,\theta\left(1 + \sin(i\,\tfrac{\pi}{2})\right) = 10 + \tfrac{30}{\pi}\,\theta\left(1 + i\sinh\tfrac{\pi}{2}\right) = 10 + \tfrac{30}{\pi}\tan^{-1}\left(\sinh\tfrac{\pi}{2}\right) \cong 21.1^\circ\,.$$

5.29 Example: Find the steady state temperature $v(z)$ at each point on a plate modelled by the half-plane $H = \{x + i\,y | y > 0\}$ given that the temperature along the boundary is held constant at a° for $x > 1$, b° for $-1 < x < 1$ and at c° for $x < -1$.

Solution: We can use the fact that the sum of two harmonic maps will also be harmonic. We use the solution from example 5.26 to get one harmonic map v_1 in H with $v_1 = a$ along the portion of the x-axis with $x > 1$ and $v_1 = b$ along the portion with $x < 1$, and we get another harmonic map v_2 in H with $v_2 = 0$ along the portion of the x-axis with $x > 1$ and $v_2 = c - b$ along the portion with $x < -1$. Then we add them to get $v = v_1 + v_2$. To be explicit, $v_1(z) = a + \frac{b-a}{\pi}\,\theta(z-1)$ and $v_2(z) = \frac{c-b}{\pi}\,\theta(z+1)$ and so

$$v(z) = a + \frac{b-a}{\pi}\,\theta(z-1) + \frac{c-b}{\pi}\,\theta(z+1)\,,$$

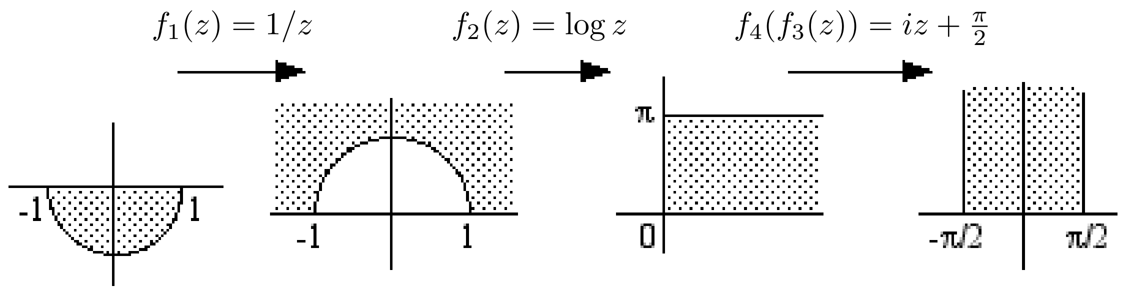
where $0 \le \theta(z-1), \theta(z+1) \le \pi$.

5.30 Example: Find the steady-state temperature $u(z)$ in the semi-circular plate modelled by $U = \{x + i\,y | x^2 + y^2 < 1, y < 0\}$ given that the temperature along the boundary is held constant at $a^\circ = 5^\circ$ when $y = 0$ and $x > 0$, and at $b^\circ = 10^\circ$ when $y = -\sqrt{1-x^2}$ and at $c^\circ = 20^\circ$ when $y = 0$ and $x < 0$. In particular, find the temperature at $z = -\frac{1}{2}\,i$.

Solution: The map $f_1 = 1/z$ sends U to the region V above the x-axis and outside the unit circle $V = \{x + i\,y | x^2 + y^2 > 1, y > 0\}$. Then $f_2(z) = \log z$, the branch of the logarithm with $0 \le \theta \le \pi$, maps V to the semi-infinite strip $W = \{x + i\,y | x > 0, 0 < y < \pi\}$. We rotate the strip by 90° using $f_3(z) = i\,z$ then shift it to the right by $\frac{\pi}{2}$ using the map $f_4(z) = z + \frac{\pi}{2}$ (so that its base is centred at the origin), and then we use the map $f_5(z) = \sin z$ to map the strip to the half-plane $H = \{x + i\,y | y > 0\}$. The portions of the boundary which are to be held constant at a°, b° and c° are mapped to the portions of the x-axis with $x > 1$, $-1 < x < 1$ and $x < -1$ respectively, so we can use our solution $v(z)$ from the previous example. Our solution is $u = v \circ f_5 \circ \ldots \circ f_1$. To be explicit, we have $f_5(f_4(f_3(z))) = \sin(i\,z + \frac{\pi}{2}) = \cos(i\,z) = \cosh z$, and $f_5(f_4(f_3(f_2(z)))) = \cosh(\log z) = \frac{e^{\log z} + e^{-\log z}}{2} = \frac{z + \frac{1}{z}}{2}$, and so $(f_5 \circ \ldots \circ f_1)(z) = \frac{\frac{1}{z} + z}{2} = \frac{1+z^2}{2z}$. Our solution is

$$\begin{aligned} u(z) &= a + \frac{b-a}{\pi}\,\theta\left(\frac{1+z^2}{2z} - 1\right) + \frac{c-b}{\pi}\,\theta\left(\frac{1+z^2}{2z} + 1\right) \\ &= 5 + \frac{5}{\pi}\,\theta\left(\frac{1+z^2}{2z} - 1\right) + \frac{10}{\pi}\,\theta\left(\frac{1+z^2}{2z} + 1\right)\,. \end{aligned}$$

In particular, $u(-i/2) = 5 + \frac{5}{\pi}\theta\big(\frac{3/4}{-i} - 1\big) + \frac{10}{\pi}\,\theta\big(\frac{3/4}{-i} + 1\big) = 5 + \frac{5}{\pi}\,\theta\big(-1 + i\,\frac{3}{4}\big) + \frac{10}{\pi}\,\theta(1 + i\,\frac{3}{4}) = 5 + \frac{5}{\pi}\big(\pi - \tan^{-1}\frac{3}{4}\big) + \frac{10}{\pi}\tan^{-1}\frac{3}{4} = 10 + \frac{5}{\pi}\tan^{-1}\frac{3}{4} \cong 11.0^\circ$.

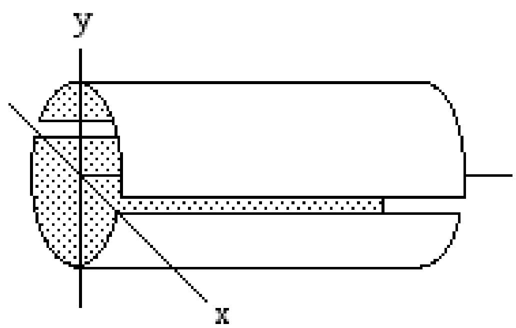


5.31 Note: All of the above examples can be re-worded so that they are asking us to find the electrostatic potential in a certain region given that the voltage along the boundary is held constant. If u is the electrostatic potential in a region, then the **electric field** E is defined by

$$E = -\nabla u\,.$$

If f is holomorphic and $u = \text{Re}\,(f)$ and $v = \text{Im}\,(f)$, then we have $\nabla u = u_x + i\,u_y = \overline{f_z}$ and $\nabla v = v_x + i\,v_y = -u_y + i\,u_x = i(u_x + i\,u_y) = i\,\nabla u = i\,\overline{f_z}$.

5.32 Example: Find the electostatic potential and the electric field at each point inside a long hollow metal cylinder, with unit radius, made up of two semi-cylindrical pieces separated by thin strips of insulating material, with one piece held at a potential of 1 Volt, and the other at 5 Volts. In particular, find the electrostatic potential and the electric field at points along the centre of the cylinder.



Solution: The cross-section of the cylinder is modelled by the unit disc $U = D(0,1)$. As in example 5.27, the electric potential is $u(z) = 3 + \frac{4}{\pi}\,\theta\left(\frac{1-z}{1+z}\right)$. Note that $u = \text{Re}\,(f)$, where $f(z) = 3 - \frac{4}{\pi}\,i\,\log\left(\frac{1-z}{1+z}\right)$. The electric field is $E = -\nabla u = -\overline{f_z} = \overline{\frac{4}{\pi}\,i\,\frac{1+z}{1-z}\,\frac{-2}{(1+z)^2}} = \frac{8\,i}{\pi(1-\overline{z}^2)}$. In particular, we have $u(0) = 3$ and $E(0) = \frac{8}{\pi}\,i$.

5.33 Example: Find all solutions $v(z)$ to Laplace's equation in $\mathbf{C}^*$ such that $v(r\,e^{i\,\theta}) = f(r)$ for some function f (the solution will model the electrostatic potential at each point around a long charged rod).

Solution: The exponential function maps $\mathbf{C}$ onto $\mathbf{C}^*$. If $v(z)$ is harmonic in $\mathbf{C}^*$ then $u(z) = v(e^z)$ will be harmonic in $\mathbf{C}$. If v is of the form $v(r\,e^{i\,\theta}) = f(r)$ then we have $u(x + i\,y) = v(e^x e^{iy}) = f(e^x)$. Since u is independent of y, Laplace's equation becomes $u_{xx} = 0$, and the only solutions are of the form $u(x + i\,y) = ax + b = \text{Re}\,(az + b)$ for some $a, b \in \mathbf{R}$. Thus we have $v(z) = u(\log z) = \text{Re}\,(a\log z + b) = a\ln|z| + b$.

5.34 Example: Find the electrostatic potential $v(z)$ and the electric field $E(z)$ at each point inside a long grounded cylinder, of unit radius, which encloses a charged wire centred inside the cylinder.

Solution: We look for a harmonic map $v(z)$ defined on the punctured disc $U = D^*(0,1)$ with $v(z) = 0$ when $|z| = 1$. From the previous example, we can take $v(z) = a\ln|z|$. The constant a depends on the charge per unit length and on the choice of units. In fact

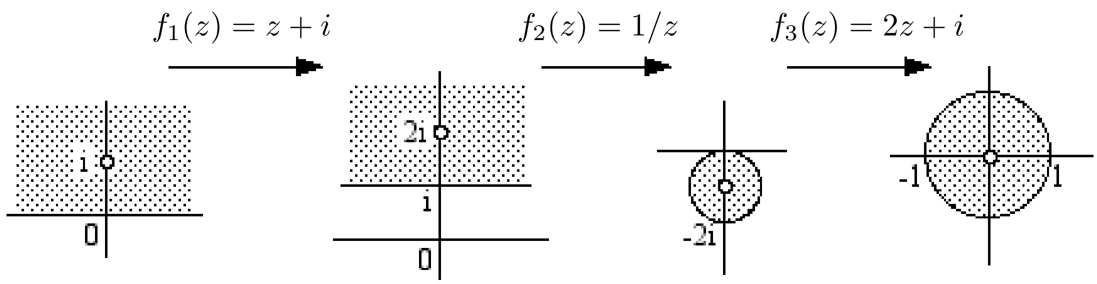
$$v(z) = -2kq\ln|z|\,,$$

where q is the charge on the rod in coulombs per meter and $k \cong 9 \times 10^9\ \frac{N\,m^2}{C^2}$. Since $v = \text{Re}\,(f)$, where $f(z) = -2kq\log(z)$, we have $E(z) = -\overline{f_z} = 2kq/\overline{z}$.

5.35 Example: A charged wire at $x=0, y=1$ lies inside the region in space given by $y>0$, and the boundary of the region is grounded. Find the potential $u(z)$ at each point in the region and around the wire.

Solution: Let U be the punctured half-plane $U=\{z|\text{Im}\, z>0, z\neq i\}$. The map $f_1(z)=z+i$ maps U to the set $V=\{z|\text{Im}\, z>1, z\neq 2i\}$, the map $f_2(z)=1/z$ maps V to the punctured disc $W=D^*(-\frac{1}{2}i,\frac{1}{2})$, and the map $f_3(z)=2z+i$ maps W to the punctured disc $D^*(0,1)$. So our solution is $u=v\circ f_3\circ f_2\circ f_1$ where $v(z)=-2kq\ln|z|$ is the solution from the previous example. Check that

$$u(z)=-2kq\ln\left|\frac{z-i}{z+i}\right|.$$



5.36 Note: The velocity field F of a flow (of perfect fluid) and the velocity potential v are related (like the electric field and electric potential) by

$$F=-\nabla v.$$

5.37 Example: Find the velocity potential $v(z)$ of the constant flow with velocity field $F(x+i\,y)=c$ in the upper half plane $H=\{x+i\,y|y>0\}$.

Solution: We must have $F=-\nabla v$ so we need $c=-(v_x+i\,v_y)$, that is $v_x=-c$ and $v_y=0$. Since $v_y=0$, v is independent of y, and since $v_x=-c$ we have

$$v=-c\,x=\text{Re}\,(-c\,z).$$

We could add a constant to this solution.

5.38 Example: Use the previous example to find the velocity potential for the region $U=\{x+i\,y|x^2+y^2>1, y>0\}$ given that as $z\to\infty$ the flow tends to the constant flow $F=k$. Also, determine the speed of the flow near $z=i$, that is, at the top of the bump.

Solution: As in example 5.32, the map $f(z)=\cosh(\log z)=\frac{1}{2}(z+1/z)$ sends the region U to the upper half-plane $H=\{x+i\,y|y>0\}$. We use the potential v from the previous example, and we take $u(z)=v(f(z))=\text{Re}\, g(z)$, where $g(z)=-\frac{c}{2}(z+1/z)$. The velocity field is $F=-\overline{g_z}=\frac{c}{2}(1-1/\overline{z}^2)$. As $z\to\infty$ we have $F(z)\to c/2$ so we must take $c=2k$. Thus our solution is

$$v(z)=\text{Re}\big(-k(z+z^{-1})\big)\quad,\quad F(z)=k(1-1/\overline{z}^2).$$

We have $F(i)=2k$, so the velocity at the top of the bump is twice the velocity at ∞.

Chapter 6. Integration

6.1 Definition: Let I be an interval in $\mathbf{R}$ (I could be open, closed or half-open). Say $I = \langle a, b \rangle$ where $a, b \in \mathbf{R} \cup \{\pm\infty\}$ with $a < b$ and where $\langle$ and $\rangle$ denote either open or closed brackets depending on whether a and b are open or closed endpoints of I. A function $f : I \to \mathbf{C}$ is called **piecewise continuous** when there exist points $s_i \in \mathbf{R} \cup \{\pm\infty\}$ with $a = s_0 < s_1 < \cdots < s_n = b$ and there exist continuous functions

$$g_1 : \langle s_0, s_1] \to \mathbf{C}\ ,\ g_i : [s_{i-1} s_i] \to \mathbf{C} \text{ for } 1 < i < n\ , \text{ and } g_n : [s_{n-1}, s_n \rangle \to \mathbf{C}$$

such that $f(t) = g(t)$ for all $t \in (s_{i-1}, s_i)$. A function $f : I \to \mathbf{C}$ is called **piecewise $\mathcal{C}^1$** when f is differentiable and f' is piecewise continuous on I.

6.2 Definition: Let $f : I \subseteq \mathbf{R} \to \mathbf{C}$ be piecewise continuous, where I is an interval, let $u(t) = \operatorname{Re} f(t)$ and $v(t) = \operatorname{Im} f(t)$, and let $t_1, t_2 \in I$. We define the **integral** of f from t_1 to t_2 to be

$$\int_{t_1}^{t_2} f = \int_{t_1}^{t_2} f(t)\, dt := \int_{t_1}^{t_2} u(t)\, dt \ + i \int_{t_1}^{t_2} v(t)\, dt\, .$$

6.3 Remark: It is also possible to define the definite integral as a limit of Riemann sums, but we shall not do this here.

6.4 Example: Let $f(t) = e^{i\,t}$ for $t \in \mathbf{R}$. For $\theta \in \mathbf{R}$, find $\displaystyle\int_0^\theta \alpha(t)\, dt$.

Solution: We have

$$\begin{aligned} \int_0^\theta f(t)\, dt &= \int_0^\theta \big(\cos t + i\, \sin t\big) dt = \int_0^\theta \cos t\, dt + i \int_0^\theta \sin t\, dt \\ &= \Big[\sin t\Big]_0^\theta + i \Big[-\cos t\Big]_0^\theta = \sin\theta + i\,(1 - \cos\theta) = i - i\, e^{i\,\theta}\, . \end{aligned}$$

Note that as θ varies, this traces out the circle centered at i of radius 1.

6.5 Theorem: *(Linearity) If $f, g : I \subseteq \mathbf{R} \to \mathbf{C}$ are piecewise continuous, where I is an interval with $t_1, t_2 \in I$, and $c \in \mathbf{C}$ then*

$$\int_{t_1}^{t_2} c\, f = c \int_{t_1}^{t_2} f \ \text{ and } \int_{t_1}^{t_2} (f + g) = \int_{t_1}^{t_2} f + \int_{t_1}^{t_2} g\, .$$

Proof: This follows from Linearity for real-valued functions.

6.6 Theorem: *(Decomposition) If $f : I \subseteq \mathbf{R} \to \mathbf{C}$ is piecewise continuous, where I is an interval with $t_1, t_2, t_3, \cdots, t_n \in I$ then*

$$\int_{t_1}^{t_n} f = \int_{t_1}^{t_2} f + \int_{t_2}^{t_3} f + \cdots + \int_{t_{n-1}}^{t_n} f\, .$$

Proof: This follows from the Decomposition Theorem for real-valued functions.

6.7 Theorem: *(Change of Parameter) Let $s : I \subseteq \mathbf{R} \to \mathbf{R}$ be piecewise $\mathcal{C}^1$ (this means that $s = s(t)$ is continuous and $s'(t)$ is piecewise continuous) where I is an interval, let $J = s(I)$, which is also an interval, let $f : J \subseteq \mathbf{R} \to \mathbf{R}$ be piecewise continuous, and let $t_1, t_2 \in I$. Then*

$$\int_{t_1}^{t_2} f(s(t))s'(t)\,dt = \int_{s(t_1)}^{s(t_2)} f(s)\,ds\,.$$

Proof: This follows from the Change of Variables Theorem for real-valued functions.

6.8 Theorem: *(Estimation) Let $f : [t_1, t_2] \subset \mathbf{R} \to \mathbf{C}$ be piecewise continuous. Then*

$$\left|\int_{t_1}^{t_2} f(t)\,dt\right| \leq \int_{t_1}^{t_2} |f(t)|\,dt\,.$$

Proof: Write $\displaystyle\int_{t_1}^{t_2} f(t)\,dt$ in polar coordinates as $\displaystyle\int_{t_1}^{t_2} f(t)\,dt = \left|\int_{t_1}^{t_2} f(t)\,dt\right| e^{i\,\theta}$. Then

$$\left|\int_{t_1}^{t_2} f(t)\,dt\right| = e^{-i\,\theta}\int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} e^{-i\,\theta} f(t)\,dt = \left|\text{Re}\int_{t_1}^{t_2}\left(e^{-i\theta}f(t)\right)dt\right|\,,$$

where the last equality holds since for $r \geq 0$ we have $r = |\text{Re}\,(r)|$. We can then use the Estimation Theorem for real-valued functions to obtain

$$\left|\text{Re}\int_{t_1}^{t_2}\left(e^{-i\theta}f(t)\right)dt\right| = \left|\int_{t_1}^{t_2}\text{Re}\left(e^{-i\theta}f(t)\right)dt\right| \leq \int_{t_1}^{t_2}\left|\text{Re}\left(e^{i\,\theta}f(t)\right)\right|\,dt \leq \int_{t_1}^{t_2}\left|f(t)\right|dt\,,$$

since $|\text{Re}\,(e^{i\,\theta}f(t))| \leq |e^{i\theta}f(t)| = |f(t)|$.

6.9 Theorem: *(The Fundamental Theorem of Calculus) Let $f, g : I \subseteq \mathbf{R} \to \mathbf{C}$ where I is an interval with $t_1, t_2 \in I$, f is piecewise continuous, g is differentiable, and $g' = f$. Then*

$$\int_{t_1}^{t_2} f(t)\,dt = g(t_2) - g(t_1)\,.$$

Proof: Let $u(t) = \text{Re}\,g(t)$ and $v(t) = \text{Im}\,g(t)$ so that $g(t) = u(t) + i\,v(t)$ and $f(t) = g'(t) = u'(t) + i\,v'(t)$. Then, using the Fundamental Theorem of Calculus for real-valued functions, we have

$$\begin{aligned}\int_{t_1}^{t_2} f(t)\,dt &= \int_{t_1}^{t_2} u'(t) + i\,v'(t)\;dt = \int_{t_1}^{t_2} u'(t)\,dt\; + i\int_{t_1}^{t_2} v'(t)\,dt \\ &= \big(u(t_2) - u(t_1)\big) + i\,\big(v(t_2) - v(t_1)\big) = g(t_2) - g(t_1)\,.\end{aligned}$$

6.10 Example: With the help of the Fundamental Theorem of Calculus, we can find the integral of example 6.4 as follows

$$\int_0^\theta e^{i\,t}\,dt = \Big[-i\,e^{i\,t}\Big]_{t=0}^{\theta} = -i\,e^{i\,\theta} + i\,.$$

6.11 Definition: Let $a, b \in U \subseteq \mathbf{C}$. A **path** (or **curve**) from a to b in U is a piecewise $\mathcal{C}^1$ function $\alpha : [t_1, t_2] \subseteq \mathbf{R} \to U$ with $\alpha(t_1) = a$ and $\alpha(t_2) = b$. In the case that $a = b$, we say that α is a **loop** at a in U.

6.12 Example: For $a, b \in \mathbf{C}$, the path

$$\alpha(t) = a + (b - a)\,t \ , \text{ for } 0 \leq t \leq 1$$

traces the line segment from a to b.

6.13 Example: For $a \in \mathbf{C}$ and $0 < r \in \mathbf{R}$, the loop

$$\alpha(t) = a + r\,e^{i\,t} \ , \text{ for } 0 \leq t \leq 2\pi$$

traces the circle of radius r centered at a.

6.14 Definition: The **arclength** of a path $\alpha : [t_1, t_2] \to U \subseteq \mathbf{C}$ is given by

$$L(\alpha) = \int_{t_1}^{t_2} |\alpha'(t)|\,dt\,.$$

We remark that the arclength $L(\alpha)$ exists and is finite, since α' is piecewise continuous.

6.15 Example: Find the arclength of the path $\alpha(t) = t^2 + i\,t^3$, $0 \leq t \leq 1$.

Solution: We have

$$\begin{aligned} L(\alpha) &= \int_0^1 |\alpha'(t)|\,dt = \int_0^1 |2t + i\,3t^2|\,dt = \int_0^1 \sqrt{4t^2 + 9t^4}\,dt \\ &= \int_0^1 t\sqrt{4 + 9t^2}\,dt = \int_4^{13} \tfrac{1}{18}\sqrt{u}\,du \\ &= \Big[\tfrac{1}{27}u\sqrt{u}\Big]_4^{13} = \tfrac{1}{27}\big(13\sqrt{13} - 8\big)\,. \end{aligned}$$

6.16 Definition: Given a path $\alpha : [t_1, t_2] \to U \subseteq \mathbf{C}$ and a continuous function $f : U \to \mathbf{C}$ we define the **path integral** of f along α to be

$$\int_\alpha f = \int_\alpha f(z)\,dz := \int_{t_1}^{t_2} f(\alpha(t))\alpha'(t)\,dt\,.$$

6.17 Remark: It is possible to define the path integral as a limit of Riemann sums, but we shall not do this here.

6.18 Remark: The complex path integral is related to real path integrals in the following way. Write $z = \alpha(t) = x(t) + i\,y(t)$ and $f(z) = u(z) + i\,v(z)$ with $x, y, u, v \in \mathbf{R}$. Then

$$\begin{aligned} \int_\alpha f(z)\,dz &= \int_{t_1}^{t_2} f(\alpha(t))\alpha'(t)\,dt = \int_{t_1}^{t_2} \big(u(\alpha(t)) + i\,v(\alpha(t))\big)\big(x'(t) + i\,y'(t)\big)\,dt \\ &= \int_{t_1}^{t_2} u(\alpha(t))x'(t) - v(\alpha(t))y'(t)\ dt + i\int_{t_1}^{t_2} v(\alpha(t))x'(t) + u(\alpha(t))y'(t)\ dt \\ &= \int_\alpha (u\,dx - v\,dy) + i\int_\alpha (v\,dx + u\,dy)\,. \end{aligned}$$

This can easily be remembered by defining $dz = dx + i\,dy$ and then writing

$$\int_\alpha f(z)\,dz = \int_\alpha (u + i\,v)(dx + i\,dy) = \int_\alpha (u\,dx - v\,dy) + i\int_\alpha (v\,dx + u\,dy)\,.$$

In a similar way we could define the path integral $\displaystyle\int_\alpha f(z)\,d\overline{z}$, where $d\overline{z} = dx - i\,dy$

6.19 Example: Find $\displaystyle\int_\alpha c$ where $a,b,c\in\mathbf{C}$ and $\alpha(t)=a+(b-a)\,t$ for $0\le t\le 1$.

Solution: We have

$$\int_\alpha c\,dz=\int_0^1 c\,\alpha'(t)\,dt=\int_0^1 c\,(b-a)\,dt=\Big[c\,(b-a)\,t\Big]_{t=0}^1=c\,(b-a)\,.$$

6.20 Example: Find $\displaystyle\int_\alpha z^2\,dz$ where $\alpha(t)=2+(-1+i)\,t$ for $0\le t\le 1$.

Solution: We have

$$\begin{aligned}\int_\alpha z^2\,dz&=\int_0^1 \big(\alpha(t)\big)^2\alpha'(t)\,dt=\int_0^1 \big(2+(-1+i)\,t\big)^2(-1+i)\;dt\\&=\Big[\tfrac{1}{3}\big(2+(-1+i)\,t\big)^3\Big]_0^1=\tfrac{1}{3}\big((1+i)^3-2^3\big)=\tfrac{1}{3}(-2+2i-8)=-\tfrac{10}{3}+\tfrac{2}{3}\,i\,.\end{aligned}$$

6.21 Theorem: *(Linearity) Let α be a path in $U\subseteq\mathbf{C}$ let $f,g:U\to\mathbf{C}$ be continuous, and let $c\in\mathbf{C}$. Then*

$$\int_\alpha c\,f=c\int_\alpha f\ \text{ and }\ \int_\alpha (f+g)=\int_\alpha f+\int_\alpha g\,.$$

Proof: This follows from the Linearity Theorem 6.5.

6.22 Theorem: *(Decomposition) Let $\alpha:[t_1,t_2]\to U\subseteq\mathbf{C}$ be a path, let n be a positive integer, let $t_1=s_0<s_1<s_2<\cdots<s_n=t_1$, for each $i=1,2,\cdots,n$, let α_i be the restriction of α to the interval $[s_{i-1},s_i]$, and let $f:U\to\mathbf{C}$ be continuous. Then*

$$\int_\alpha f=\int_{\alpha_1} f+\int_{\alpha_2} f+\cdots+\int_{\alpha_n} f\,.$$

Proof: This follows from the Decomposition Theorem 6.6.

6.23 Definition: For a path $\alpha:[t_1,t_2]\to U\subset\mathbf{C}$, we define the **inverse path** α^{-1} by

$$\alpha^{-1}(t)=\alpha(t_1+t_2-t)\ ,\ \text{for } t_1\le t\le t_2\,.$$

Note that α^{-1} has the same image as α, but it traces this image in the opposite direction with $\alpha^{-1}(t_1)=\alpha(t_2)$ and $\alpha^{-1}(t_2)=\alpha(t_1)$.

6.24 Theorem: *(Change of Direction) Let $\alpha:[t_1,t_2]\to U\subseteq\mathbf{C}$ be a path, and let $f:U\to\mathbf{C}$ be continuous. Then*

$$\int_{\alpha^{-1}} f=-\int_\alpha f\,.$$

Proof: This theorem is a special case of the following more general theorem.

6.25 Theorem: *(Change of Parameter) Let $s : [t_1, t_2] \subset \mathbf{R} \to [s_1, s_2] \subset \mathbf{R}$ be invertible and piecewise $\mathcal{C}^1$. Note that s must be monotonic, and if s is increasing then we have $s_1 = s(t_1)$ and $s_2 = s(t_2)$, while if s is decreasing then we have $s_1 = s(t_2)$ and $s_2 = s(t_1)$. Let $\alpha : [s_1, s_2] \to U \subseteq \mathbf{C}$ be a path, let $\beta(t) = \alpha(s(t))$ for $t_1 \le t \le t_2$, and let $f : U \to \mathbf{C}$ be continuous. Then*

$$\int_\beta f(z)\,dz = \pm \int_\alpha f(z)\,dz\,,$$

where we use $+$ when s is increasing and we use $-$ when s is decreasing.

Proof: Since $\beta(t) = \alpha(s(t))$, the Chain Rule gives $\beta'(t) = \alpha'(s(t))s'(t)$, and so

$$\int_\beta f = \int_{t_1}^{t_2} f(\beta(t))\beta'(t)\,dt = \int_{t_1}^{t_2} f(\alpha(s(t)))\alpha'(s(t))s'(t)\,dt = \int_{s(t_1)}^{s(t_2)} f(\alpha(s))\alpha'(s)\,ds$$

by the Change of Parameter Theorem 6.7. When s is increasing, the integral on the right is equal to $\displaystyle\int_{s_1}^{s_2} f(\alpha(s))\alpha'(s)\,ds = \int_\alpha f$, but when s is decreasing, the integral on the right is equal to $\displaystyle\int_{s_2}^{s_1} f(\alpha(s))\alpha'(s)\,ds = -\int_{s_1}^{s_2} f(\alpha(s))\alpha'(s)\,ds = -\int_\alpha f$.

6.26 Remark: We use the Decomposition Theorem, the Change of Direction Theorem and the Change of Parameter Theorem implicitly when we join two or more paths together to form a single path or loop. For example, let $a, b \in U \subseteq \mathbf{C}$, let $f : U \to \mathbf{C}$ be continuous, let α and β be two paths from a to b in U, and let γ be a loop which follows α then β^{-1}. Then no matter how we choose to parametrize γ, we have

$$\int_\gamma f = \int_\alpha f - \int_\beta f\,.$$

This fact makes it unnecessary to find an explicit formula for $\gamma(t)$, such as the following: if $\alpha : [t_1, t_2] \to U$ and $\beta : [t_3, t_4] \to U$, then one specific parametrization for a loop γ which follows α then β^{-1} is given by

$$\gamma(t) = \begin{cases} \alpha(t) & , \quad t_1 \le t \le t_2 \\ \beta(t_2 + t_4 - t)\ , & \quad t_2 \le t \le t_2 + t_4 - t_3\,. \end{cases}$$

6.27 Theorem: *(Estimation) Let $\alpha : [t_1, t_2] \to U \subseteq \mathbf{C}$ be a path, let $L = L(\alpha)$, and let $M = \max\limits_{z=\alpha(t)} \big|f(z)\big|$. Then*

$$\Big|\int_\alpha f(z)\,dz\Big| \le \int_{t_1}^{t_2} \big|f(\alpha(t))\alpha'(t)\big|\,dt \le ML\,.$$

Proof: This follows from the Estimation Theorem 6.8.

6.28 Definition: Let $f, g : U \subseteq \mathbf{C} \to \mathbf{C}$. If $g'(z) = f(z)$ for all $z \in U$ then we write

$$g = \int f$$

and we say that g is an **antiderivative** of f in U.

6.29 Note: Note that since complex functions have the same derivative formulas as real functions, they also have the same antiderivative formulas. For example, we can use Integration by Parts or the Substitution Rule to find an antiderivative.

6.30 Example: Find an antiderivative for $f(z) = z^3\, e^{z^2}$.

Solution: Make the substitution $w = z^2$, $dw = 2z\,dz$, then use Integration by Parts with $u = \frac{1}{2}w$, $du = \frac{1}{2}dw$, $v = e^w$ and $dv = e^w\,dw$ to get

$$\begin{aligned}\int z^3\, e^{z^2}\, dz &= \int \tfrac{1}{2} w\, e^w\, dw = \tfrac{1}{2} w\, e^w - \int \tfrac{1}{2} e^w\, dw = \tfrac{1}{2} w\, e^w - \tfrac{1}{2} e^w \\ &= \tfrac{1}{2}\,(w-1)\, e^w = \tfrac{1}{2}\big(z^2-1\big)\, e^{z^2}\,.\end{aligned}$$

6.31 Note: Let $U \subseteq \mathbf{C}$ be a non-empty connected open set. If $V \subset U$ is non-empty and open, and $U \setminus V$ is also open, then we must have $V = U$ since otherwise U would be separated by the open sets V and $U \setminus V$.

6.32 Theorem: *Let $U \subseteq \mathbf{C}$ be a non-empty connected open set. Let $f, g : U \to \mathbf{C}$ be holomorphic with $f' = g'$ in U. Then there is a constant $c \in \mathbf{C}$ such that $f = g + c$ in U.*

Proof: Let $a \in U$, let $c = f(a) - g(a)$, let $h = f - g$, and let

$$V = \big\{z \in U \big| f(z) = g(z) + c\big\} = \big\{z \in U \big| h(z) = c\big\}\,.$$

We must show that $V = U$. By the above note, we can do this by showing that V is non-empty and that both V and $U \setminus V$ are open. The set V is clearly non-empty since $a \in V$. To see that $V \setminus U$ is open, note that $V = h^{-1}(c)$ so we have $U \setminus V = h^{-1}\big(\mathbf{C} \setminus \{c\}\big)$. Since h is continuous and $\mathbf{C} \setminus \{c\}$ is open, the set $h^{-1}\big(\mathbf{C} \setminus \{c\}\big)$ is open by Theorem 3.29. It remains to show that V is open.

Let $w \in V$, so we have $h(w) = c$, and choose $r > 0$ so that $D(w, r) \subseteq U$. We claim that $D(w, r) \subseteq V$. Let $z \in D(w, r)$. We must show that $f(z) = g(z) + c$, or equivalently that $h(z) = c$. Note that since $f' = g'$ and $h = f - g$ we have $h' = 0$. Let p be the point with $\mathrm{Re}\,(p) = \mathrm{Re}\,(z)$ and $\mathrm{Im}\,(p) = \mathrm{Im}\,(w)$. Let $u = \mathrm{Re}\,(h)$ and $v = \mathrm{Im}\,(h)$ so that $h = u + i\,v$. On the horizontal line through w, given by $\alpha(t) = w + t$ for $t \in \mathbf{R}$, we have

$$\frac{d}{dt} u\big(\alpha(t)\big) + i\, \frac{d}{dt} v\big(\alpha(t)\big) = \frac{d}{dt} h\big(\alpha(t)\big) = h'\big(\alpha(t)\big)\alpha'(t) = 0.$$

Since $\frac{d}{dt} u\big(\alpha(t)\big) = 0$ and $\frac{d}{dt} v\big(\alpha(t)\big) = 0$, it follows that the real-valued functions $u\big(\alpha(t)\big)$ and $v\big(\alpha(t)\big)$ are both constant and so $h\big(\alpha(t)\big)$ is constant. In particular, $h(p) = h(w) = c$. A similar argument involving the vertical line through p, which is given by $\beta(t) = p + it$ for $t \in \mathbf{R}$, shows that $h(z) = h(p) = c$.

6.33 Example: Let $U_\alpha = \big\{r\, e^{i\,\theta} \big| r > 0, \alpha < \theta < \alpha + 2\pi\big\}$, and let $f(z) = 1/z$. Then the antiderivatives of f in U_α are the maps of the form $g(z) = \log z + c$ where $\log z = |z| + i\,\theta(z)$ with $\alpha < \theta(z) < \alpha + 2\pi$. However, $f(z)$ does not have an antiderivative in $\mathbf{C}^*$ because none of the maps $g(z)$ can be extended continuously to $\mathbf{C}^*$.

6.34 Theorem: *(The Fundamental Theorem of Calculus) Let $\alpha : [t_1, t_2] \to U \subseteq \mathbf{C}$ be a path in U, and let $f, g : U \to \mathbf{C}$ with f continuous and g holomorphic with $g' = f$ in U. Then*

$$\int_\alpha f = \Big[g(z)\Big]_{\alpha(t_1)}^{\alpha(t_2)} = g(\alpha(t_2)) - g(\alpha(t_1)) \,.$$

In particular, if α is a loop then

$$\int_\alpha f = 0 \,.$$

Proof: Let $h(t) = g(\alpha(t))$. By the Chain Rule, $h'(t) = Dg(\alpha(t))\alpha'(t) = g'(\alpha(t))\alpha'(t)$, and so by the Fundamental Theorem of Calculus 6.9, we have

$$\int_\alpha f = \int_\alpha g' = \int_{t_1}^{t_2} g'(\alpha(t))\alpha'(t)\, dt = \int_{t_1}^{t_2} h'(t)\, dt = h(t_2) - h(t_1) = g(\alpha(t_2)) - g(\alpha(t_1)) \,.$$

When α is a loop we have $\alpha(t_1) = \alpha(t_2)$, so $g(\alpha(t_1)) = g(\alpha(t_2))$, and hence $\displaystyle\int_\alpha f = 0$.

6.35 Example: Using the Fundamental Theorem of Calculus, we can solve example 6.20 as follows

$$\int_\alpha z^2\, dz = \Big[\tfrac{1}{3}\, z^3\Big]_{\alpha(0)}^{\alpha(1)} = \Big[\tfrac{1}{3}\, z^3\Big]_2^{1+i} = \tfrac{1}{3}\big((-2+2i) - (8)\big) = -\tfrac{10}{3} + \tfrac{2}{3}\, i \,.$$

6.36 Example: Find $\displaystyle\int_\alpha z^3\, e^{z^2}\, dz$ where $\alpha(t) = 1 + 2\, e^{i\,t}$, $\frac{2\pi}{3} \le t \le \pi$.

Solution: Since $\alpha\left(\frac{2\pi}{3}\right) = \sqrt{3}\, i$ and $\alpha(\pi) = -1$, by the Fundamental Theorem of Calculus, using the antiderivative calculated in example 6.30, we have

$$\int_\alpha z^3\, e^{z^2}\, dz = \Big[\tfrac{1}{2}(z^2 - 1)\, e^{z^2}\Big]_{\sqrt{3}\, i}^{-1} = 0 - \tfrac{1}{2}(-4)e^{-3} = \tfrac{2}{e^3} \,.$$

6.37 Example: Find $\displaystyle\int_\alpha \sin^3 z \sec^2 z\; dz$ where $\alpha(t) = i + e^{i\,t}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

Solution: We make the substitution $u = \cos z$, $du = -\sin z\, dz$ to get

$$\begin{aligned}\int \sin^3 z \sec^2 z\; dz &= \int \frac{\sin^3 z\, dz}{\cos^2 z} = \int \frac{(1 - \cos^2 z)\sin z\, dz}{\cos^2 z} = \int -\frac{1 - u^2}{u^2}\, du \\ &= \int 1 - \tfrac{1}{u^2}\; du = u + \tfrac{1}{u} = \cos z + \sec z \,,\end{aligned}$$

and so

$$\begin{aligned}\int_\alpha \sin^3 z \sec^2 z\; dz &= \Big[\cos z + \sec z\Big]_{\alpha(-\pi/2)}^{\alpha(\pi/2)} = \Big[\cos z + \sec z\Big]_0^{2i} \\ &= \left(\tfrac{e^2 + e^{-2}}{2} + \tfrac{2}{e^2 + e^{-2}}\right) - (1+1) = \tfrac{e^4+1}{2e^2} + \tfrac{2e^2}{e^4+1} - 2 \,.\end{aligned}$$

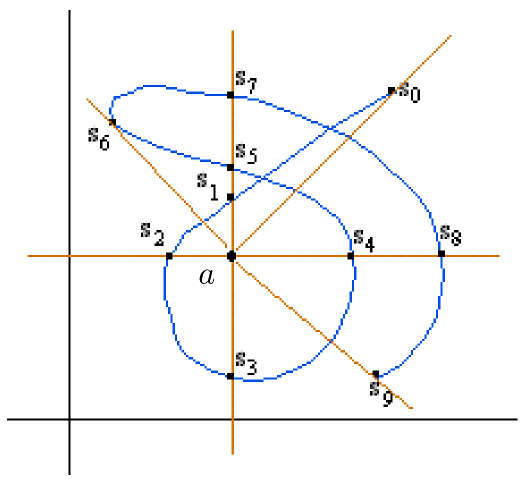
6.38 Definition: For a path $\alpha : [t_1, t_2] \to U \subseteq \mathbf{C}$ and a point $a \in \mathbf{C}$ which does not lie on the path α, we define the **winding number** $\eta(\alpha, a)$ of α about a as follows. We write $\alpha(t) = a + r(t)e^{i\,\theta(t)}$ where $r(t) = |\alpha(t) - a|$ and $\theta(t)$ is chosen continuously with $0 \le \theta(t_1) < 2\pi$ (it can be shown that the map $\theta(t)$ exists and is uniquely determined), and then we set

$$\eta(\alpha, a) = \frac{\theta(t_2) - \theta(t_1)}{2\pi}\,.$$

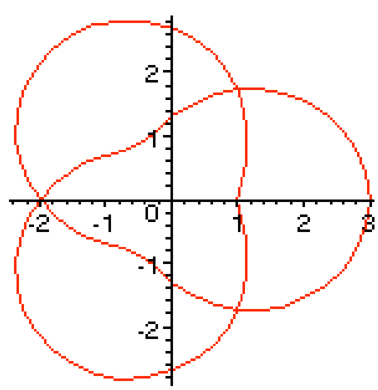
If α is a loop then we have $\alpha(t_1) = \alpha(t_2)$ and so $e^{i\,\theta(t_1)} = e^{i\,\theta(t_2)}$ and hence $\theta(t_2) - \theta(t_1)$ will be a multiple of 2π. Thus for a loop α, we have $\eta(\alpha, a) \in \mathbf{Z}$.

6.39 Example: It is not hard to find the winding number $\eta(\alpha, a)$ from a picture of the path α. For example, for α and a as shown below, we can choose values $t = s_i$ (as shown). Then $\theta(s_0) \cong \frac{\pi}{4}$, and then $\theta(t)$ increases (since we move counterclockwise around a) with $\theta(s_1) = \frac{\pi}{4}$, $\theta(s_2) = \pi$, $\theta(s_3) = \frac{3\pi}{2}$, $\theta(s_4) = 2\pi$ and $\theta(s_5) = \frac{5\pi}{2}$, and then $\theta(t)$ reaches its maximum at $\theta(s_6) \cong \frac{11\pi}{4}$ and begins to decrease (since we now begin moving clockwise around a) with $\theta(s_7) = \frac{5\pi}{2}$, $\theta(s_8) = 2\pi$ and finally $\theta(s_9) \cong \frac{7\pi}{4}$. Thus we have

$$\eta(\alpha, a) = \frac{\theta(s_9) - \theta(s_0)}{2\pi} \cong \frac{\frac{7\pi}{4} - \frac{\pi}{4}}{2\pi} = \tfrac{3}{4}\,.$$



6.40 Example: If α is the pretzel curve $\alpha(t) = r(t)e^{i\,\theta(t)}$, where $r(t) = (2 + \cos 3t)$ and $\theta(t) = 2t$ with $0 \le t \le 2\pi$ (as shown below), then the winding number of α about 0 is $\eta(\alpha, 0) = \frac{\theta(2\pi) - \theta(0)}{2\pi} = \frac{4\pi - 0}{2\pi} = 2$. The winding number about other points is hard to compute from the given formula for α, but is easy to find using a sketch of the curve. For example we have $\eta(\alpha, 2) = \eta(\alpha, 2e^{i\,2\pi/3}) = \eta(\alpha, 2e^{i\,4\pi/3}) = 1$ and $\eta(\alpha, 4) = 0$.



6.41 Theorem: *(Winding Number) For the path $\alpha(t) = a + r(t)e^{i\,\theta(t)}$ with $r(t) > 0$ and $t_1 \le t \le t_t$, we have*

$$\int_\alpha \frac{dz}{z-a} = \ln\frac{r(t_2)}{r(t_1)} + 2\pi\,i\,\eta(\alpha,a) = \Big[\ln|z-a|\Big]_{\alpha(t_1)}^{\alpha(t_2)} + 2\pi\,i\,\eta(\alpha,a)\,.$$

In particular, when α is a loop we have $r(t_1) = r(t_2)$ so

$$2\pi\,i\,\eta(\alpha,a) = \int_\alpha \frac{dz}{z-a}$$

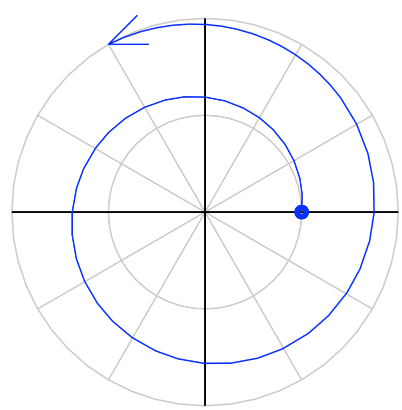
Proof: We have

$$\begin{aligned}\int_\alpha \frac{dz}{z-a} &= \int_{t_1}^{t_2} \frac{\alpha'(t)}{\alpha(t)-a}\,dt = \int_{t_1}^{t_2} \frac{r'e^{i\,\theta} + i\,r\,\theta' e^{i\,\theta}}{r\,e^{i\,\theta}}\,dt = \int_{t_1}^{t_2} \frac{r'}{r}\,dt + i\int_{t_1}^{t_2} \theta'\;dt \\ &= \Big[\ln r(t)\Big]_{t_1}^{t_2} + i\Big[\theta(t)\Big]_{t_1}^{t_2} = \ln r(t_2) - \ln r(t_1) + i\big(\theta(t_2) - \theta(t_1)\big) \\ &= \ln\frac{r(t_2)}{r(t_1)} + 2\pi\,i\,\eta(\alpha,a)\,.\end{aligned}$$

6.42 Example: Find $\displaystyle\int_\alpha \frac{5z+2}{z^2(z+1)^2}\,dz$ where $\alpha(t) = \big(1 + \tfrac{3}{8}t\big)e^{i\,\pi t}$ for $0 \le t \le \frac{8}{3}$.

Solution: First we sketch the path α by making a table of values and plotting points on a polar grid.

t	$\theta = \pi\,t$	$r = 1 + \frac{3}{8}\,t$
0	0	1
1/3	$\pi/3$	9/8
2/3	$2\pi/3$	5/4
1	π	11/8
4/3	$4\pi/3$	3/2
5/3	$5\pi/3$	13/8
2	2π	7/4
7/3	$7\pi/3$	15/8
8/3	$8\pi/3$	2



Notice that $\alpha(0) = 1$ and $\alpha\left(\frac{8}{3}\right) = -1 + \sqrt{3}\,i$. Now we decompose the function $\dfrac{5z+2}{z^2(z+1)^2}$ into partial fractions. In order to get $\dfrac{A}{z} + \dfrac{B}{z^2} + \dfrac{C}{z+1} + \dfrac{D}{(z+1)^2} = \dfrac{5z+2}{z^2(z+1)^2}$ we need $Az(z+1)^2 + B(z+1)^2 + Cz^2(z+1) + Dz^2 = 5z+2$. Equate coefficients to get the equations $A + C = 0$, $2A + B + C + D = 0$, $A + 2B = 5$ and $B = 2$. Solve these to get $A = 1$, $B = 2$, $C = -1$ and $D = -3$. Using the Winding Number Theorem, we have

$$\begin{aligned}\int_\alpha \frac{5z+2}{z^2(z+1)^2}\,dz &= \int_\alpha \frac{1}{z} + \frac{2}{z^2} - \frac{1}{z+1} - \frac{3}{(z+1)^2}\;dz \\ &= \ln\tfrac{2}{1} + 2\pi\,i\,\eta(\alpha,0) - \Big[\tfrac{2}{z}\Big]_1^{-1+\sqrt{3}\,i} - \ln\tfrac{\sqrt{3}}{2} - 2\pi\,i\,\eta(\alpha,-1) + \Big[\tfrac{3}{z+1}\Big]_1^{-1+\sqrt{3}\,i} \\ &= \ln 2 + 2\pi\,i\,\tfrac{4}{3} - \tfrac{2}{-1+\sqrt{3}\,i} + 2 - \ln\tfrac{\sqrt{3}}{2} - 2\pi\,i\,\tfrac{5}{4} + \tfrac{3}{-1+\sqrt{3}\,i} - \tfrac{3}{2} \\ &= \ln 2 + \tfrac{8\pi}{3}\,i + \tfrac{1}{2}(1+\sqrt{3}\,i) + 2 - \ln\tfrac{\sqrt{3}}{2} - \tfrac{5\pi}{2}\,i - \tfrac{3}{4}(1+\sqrt{3}\,i) - \tfrac{3}{2} \\ &= \left(\ln\tfrac{4}{\sqrt{3}} + \tfrac{1}{4}\right) - \left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}\right)\,i\,.\end{aligned}$$

Chapter 7. Cauchy's Integral Formulas

7.1 Remark: Let $U \subseteq \mathbf{C}$ be open, and let α be a path which runs counterclockwise around the boundary of a closed convex set $E \subset U$. Recall that Green's theorem (for real path integrals) states that if $u, v : U \to \mathbf{R}$ are $\mathcal{C}^1$ maps, then

$$\int_\alpha u\,dx + v\,dy = \int\!\!\int_E (v_x - u_y)\,dx\,dy\,.$$

Let $f : U \to \mathbf{C}$ be holomorphic, and let $u = \mathrm{Re}\,(f)$ and $v = \mathrm{Im}\,(f)$. If we suppose that u and v are $\mathcal{C}^1$, then Green's Theorem and the Cauchy-Riemann equations imply that

$$\begin{aligned}\int_\alpha f(z)\,dz = \int_\alpha (u + i\,v)(dx + i\,dy) &= \int_\alpha u\,dx - v\,dy + i\int_\alpha v\,dx + u\,dy \\ &= \int\!\!\int_E (-v_x - u_y)dx\,dy + i\int\!\!\int_E (u_x - v_y)dx\,dy = 0\,.\end{aligned}$$

We shall now prove a series of theorems which generalize this result (which is known as **Cauchy's Theorem**) and which do not require the assumption that u and v are $\mathcal{C}^1$. Indeed, we shall be able to show that every holomorphic map is $\mathcal{C}^\infty$.

7.2 Lemma: *Let $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ be non-empty compact sets. Then $\bigcap\limits_{n=0}^{\infty} K_n \neq \emptyset$.*

Proof: Suppose, for a contradiction, that $\bigcap\limits_{n=0}^{\infty} K_n = \emptyset$. For $n \in \mathbf{Z}^+$, let $U_n = {K_n}^c$. Note that each U_n is open and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$. We have $\emptyset = K_0 \cap \bigcap\limits_{n=1}^{\infty} K_n = K_0 \cap \Big(\bigcup\limits_{n=1}^{\infty} U_n\Big)^c$ and so $K_0 \subseteq \bigcup\limits_{n=1}^{\infty} U_n$. Thus $\big\{U_1, U_2, U_3, \cdots\big\}$ is an open cover of K_0. Since K_0 is compact, we can choose a finite subcover, say $K_0 \subseteq U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_l}$ with $n_1 < n_2 < \cdots < n_l$. Since $U_{n_1} \subseteq U_{n_2} \subseteq \cdots \subseteq U_{n_l}$ we have $K_0 \subseteq U_{n_l}$. But then $K_0 \cap {U_{n_l}}^c = \emptyset$, that is $K_0 \cap K_{n_l} = \emptyset$, and this is not possible since $\emptyset \neq K_{n_l} \subseteq K_0$.

7.3 Theorem: *(Cauchy's Theorem in a Triangle) Suppose that $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is holomorphic in U. Let Δ be a closed solid triangle in U and let α be a loop which goes once around the boundary of the triangle. Then $\int_\alpha f = 0$.*

Proof: Let $I = \left|\int_\alpha f\right|$ and set $I_0 = I$, $\Delta_0 = \Delta$, $\alpha_0 = \alpha$ and $L_0 = L(\alpha)$. Divide Δ into four similar triangles Δ_{01}, Δ_{02}, Δ_{03} and Δ_{04}, let $\alpha_{01}, \ldots, \alpha_{04}$ be loops around these triangles, and let $I_{0j} = \left|\int_{\alpha_{0j}} f\right|$ for $j = 1, 2, 3, 4$. Choose k so that I_{0k} is the largest of these, and then set $I_1 = I_{0k}$, $\Delta_1 = \Delta_{0k}$, $\alpha_1 = \alpha_{0k}$ and $L_1 = L(\alpha_1)$. Since the triangles Δ_{0j} are half as big as Δ_0 we have $L_0 = 2L_1$. Also, since $I_1 \geq I_{0j}$ for all j, we have

$$I_0 = \left|\int_{\alpha_0} f\right| = \left|\sum_{j=1}^{4} \int_{\alpha_{0j}} f\right| \leq \sum_{j=1}^{4} \left|\int_{\alpha_{0j}} f\right| = \sum_{j=1}^{n} I_{0j} \leq 4I_1\,.$$

Next we subdivide Δ_1 into four similar triangles Δ_{11}, Δ_{12}, Δ_{13} and Δ_{14}, and repeat the procedure. In this way we obtain a sequence of similar triangles $\Delta_0 \supset \Delta_1 \supset \ldots$ with a loop α_k around each triangle, and we have $I_0 \leq 4I_1 \leq 4^2 I_2 \leq \ldots$ and $L_0 = 2L_1 = 2^2 L_2 = \ldots$, where $I_k = \left|\int_{\alpha_k} f\right|$ and $L_k = L(\alpha_k)$. By the above lemma, we can choose a point a which lies in all of the compact sets Δ_k.

Now let $\epsilon > 0$. Since f is holomorphic at a, we can choose δ so that for $|z-a| < \delta$ we have $\left|f(z) - \big(f(a) + f'(a)(z-a)\big)\right| \leq \epsilon|z-a|$. Choose N so that for $n \geq N$ we have $\Delta_n \subset D(a,\delta)$, and note that for all $z \in \Delta_n$ we have $|z-a| \leq L_n$. So for $z \in \Delta_n$ we have $|z-a| < \delta$ which implies $\left|f(z) - \big(f(a) + f'(a)(z-a)\big)\right| \leq \epsilon|z-a| < \epsilon L_n$. Since $f(a) + f'(a)(z-a)$ has an antiderivative, namely $f(a)\,z + f'(a)\left(\frac{1}{2}z^2 - az\right)$, we know that $\int_{\alpha_n} f(a) + f'(a)(z-a)\,dz = 0$ so we have $\int_\alpha f(z)\,dz = \int_\alpha f(z) - \big(f(a) + f'(a)(z-a)\big)\,dz$. Writing $M_n = \max\limits_{z=\alpha(t)} \big(f(z) - (f(a) + f'(a)(z-a)\big) < \epsilon L_n$, the Estimation Theorem gives

$$I_n = \left|\int_{\alpha_n} f(z)\,dz\right| = \left|\int_{\alpha_n} f(z) - \big(f(a) + f'(a)(z-a)\big)\,dz\right| \leq M_n L_n \leq \epsilon\, {L_n}^2 = \epsilon\,\frac{{L_0}^2}{4^n}\,.$$

Thus $I_0 \leq 4^n I_n \leq \epsilon {L_0}^2$. Since ϵ was arbitrary, we must have $I_0 = 0$.

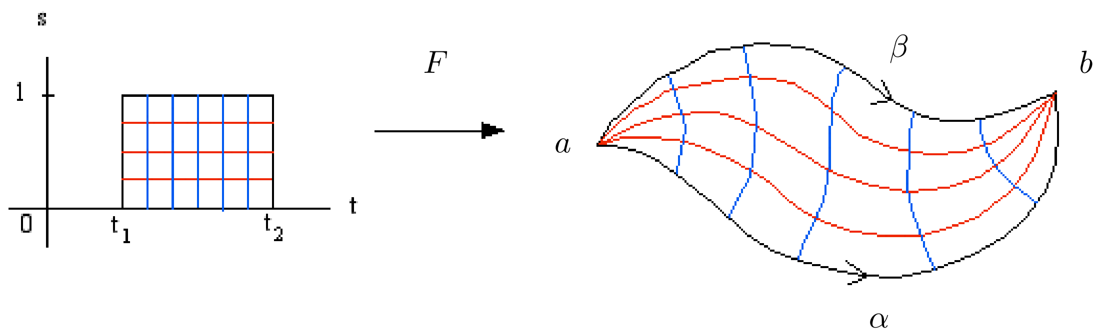
7.4 Theorem: *(Cauchy's Theorem in a Convex Region) Suppose that $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is holomorphic in U, where U is open and convex. Then f has an antiderivative in U. Consequently, $\displaystyle\int_\alpha f = 0$ for all loops α in U.*

Proof: Choose any point $a \in U$. For each $z \in U$ set $g(z) = \displaystyle\int_\alpha f$ where α is the line segment from a to z (that is $\alpha(t) = a + (z-a)\,t$, $0 \leq t \leq 1$). We claim that $g'(z) = f(z)$ for all $z \in U$. Indeed, given $h \in \mathbf{C}$ (small enough so that $z + h \in U$) we let β be the line segment from z to $z+h$ and we let γ be the line segment from $z+h$ to a, so by the definition of g we have $g(z+h) = \int_{\gamma^{-1}} f = -\int_\gamma f$, and by Cauchy's Theorem in a Triangle we have $\int_\alpha f + \int_\beta f + \int_\gamma f = 0$, and so

$$\begin{aligned}\left|f(z) - \frac{g(z+h) - g(z)}{h}\right| &= \left|f(z) + \frac{1}{h}\Big(\int_\alpha f(w)\,dw + \int_\gamma f(w)\,dw\Big)\right| \\ &= \left|f(z) - \frac{1}{h}\int_\beta f(w)\,dw\right| \\ &= \left|\frac{1}{h}\int_\beta f(z)\,dw - \frac{1}{h}\int_\beta f(w)\,dw\right| \\ &= \left|\frac{1}{h}\int_\beta f(z) - f(w)\;dw\right| \\ &\leq \max_{w=\beta(t)} \big|f(z) - f(w)\big|\,.\end{aligned}$$

As $h \to 0$ we have $w = \beta(t) \to z$ and so $|f(z) - f(w)| \to 0$, since f is continuous.

7.5 Definition: Let $\alpha, \beta : [t_1, t_2] \to U \subseteq \mathbf{C}$ be paths with $\alpha(t_1) = \beta(t_1) = a$ and $\alpha(t_2) = \beta(t_2) = b$. A **path-homotopy** (or **deformation of paths**) from α to β in U is a continuous map $F : [t_1, t_2] \times [0, 1] \to U$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all t, and also $F(t_1, s) = a$ and $F(t_2, s) = b$ for all s. If such a homotopy exists, then we say that α is (path)-**homotopic** to β in U and we write $\alpha \cong \beta$ in U. Note that for each fixed $s \in [0, 1]$, the map $F_s(t) := F(t, s)$ is a continuous curve from a to b.



7.6 Example: In a convex set $U \subseteq \mathbf{C}$ we can find a path-homotopy between any two paths $\alpha, \beta : [t_1, t_2] \to U$ with $\alpha(t_1) = \beta(t_1)$ and $\alpha(t_2) = \beta(t_2)$. Indeed, we can take

$$F(t, s) = \alpha(t) + s\left(\beta(t) - \alpha(t)\right).$$

7.7 Theorem: *(Cauchy's Theorem for Paths) If $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is holomorphic and α and β are homotopic paths in U then $\displaystyle\int_\alpha f = \int_\beta f$.*

Proof: Say $\alpha, \beta : [p, q] \to U$. Choose a path-homotopy $F : [p, q] \times [0, 1] \to U$ from α to β in U. Choose partitions $p = t_0 < t_1 < \ldots t_k = q$ and $0 = s_1 < s_2 < \ldots s_l = 1$ with the property that for each i, j the image $F\big([t_{i-1}, t_i] \times [s_{j-1}, s_j]\big)$ is contained in a convex set which lies in U. (To *prove* that such partitions can be found, you must use the fact that $[p, q] \times [0, 1]$ is compact). For each i and j, let α_i be the restriction of α to $[t_{i-1}, t_i]$, let β_i be the restriction of β to $[t_{i-1}, t_i]$, let a_i be the line segment from $\alpha(t_{i-1})$ to $\alpha(t_i)$, let b_i be the line segment from $\beta(t_{i-1})$ to $\beta(t_i)$, and let γ_{ij} be the polygonal loop around the polygon with vertices at $F(t_{i-1}, s_{j-1})$, $F(t_i, s_{j-1})$, $F(t_i, s_j)$ and $F(t_{i-1}, s_j)$. Then by Cauchy's Theorem for convex sets, we have

$$\int_{\alpha_i} f = \int_{a_i} f \quad , \quad \int_{\beta_i} f = \int_{b_i} f \quad \text{and} \quad \int_{\gamma_{ij}} f = 0\ .$$

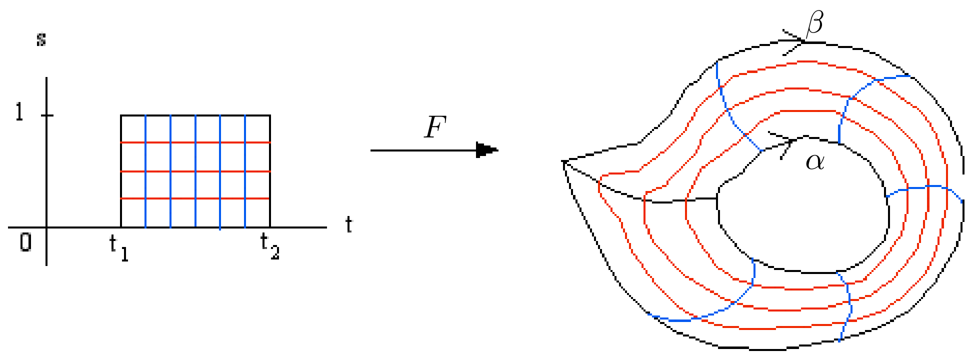
When we consider all of the paths ${a_i}^{-1}$, b_i and γ_{ij}, every line segment occurs twice, once in each direction, and so the path integrals all cancel with each other to give

$$\begin{aligned} 0 &= \sum_i \int_{b_i} f - \sum_i \int_{a_i} f + \sum_{i,j} \int_{\gamma_{ij}} f \\ &= \sum_i \int_{\beta_i} f - \sum_i \int_{\alpha_i} f \\ &= \int_\beta f - \int_\alpha f \end{aligned}$$

7.8 Example: Let $\alpha, \beta : [0, \pi] \to \mathbf{C}^*$ be given by $\alpha(t) = e^{i\,t}$ and $\beta(t) = e^{-i\,t}$. Show that α and β are not homotopic in $\mathbf{C}^*$.

Solution: Let $f(z) = 1/z$. Then f is holomorphic in $\mathbf{C}^*$ and we have $\int_\alpha f = i\,\pi$ and $\int_\beta f = -i\,\pi$. Since $\int_\alpha f \neq \int_\beta f$ we know that α is not homotopic to β.

7.9 Definition: Let $\alpha, \beta : [t_1, t_2] \to U \subseteq \mathbf{C}$ be loops in U. A **loop-homotopy** (or **deformation of loops**) from α to β in U is a continuous map $F : [t_1, t_2] \times [0, 1] \to U$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all t and $F(t_1, s) = F(t_2, s)$ for all s. If such a homotopy exists, we say that α is (loop)-**homotopic** to β in U and we write $\alpha \sim \beta$ in U.



7.10 Example: In a convex set $U \subseteq \mathbf{C}$, any two loops are homotopic. Indeed, given loops $\alpha, \beta : [t_1, t_2] \to U$ we can take $F(t, s) = \alpha(t) + s\,\big(\beta(t) - \alpha(t)\big)$.

7.11 Theorem: *(Cauchy's Theorem for Loops) If $f : U \subseteq \mathbf{C} \to \mathbf{C}$ is holomorphic and α and β are homotopic loops in U, then* $\displaystyle\int_\alpha f = \int_\beta f$.

Proof: The proof is the same as the proof of Cauchy's Theorem for Paths.

7.12 Example: Let $\alpha, \beta : [t_1, t_2] \to \mathbf{C}^*$ be loops. Show that if $\eta(\alpha, 0) \neq \eta(\beta, 0)$ then α and β are not homotopic in $\mathbf{C}^*$.

Solution: Let $f(z) = 1/z$. Then f is holomorphic and we have $\int_\alpha f = 2\pi\,i\;\eta(\alpha, 0)$ and $\int_\beta f = 2\pi\,i\;\eta(\beta, 0)$, so if $\eta(\alpha, 0) \neq \eta(\beta, 0)$ then α and β cannot be homotopic in $\mathbf{C}^*$.

7.13 Definition: A set $U \subset \mathbf{C}$ is called **simply connected** when U is connected and any two loops $\alpha, \beta : [t_1, t_2] \to U$ are homotopic in U. Roughly speaking, a connected set will be simply connected if it doesn't have any holes in it.

7.14 Example: Any convex set is simply connected, but $\mathbf{C}^*$ is not.

7.15 Theorem: *(Cauchy's Theorem in a Simply Connected Region) If $U \subseteq \mathbf{C}$ is a simply connected open set and if $f : U \to \mathbf{C}$ is holomorphic, then* $\displaystyle\int_\alpha f = 0$ *for every loop α in U.*

Proof: Since U is simply connected, any loop $\alpha : [t_1, t_2] \to U$ will be homotopic to the constant loop κ given by $\kappa(t) = \alpha(t_1)$ for all t, so $\displaystyle\int_\alpha f = \int_\kappa f = \int_{t_1}^{t_2} f(\alpha(a))\kappa'(t)\,dt = 0$ since $\kappa'(t) = 0$.

7.16 Theorem: *(Cauchy's Integral Formulas) Let $U \subseteq \mathbf{C}$ be a convex open set, let $a \in U$, let f be holomorphic in U and let α be a loop in $U \setminus \{a\}$. Then*

(a) $2\pi i\ \eta(\alpha, a)\, f(a) = \displaystyle\int_\alpha \frac{f(z)}{z-a}\, dz$.

(b) All the derivatives $f^{(n)}(a)$ exist, and $2\pi i\ \eta(\alpha,a)\, f^{(n)}(a) = n! \displaystyle\int_\alpha \frac{f(z)}{(z-a)^{n+1}}\, dz$.

Proof: First we prove part (a). For any $\epsilon > 0$, let α_ϵ denote the path $\alpha_\epsilon(t) = a + \epsilon(\alpha(t) - a)$. Note that $\alpha \sim \alpha_\epsilon$ in $U \setminus \{a\}$, indeed a homotopy is given by $F(t,s) = \alpha(t) + s(\alpha_\epsilon(t) - \alpha(t))$. Also note that the map $\dfrac{f(z) - f(a)}{z-a}$ is holomorphic in $U \setminus \{a\}$. So we have

$$\begin{aligned}\left|\int_\alpha \frac{f(z)}{z-a}\, dz - 2\pi i\, \eta(\alpha,a)\, f(a)\right| &= \left|\int_\alpha \frac{f(z)}{z-a}\, dz - \int_\alpha \frac{f(a)}{z-a}\, dz\right| \\ &= \left|\int_\alpha \frac{f(z)-f(a)}{z-a}\, dz\right| = \left|\int_{\alpha_\epsilon} \frac{f(z)-f(a)}{z-a}\, dz\right| \le M_\epsilon\, L(\alpha_\epsilon)\,,\end{aligned}$$

where $M_\epsilon = \displaystyle\max_{z=\alpha_\epsilon(t)} \left|\frac{f(z)-f(a)}{z-a}\right|$. As $\epsilon \to 0$ we have $\dfrac{f(z)-f(a)}{z-a} \to f'(a)$ so $M_\epsilon \to |f'(a)|$, and also $L(\alpha_\epsilon) = \epsilon\, L(\alpha) \to 0$

We prove part (b) inductively. Suppose $2\pi i\ \eta(\alpha,a)\, f^{(n)}(a) = n! \displaystyle\int_\alpha \frac{f(z)}{(z-a)^{n+1}}\, dz$. Then we have

$$\begin{aligned}2\pi i\ \eta(\alpha,a)\left(\frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}\right) &= \frac{n!}{h}\int_\alpha \frac{f(z)}{(z-(a+h))^{n+1}} - \frac{f(z)}{(z-a)^{n+1}}\, dz \\ &= \frac{n!}{h}\int_\alpha f(z)\left(\frac{1}{(z-(a+h))^{n+1}} - \frac{1}{(z-a)^{n+1}}\right) dz \\ &= \frac{(n+1)!}{h}\int_\alpha f(z)\int_\lambda \frac{1}{(z-w)^{n+2}}\, dw\ dz\,,\end{aligned}$$

where λ is the line segment from a to $a+h$. So we have

$$\begin{aligned}L :&= \left|2\pi i\ \eta(\alpha,a)\,\frac{f^{(n)}(a+h) - f^{(n)}(a)}{h} - (n+1)!\int_\alpha \frac{f(z)}{(z-a)^{n+2}}\, dz\right| \\ &= \left|\frac{(n+1)!}{h}\left(\int_\alpha f(z)\int_\lambda \frac{1}{(z-w)^{n+2}}\, dw\ dz - h\int_\alpha \frac{f(z)}{(z-a)^{n+2}}\right) dz\right| \\ &= \left|\frac{(n+1)!}{h}\int_\alpha f(z)\left(\int_\lambda \frac{1}{(z-w)^{n+2}}\, dw - h\,\frac{1}{(z-a)^{n+2}}\right) dz\right| \\ &= \left|\frac{(n+1)!}{h}\int_\alpha f(z)\int_\lambda \frac{1}{(z-w)^{n+2}} - \frac{1}{(z-a)^{n+2}}\, dw\ dz\right| \\ &= \left|\frac{(n+2)!}{h}\int_\alpha f(z)\int_\lambda\int_\tau \frac{1}{(z-u)^{n+3}}\, du\ dw\ dz\right|,\end{aligned}$$

where τ is the line segment from a to w. Choose $r > 0$ so that $D(a, 2r) \subseteq U \setminus \text{Image}\,(\alpha)$, and let $|h| < r$. For $w \in \text{Image}\,(\lambda)$ and $u \in \text{Image}\,(\tau)$ we have w between a and $a+h$, and u between a and w, so $u \in D(a,r)$, and so we have $|z-u| \ge r$ hence $\dfrac{1}{|z-u|} \le \dfrac{1}{r}$. By the Estimation Theorem, $L \le \dfrac{(n+2)!}{|h|}\, L(\alpha) \displaystyle\max_{z=\alpha(t)} |f(z)|\, |h|\, |h|\, \frac{1}{r^{n+3}} \to 0$ as $|h| \to 0$.

7.17 Example: Let $\alpha(t) = 2\,e^{i\,t}$ for $0 \le t \le 2\pi$ and let $f(z) = \dfrac{z+1}{z^2+1}$. Find $\displaystyle\int_\alpha f(z)\,dz$.

Solution: We shall find the integral in several ways. First, we shall use partial fractions. To write $\dfrac{z+1}{z^2+1} = \dfrac{z+1}{(z+i)(z-i)}$ in the form $\dfrac{A}{z+i} + \dfrac{B}{z-i}$ we need $A(z-i)+B(z+i) = z+1$ for all z. Setting $z = i$ gives $B(2i) = i+1$ so $B = \dfrac{i+1}{2i} = \dfrac{i-1}{2}$. Setting $z = -i$ gives $A(-2i) = -i+1$ so $A = \dfrac{-i+1}{-2i} = \dfrac{1+i}{2}$. And so we have

$$\int_\alpha \frac{z+1}{z^2+1}\,dz = \frac{1+i}{2}\int_\alpha \frac{dz}{z+i} + \frac{1-i}{2}\int_\alpha \frac{dz}{z-i} = \frac{1+i}{2}\,2\pi\,i\;\eta(\alpha,-i) + \frac{1-i}{2}\,2\pi\,i\;\eta(\alpha,i)$$
$$= \frac{1+i}{2}\,2\pi\,i + \frac{1-i}{2}\,2\pi\,i = \pi\,(i-1) + \pi\,(i+1) = 2\pi\,i\,.$$

Now we shall find the integral again by immitating the proof of Cauchy's integral formula. Notice that f is holomorphic except at $z = \pm i$. Let α_1 be the loop around the top half of the circle, and let α_2 be the loop around the bottom half, to be explicit, we take $\alpha_1(t) = \begin{cases} 2\,e^{i\,t} & \text{for } 0 \le t \le \pi \\ \frac{2}{\pi}t - 3 & \text{for } \pi \le t \le 2\pi \end{cases}$ and $\alpha_2(t) = \begin{cases} 1 - \frac{2}{\pi}\,t & \text{for } 0 \le t \le \pi \\ 2\,e^{i\,\pi} & \text{for } \pi \le t \le 2\pi \end{cases}$ and then we will have $\int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_2} f$. Next we deform the paths α_1 and α_2 into the circular paths σ_1 and σ_2, where $\sigma_1(t) = i + r\,e^{i\,t}$ and $\sigma_2(t) = -i + r\,e^{i\,t}$ for $0 \le t \le 2\pi$, where $0 < r < 1$. We have

$$\int_{\alpha_1} f = \int_{\sigma_1} f = \int_0^{2\pi} f(\sigma_1(t))\,\sigma_1{}'(t)\,dt = \int_0^{2\pi} \frac{1+i+r\,e^{i\,t}}{-1+2i\,re^{it}+r^2e^{i\,2t}+1}\,i\,re^{it}\,dt$$
$$\to \int_0^{2\pi} \frac{1+i}{2}\,dt = \pi(1-i) \text{ as } r \to 0\,,$$

and we have

$$\int_{\alpha_2} f = \int_{\sigma_2} f = \int_0^{2\pi} f(\sigma_2(t))\,\sigma_2{}'(t)\,dt = \int_0^{2\pi} \frac{1-i+r\,e^{it}}{-2i\,re^{it}+r^2e^{i\,2t}}\,i\,re^{it}\,dt$$
$$\longrightarrow \int_0^{2\pi} \frac{1-i}{-2}\,dt = \pi\,(i-1) \text{ as } r \to 0\,.$$

Thus $\displaystyle\int_\alpha f = 2\pi\,i$.

Finally, we shall compute the integral a third time using Cauchy's formula. Taking α_1 and α_2 as above, we have

$$\int_{\alpha_1} f = \int_{\alpha_1} \frac{(z+1)/(z+i)}{z-i}\,dz = \int_{\alpha_1} \frac{F(z)}{z-i}\,dz = 2\pi\,i\,F(i) = 2\pi\,i\,\frac{i+1}{2i} = \pi(i+1)\,,$$

where $F(z) = (z+1)/(z+i)$, and

$$\int_{\alpha_2} f = \int_{\alpha_2} \frac{(z+1)/(z-i)}{z+i}\,dz = \int_{\alpha_2} \frac{G(z)}{z+i}\,dz = 2\pi\,i\,G(-i) = 2\pi\,i\,\frac{1-i}{-2i} = \pi\,(i-1)\,,$$

where $G(z) = (z+1)/(z-i)$. Again we obtain $\displaystyle\int_\alpha f = 2\pi\,i$.

7.18 Example: Let $\alpha(t) = 2\,e^{i\,t}$ for $0 \le t \le 2\pi$, and let $f(z) = \dfrac{e^z}{z^2-1}$. Find $\int_\alpha f$.

Solution: Of the three methods we used above, only the third works easily here. Notice that f is holomorphic except at $z = \pm 1$. Let α_1 be the loop around the right half of the circle, and let α_2 be the loop around the left half, so we have $\int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_2} f$. Then we have

$$\int_{\alpha_1} f = \int_{\alpha_1} \frac{e^z/(z+1)}{z-1}\,dz = \int_{\alpha_1} \frac{F(z)}{z-1}\,dz = 2\pi\,i\,F(1) = 2\pi\,i\,\frac{e}{2} = i\,\pi\,e\,,$$

and

$$\int_{\alpha_2} f = \int_{\alpha_2} \frac{e^z/(z-1)}{z+1}\,dz = \int_{\alpha_2} \frac{G(z)}{z+1}\,dz = 2\pi\,i\,G(-1) = 2\pi\,i\frac{e^{-1}}{-2} = -i\pi e^{-1}\,.$$

So the integral of f over α is equal to $i\,\pi\left(e - \frac{1}{e}\right)$.

7.19 Example: Let $\alpha(t) = 2\,e^{it}$ for $0 \le t \le 2\pi$ and let $f(z) = \dfrac{z+1}{z^3(z-1)^2}$. Find $\displaystyle\int_\alpha f$.

Solution: We shall solve this integral using two methods. First we use partial fractions. To write f in the form $\dfrac{z+1}{z^3(z-1)^2} = \dfrac{A}{z} + \dfrac{B}{z^2} + \dfrac{C}{z^3} + \dfrac{D}{z-1} + \dfrac{E}{(z-1)^2}$ we need to have $Az^2(z-1)^2 + Bz(z-1)^2 + C(z-1)^2 + Dz^3(z-1) + Ez^3 = z+1$ for all z. Equating coefficients gives fives 5 equations: $A + D = 0$, $-2A + B - D + E = 0$, $A - 2B + C = 0$, $B - 2C = 1$ and $C = 1$. Solving these gives $A = 5$, $B = 3$, $C = 1$, $D = -5$ and $E = 2$. So

$$\begin{aligned}\int_\alpha f &= \int_\alpha \frac{5}{z} + \frac{3}{z^2} + \frac{1}{z^3} - \frac{5}{z-1} + \frac{2}{(z-1)^2}\,dz \\ &= 2\pi\,i\big(5\eta(\alpha,0) - 5\eta(\alpha,1)\big) = 2\pi\,i(5-5) = 0\,.\end{aligned}$$

Now we compute the integral again using Cauchy's formulas. Notice that f is holomorphic except at $z = 0, 1$. Let α_1 be the loop around the portion of the circle which lies to the right of the line $y = \frac{1}{2}$ and let α_0 be the loop around the portion to the left of $y = \frac{1}{2}$, so that $\int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_0} f$. We have

$$\int_{\alpha_0} f = \int_{\alpha_0} \frac{(z+1)/z^3}{(z-1)^2}\,dz = \int_{\alpha_0} \frac{F(z)}{z^3}\,dz = \frac{2\pi\,i}{2!}\,F''(0)\,.$$

From $F(z) = \dfrac{z+1}{(z-1)^3}$, we calculate $F'(z) = \dfrac{-z-3}{(z-1)^3}$ and $F''(z) = \dfrac{2z+10}{(z-1)4}$ to get $F''(0) = 10$, so we have $\displaystyle\int_{\alpha_1} f = 10\pi\,i$. Also,

$$\int_{\alpha_1} f = \int_{\alpha_1} \frac{(z+1)/z^3}{(z-1)^2}\,dz = \int_{\alpha_1} \frac{G(z)}{(z-1)^2}\,dz = \frac{2\pi\,i}{1!}G'(1)\,.$$

From $G(z) = \dfrac{z+1}{z^3}$ we find $G'(z) = \dfrac{-2z-3}{z^4}$ to get $G'(1) = -5$, so we have $\displaystyle\int_{\alpha_2} f = -10\pi\,i$. Again we obtain $\displaystyle\int_\alpha f = 0$.

7.20 Note: If $\alpha(t)=e^{i\,t}$ for $0\le t\le 2\pi$, then $\sin t=\frac{1}{2i}\left(e^{i\,t}-e^{-i\,t}\right)=\frac{1}{2i}\Big(\alpha(t)-\frac{1}{\alpha(t)}\Big)$ and $\cos t=\frac{1}{2}\left(e^{i\,t}+e^{-i\,t}\right)=\frac{1}{2}\Big(\alpha(t)+\frac{1}{\alpha(t)}\Big)$, and $\alpha'(t)=i\,e^{i\,t}=i\alpha(t)$. It follows that

$$\int_0^{2\pi} f(\sin t,\cos,t)\,dt=\int_\alpha f\left(\tfrac{1}{2i}\left(z-\tfrac{1}{z}\right),\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)\right)\,\frac{1}{iz}\;dz\,.$$

Sometimes we can use this equality to solve a real integral involving trigonometric (or hyperbolic) functions, by converting it to a path integral.

7.21 Example: Find $\displaystyle\int_0^\pi \frac{dt}{2+\cos t}$.

Solution: Let $\alpha(t)=e^{i\,t}$ for $0\le t\le 2\pi$. Then

$$\begin{aligned}\int_0^\pi \frac{dt}{2+\cos t}&=\frac{1}{2}\int_0^{2\pi}\frac{dt}{2+\cos t}=\frac{1}{2}\int_\alpha\frac{\frac{1}{i\,z}\,dz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}=\int_\alpha\frac{-i\,dz}{z^2+4z+1}\\&=\int_\alpha\frac{-i\,dz}{\left(z-(-2+\sqrt3)\right)\left(z-(-2-\sqrt3)\right)}=\int_\alpha\frac{F(z)\,dz}{z-(-2+\sqrt3)}\\&=2\pi\,i\,F(-2+\sqrt3)=2\pi\,i\,\tfrac{-i}{2\sqrt3}=\tfrac{\pi}{\sqrt3}\,,\end{aligned}$$

where $F(z)=\dfrac{-i}{z-(-2-\sqrt3)}$.

7.22 Example: Find $\displaystyle\int_0^{2\pi}\frac{dt}{3+\sin^2 t}$.

Solution: Let $\alpha(t)=e^{i\,t}$ for $0\le t\le 2\pi$. Then

$$\begin{aligned}\int_0^{2\pi}\frac{dt}{3+\sin^2 t}&=\int_\alpha\frac{\frac{1}{iz}\,dz}{3-\frac{1}{4}\left(z-\frac{1}{z}\right)^2}=\int_\alpha\frac{4i\,z\;dz}{-12z^2+(z^4-2z^2+1)}=\int_\alpha\frac{4i\,z\;dz}{z^4-14z^2+1}\\&=\int_\alpha\frac{4i\,z\;dz}{\left(z-(2+\sqrt3)\right)\left(z+(2+\sqrt3)\right)\left(z-(2-\sqrt3)\right)\left(z+(2-\sqrt3)\right)}\end{aligned}$$

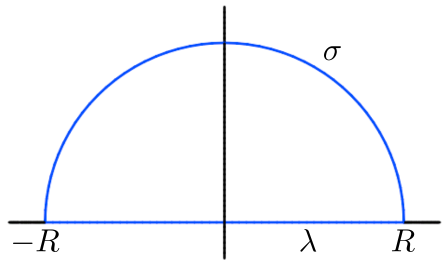
Let α_1 be the loop around the right side of the unit circle and let α_2 be the loop around the left side so that $2-\sqrt3$ lies inside α_1 and $-2+\sqrt3$ lies inside α_2. Then for

$$\begin{aligned}F(z)&=\frac{4i\,z}{\left(z-(2+\sqrt3)\right)\left(z+(2+\sqrt3)\right)\left(z+(2-\sqrt3)\right)}\quad\text{and}\\G(z)&=\frac{4i\,z}{\left(z-(2+\sqrt3)\right)\left(z+(2+\sqrt3)\right)\left(z-(2-\sqrt3)\right)}\,,\end{aligned}$$

we have

$$\begin{aligned}\int_0^{2\pi}\frac{dt}{3+\sin^2 t}&=\int_{\alpha_1}\frac{F(z)\,dz}{z-(2-\sqrt3)}+\int_{\alpha_2}\frac{G(z)}{z+(2-\sqrt3)}\\&=2\pi\,i\Big(F(2-\sqrt3)+H(-2+\sqrt3)\Big)\\&=2\pi\,i\left(\frac{(4i)(2-\sqrt3)}{((-2\sqrt3)(4)(4-2\sqrt3)}+\frac{(4i)(-2+\sqrt3)}{(-4)(2\sqrt3)(-4+2\sqrt3)}\right)\\&=-\pi\left(\tfrac{1}{-2\sqrt3}+\tfrac{1}{-2\sqrt3}\right)=\tfrac{\pi}{\sqrt3}\,.\end{aligned}$$

7.23 Note: Sometimes we can solve an improper real integral of the form $\int_0^\infty u(x)\,dx$ by finding a complex-valued function $f(z)$ of a complex variable, whose real (or imaginary) part extends $u(x)$, and then integrating $f(z)$ around a large loop which follows the positive x-axis, for example the loop which follows the line $\lambda(t) = t$ for $-R \le t \le R$, then the semicircle $\sigma(t) = Re^{i\,t}$ for $0 \le t \le \pi$, where R is some large positive real number.



7.24 Example: Find $I = \displaystyle\int_0^\infty \frac{dx}{x^4+1}$.

Solution: Let γ be the loop which follows the line $\lambda(t) = t$ for $-R \le t \le R$ then the semicircle $\sigma(t) = e^{i\,t}$ for $0 \le t \le \pi$. Then

$$\int_\gamma \frac{dz}{z^4+1} = \int_\lambda \frac{dz}{z^4+1} + \int_\sigma \frac{dz}{z^4+1}\,.$$

We have

$$\begin{aligned}\int_\gamma \frac{dz}{z^4+1} &= \int_\gamma \frac{dz}{\left(z-e^{i\,\pi/4}\right)\left(z+e^{i\,\pi/4}\right)\left(z-e^{i\,3\pi/4}\right)\left(z+e^{i\,3\pi/4}\right)} \\ &= \int_{\gamma_1} \frac{F(z)\,dz}{z-e^{i\,\pi/4}} + \int_{\gamma_2} \frac{G(z)\,dz}{z-e^{i\,3\pi/4}} = 2\pi\,i\left(F\big(e^{i\,\pi/4}\big) + G\big(e^{i\,3\pi/4}\big)\right) \\ &= 2\pi\,i\left(\frac{1}{(2e^{i\pi/4})(\sqrt{2})(\sqrt{2}\,i)} + \frac{1}{(-\sqrt{2})(\sqrt{2}\,i)(2e^{i\,3\pi/4})}\right) \\ &= \tfrac{\pi}{2}\left(\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\,i\right) + \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\,i\right)\right) = \tfrac{\pi}{\sqrt{2}}\,,\end{aligned}$$

where γ_1 is a loop around the right half of γ, and γ_2 is a loop around the left half of γ, so that $e^{i\,\pi/4}$ lies inside γ_1 and $e^{i\,3\pi/4}$ lies inside γ_2, and $F(z)$ and $G(z)$ are as expected. Also

$$\int_\lambda \frac{dz}{z^4+1} = \int_{t=-R}^{R} \frac{dt}{t^4+1} \longrightarrow 2I \text{ as } R \to \infty$$

and

$$\begin{aligned}\left|\int_\sigma \frac{dz}{z^4+1}\right| &= \left|\int_{t=0}^{\pi} \frac{iR\,e^{i\,t}\,dt}{(Re^{i\,t})^4+1}\right| \le \int_{t=0}^{\pi} \left|\frac{iR\,e^{i\,t}}{R^4e^{i\,4t}+1}\right|\,dt \\ &\le \int_{t=0}^{\pi} \frac{R\,dt}{R^4-1} = \frac{\pi R}{R^4-1} \longrightarrow 0 \text{ as } R \to \infty\,.\end{aligned}$$

Since

$$\tfrac{\pi}{\sqrt{2}} = \int_\gamma \frac{dz}{z^4+1} = \int_\lambda \frac{dz}{z^4+1} + \int_\sigma \frac{dz}{z^4+1} \longrightarrow 2I + 0 \text{ as } R \to \infty\,,$$

it follows that $I = \frac{\pi}{2\sqrt{2}}$.

7.25 Example: Find $I = \displaystyle\int_0^\infty \frac{dx}{(x^2+1)^3}$.

Solution: Let λ, σ and γ be as in the previous example. Then

$$\int_\gamma \frac{dz}{(z^2+1)^3} = \int_\gamma \frac{dz}{(z-i)^3(z+i)^3} = \int_\gamma \frac{F(z)\,dz}{(z-i)^3} = 2\pi\, i\,\frac{F''(i)}{2!} = \frac{\pi\, i}{(2i)^5} = \tfrac{3\pi}{8}$$

where $F(z) = 1/(z+i)^3$, so $F'(z) = -3/(z+i)^4$ and $F''(z) = 12/(z+i)^5$. Also,

$$\int_\lambda \frac{dz}{(z^2+1)^3} = \int_{t=-R}^{R} \frac{dt}{(t^2+1)^3} \longrightarrow \text{ as } R \to \infty$$

and

$$\left|\int_\sigma \frac{dz}{(z^2+1)^2}\right| = \left|\int_{t=0}^{\pi} \frac{iR\,e^{i\,t}\,dt}{\big((Re^{i\,t})^2+1\big)^3}\right| \le \int_0^\pi \frac{R\,dt}{(R^2-1)^3} = \frac{\pi R}{(R^2-1)^3} \longrightarrow 0 \text{ as } R \to \infty$$

Since

$$\tfrac{3\pi}{8} = \int_\gamma \frac{dz}{(z^2+1)^3} = \int_\lambda \frac{dz}{(z^2+1)^3} + \int_\sigma \frac{dz}{(z^2+1)^3} \longrightarrow 2I + 0 \text{ as } R \to \infty\,,$$

it follows that $I = \frac{3\pi}{16}$.

7.26 Example: Find $I = \displaystyle\int_0^\infty \frac{dx}{x^3+1}$.

Solution: Note that $z^3+1 = (z+1)\big(z - e^{i\,\pi/3}\big)\big(z - e^{-i\,\pi/3}\big)$. Let γ be the loop which follows the line λ then the arc σ then the line μ^{-1}, where $\lambda(t) = t$ for $0 \le t \le R$, $\sigma(t) = R\,e^{i\,t}$ for $0 \le t \le \frac{2\pi}{3}$, and $\mu(t) = t\,e^{i\,2\pi/3}$ for $0 \le t \le R$, so that the point $e^{i\pi/3}$ lies inside the loop γ. Then for $F(z) = 1/\big((z+1)(z - e^{-i\,\pi/3})\big)$ we have

$$\begin{aligned}\int_\gamma \frac{dz}{z^3+1} &= \int_\gamma \frac{F(z)}{z - e^{i\,\pi/3}} = 2\pi\, i\, F\big(e^{i\,\pi/3}\big)\\ &= \frac{2\pi\, i}{\big(\frac{3}{2} + \frac{\sqrt{3}}{2}\, i\big)(\sqrt{3}\, i)} = \frac{2\pi}{3\big(\frac{\sqrt{3}}{2} + \frac{1}{2}\, i\big)} = \tfrac{2\pi}{3}\, e^{-i\,pi/6}\,.\end{aligned}$$

Also, we have

$$\int_\lambda \frac{dz}{z^3+1} = \int_{t=0}^{R} \frac{dt}{t^3+1} \longrightarrow I \text{ as } R \to \infty\,,$$

$$\int_\mu \frac{dz}{z^3+1} = \int_{t=0}^{R} \frac{e^{i\,2\pi/3}\,dt}{t^3+1} \longrightarrow e^{i\,2\pi/3}\, I \text{ as } R \to \infty\,,$$

and

$$\left|\int_\sigma \frac{dz}{z^3+1}\right| = \left|\int_0^\pi \frac{i\,R\,e^{i\,t}\,dt}{(Re^{i\,t})^3+1}\right| \le \int_0^\pi \frac{R\,dt}{R^3-1} = \frac{\pi\, R}{R^3-1} \longrightarrow 0 \text{ as } R \to \infty\,.$$

Since

$$\tfrac{2\pi}{3}\, e^{-i\,\pi/6} = \int_\gamma \frac{dz}{z^3+1} = \int_\lambda \frac{dz}{z^3+1} + \int_\sigma \frac{dz}{z^3+1} - \int_\mu \frac{dz}{z^3+1} \longrightarrow I - e^{i\,2\pi/3} I \text{ as } R \to \infty\,,$$

it follows that $\frac{2\pi}{3}\, e^{-i\pi/6} = \big(1 - e^{i\,2\pi/3}\big)\, I = \Big(\frac{3}{2} - \frac{\sqrt{3}}{2}\, i\Big)\, I = \sqrt{3}\, e^{-i\pi/6}\, I$. Thus $I = \frac{2\pi}{3\sqrt{3}}$.

7.27 Example: Find $I = \displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$.

Solution: Write $I = \displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$. Let $f(z) = \dfrac{e^{i\,z}}{z^2+1} = \dfrac{e^{i\,z}}{(z-i)(z+i)}$, let $F(z) = \dfrac{e^{i\,z}}{z+i}$, and let α be the loop that follows the line λ given by $\lambda(t) = t$ for $-R \le t \le R$, then the semicircle σ given by $\sigma(t) = R\,e^{i\,t}$ for $0 \le t \le \pi$. Then $\int_\alpha f = \int_\lambda f + \int_\sigma f$ and we have

$$\int_\alpha f = \int_\alpha \frac{F(z)\,dz}{z-i} = 2\pi\,i\,F(i) = 2\pi\,i\cdot\frac{e^{-1}}{2i} = \tfrac{\pi}{e}\,,$$

$$\int_\lambda f = \int_{-R}^R \frac{e^{i\,t}\,dt}{t^2+1} = \int_{-R}^R \frac{\cos t}{t^2+1}\,dt + i\int_{-R}^R \frac{\sin t}{t^2+1}\,dt \longrightarrow 2I \text{ as } R\to\infty \text{ , and}$$

$$\left|\int_\sigma f\right| = \left|\int_0^\pi \frac{e^{i\,Re^{it}}\,i\,Re^{i\,t}}{(Re^{it})^2+1}\,dt\right| \le \int_0^\pi \frac{R\,dt}{R^2-1} = \frac{\pi\,R}{R^2-1} \longrightarrow 0 \text{ as } R\to\infty\,,$$

since $\left|e^{i\,Re^{it}}\right| = \left|e^{i(R\cos t + i\,R\sin t)}\right| = \left|e^{-R\sin t + i\,R\cos t}\right| = e^{-R\sin t} \le 1$ for $0 \le t \le \pi$. It folows that $2I = \frac{\pi}{e}$, so $I = \frac{\pi}{2e}$.

7.28 Example: Find $I = \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Solution: Write $I = \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$. Let $f(z) = \dfrac{e^{i\,z}}{z}$, and let α be the loop which follows first the line λ given by $\lambda(t) = t$ for $r < t < R$, then the large semicircle σ given by $\sigma(t) = R\,e^{i\,t}$ for $0 \le t \le R$, then the line κ^{-1} where $\kappa(t) = -t$ for $r \le t \le R$, and then the small semicircle ρ^{-1} where $\rho(t) = r\,e^{i\,t}$ for $0 \le t \le \pi$. Then $\int_\alpha f = \int_\lambda f + \int_\sigma f - \int_\kappa f - \int_\rho f$ and we have

$$\int_\alpha f = 0\,,$$

$$\int_\lambda f = \int_r^R \frac{e^{i\,t}}{t}\,dt = \int_r^R \frac{\cos t}{t}\,dt + i\int_r^R \frac{\sin t}{t}\,dt\,,$$

$$\int_\kappa f = -\int_r^R \frac{e^{-i\,t}}{t}\,dt = -\int_r^R \frac{\cos t}{t}\,dt + i\int_r^R \frac{\sin t}{t}\,dt\,,$$

$$\int_\lambda f - \int_\kappa f = 2i\int_r^R \frac{\sin t}{t}\,dt \longrightarrow 2i\,I \text{ as } r\to 0 \text{ and } R\to\infty\,,$$

$$\left|\int_\sigma f\right| = \left|\int_0^\pi \frac{e^{iRe^{it}}\,i\,Re^{it}}{Re^{it}}\,dt\right| \le \int_0^\pi e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt$$

$$\le 2\int_0^{\pi/2} e^{-(2R/\pi)t}\,dt = \Big[-\tfrac{\pi}{R}e^{-(2R/\pi)t}\Big]_0^{\pi/2} = \tfrac{\pi}{R}\big(1-e^{-R}\big) \le \tfrac{\pi}{R}$$

$$\longrightarrow 0 \text{ as } R\to\infty\,, \text{ and}$$

$$\int_\rho f = \int_0^\pi \frac{e^{i\,re^{it}}\,i\,re^{it}}{r\,e^{i\,t}}\,dt = \int_0^\pi i\,e^{-r\sin t + i\,r\cos t}\,dt \longrightarrow i\pi \text{ as } r\to 0\,,$$

where we used the fact that $e^{i\,Re^{it}} = e^{-R\sin t + i\,R\cos t}$ and $e^{i\,re^{it}} = e^{-r\sin t + i\,r\cos t}$. It follows that $2i\,I - i\pi = 0$, so $I = \frac{\pi}{2}$.

7.29 Example: Find $I = \displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx$.

Solution: Write $I = \displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx$. Let $f(z) = \dfrac{\log z}{z^2+1}$ and let $F(z) = \dfrac{\log z}{z+i}$, where $\log z$ is the branch of the logarithm given by $\log z = \ln|z| + i\,\theta(z)$ with $-\frac{\pi}{2} < \theta(z) < \frac{3\pi}{2}$, and let α be the loop which follows first the line λ given by $\lambda(t) = t$ for $r < t < R$, then the large semicircle σ given by $\sigma(t) = R\,e^{i\,t}$ for $0 \le t \le R$, then the line κ^{-1} where $\kappa(t) = -t$ for $r \le t \le R$, and then the small semicircle ρ^{-1} where $\rho(t) = r\,e^{i\,t}$ for $0 \le t \le \pi$. Then $\int_\alpha f = \int_\lambda f + \int_\sigma f - \int_\kappa f - \int_\rho f$ and we have

$$\begin{aligned}
\int_\alpha f &= \int_\alpha \frac{F(z)\,dz}{z-i} = 2\pi\,i\,F(i) = 2\pi\,i\,\frac{i\,\frac{\pi}{2}}{2i} = i\,\tfrac{\pi^2}{2}\,,\\
\int_\lambda f &= \int_r^R \frac{\ln t}{t^2+1}\,dt \longrightarrow I \text{ as } r\to 0 \text{ and } R\to\infty\,,\\
\int_\kappa f &= -\int_r^R \frac{\ln t + i\,\pi}{t^2+1}\,dt = -\int_r^R \frac{\ln t}{t^2+1}\,dt - i\int_r^R \frac{\pi\,dt}{t^2+1}\\
&\qquad \longrightarrow -(I+i\,J) \text{ as } r\to 0 \text{ and } R\to\infty \text{ , where } J = \int_0^\infty \frac{\pi\,dx}{x^2+1}\,,\\
\left|\int_\sigma f\right| &= \left|\int_0^\pi \frac{(\ln R + i\,t)\,i\,Re^{it}\,dt}{(Re^{it})^2+1}\,dt\right| \le \int_0^\pi \frac{(\ln R+\pi)R}{R^2-1}\,dt = \frac{\pi(\ln R+\pi)R}{R^2-1}\\
&\qquad \longrightarrow 0 \text{ as } R\to\infty \text{ , and}\\
\left|\int_\rho f\right| &= \left|\int_0^\pi \frac{(\ln r + i\,t)\,i\,re^{it}}{(re^{it})^2+1}\,dt\right| \le \int_0^\pi \frac{(\ln r+\pi)r}{1-r^2}\,dt = \frac{\pi(\ln r+\pi)r}{1-r^2}\\
&\qquad \longrightarrow 0 \text{ as } r\to 0\,.
\end{aligned}$$

It follows that $I + i\,J = i\,\frac{\pi^2}{2}$ so $I = 0$. Incidentally, we also find that $J = \frac{\pi^2}{2}$.

7.30 Example: Find $I = \displaystyle\int_0^\infty \frac{\ln x}{(x^2+1)^3}\,dx$.

Solution: Write $I = \displaystyle\int_0^\infty \frac{\ln x\;dx}{(x^2+1)^3}$ and $J = \displaystyle\int_0^\infty \frac{\pi\;dx}{(x^2+1)^3}$. Let $f(z) = \dfrac{\log z}{(z^2+1)^3}$ and $F(z) = \dfrac{\log z}{(z+i)^3}$, where $\log z$ is the branch of the logarithm given by $\log z = \ln|z| + i\,\theta(z)$ with $-\frac{\pi}{2} < \theta(z) < \frac{3\pi}{2}$. Note that

$$\begin{aligned}
F'(z) &= \frac{\frac{1}{z}\,(z+i)^3 - 3(\log z)(z+i)^2}{(z+i)^6} = \frac{(z+i) - 3z\log z}{z(z+i)^4}\ \text{, and}\\
F''(z) &= \frac{(1 - 3\log z - 3)(z)(z+i)^4 - \big((z+i) - 3z\log z\big)\big((z+i)^4 + 4z(z+i)^3\big)}{z^2(z+i)^8}\\
&= -\frac{(2+3\log z)(z)(z+i) + (z+i-3z\log z)(5z+i)}{z^2(z+i)^5}\,.
\end{aligned}$$

Let α, λ, σ, κ and ρ be as in the previous two examples. Then

$$\int_\alpha f = 2\pi i\,\frac{F''(i)}{2!} = -\pi i\cdot\frac{\left(2+i\,\frac{3\pi}{2}\right)(i)(2i)+\left(2i+\frac{3\pi}{2}\right)(6i)}{-(2i)^5} = \cdots = -\tfrac{\pi}{2}+i\,\tfrac{3\pi^2}{16}\,,$$

$$\int_\lambda f = \int_r^R \frac{\ln t\;dt}{(t^2+1)^3} \longrightarrow I \text{ as } r\to 0 \text{ and } R\to\infty\,,$$

$$\int_\kappa f = -\int_r^R \frac{\ln t + i\pi}{(t^2+1)^3}\,dt \longrightarrow -I-i\,J \text{ as } r\to 0 \text{ and } R\to\infty\,,$$

$$\left|\int_\sigma f\right| = \left|\int_0^\pi \frac{(\ln R+i\,t)\,i\,Re^{it}}{\left((Re^{it})^2+1\right)^3}\,dt\right| \le \frac{\pi(\ln R+\pi)R}{(R^2-1)^3}\,dt \longrightarrow 0 \text{ as } R\to\infty \text{ , and}$$

$$\left|\int_\rho f\right| = \left|\int_0^\pi \frac{(\ln r+i\,t)\,i\,re^{it}}{\left((re^{it})^2+1\right)^3}\,dt\right| \le \frac{\pi(\ln r+\pi)r}{(1-r^2)^3} \longrightarrow 0 \text{ as } r\to 0\,.$$

It follows that $2I+i\,J=-\frac{\pi}{2}+i\,\frac{3\pi^2}{16}$, and so $I=-\frac{\pi}{4}$. We also find that $J=\frac{3\pi^2}{16}$.

7.31 Theorem: *(Morera's Theorem) Let $f:U\subseteq\mathbf{C}\to\mathbf{C}$ be continuous. Suppose that $\int_\alpha f=0$ for every loop α in U. Then f is holomorphic in U.*

Proof: Let $a\in U$. Choose $r>0$ so that $D(a,r)\subseteq U$. Since $D(a,r)$ is convex, the proof of Cauchy's Theorem in a Convex Set shows that f has an antiderivative g in $D(a,r)$ (indeed, g may be defined by $g(z)=\int_\lambda f$ where λ is any path from a to z in U). Since g is holomorphic in $D(a,r)$, $f=g'$ is also holomorphic in $D(a,r)$ by Cauchy's Integral Formula.

7.32 Theorem: *(Liouville's Theorem) If $f:\mathbf{C}\to\mathbf{C}$ is holomorphic and bounded, then f is constant.*

Proof: Suppose that f is holomorphic in $\mathbf{C}$ with $|f(z)|\le M$ for all z. Let a and b be any two distinct points in $\mathbf{C}$. Let $\alpha(t)=a+r|b-a|e^{it}$ for $0\le t\le 2\pi$, where $r>1$. Then

$$\begin{aligned}
\left|f(a)-f(b)\right| &= \left|\frac{1}{2\pi i}\int_\alpha \frac{f(z)}{z-a}-\frac{f(z)}{z-b}\,dz\right| \\
&= \frac{1}{2\pi}\left|\int_\alpha f(z)\frac{a-b}{(z-a)(z-b)}\,dz\right| \\
&\le \frac{1}{2\pi}2\pi\,r|b-a|\,M\,|b-a|\frac{1}{r|b-a|}\,\frac{1}{(r-1)|b-a|} \\
&= \frac{M}{r-1}\to 0 \;\text{ as } r\to\infty\,.
\end{aligned}$$

7.33 Theorem: *(The Fundamental Theorem of Algebra) Every non-constant polynomial has a root in* $\mathbf{C}$.

Proof: Suppose, for a contradiction, that p is a non-constant polynomial with no roots. Since p is a non-constant polynomial, we have $p(z)\to\infty$ as $z\to\infty$, and so we can choose R large enough that when $|z|\ge R$ we have $|p(z)|\ge 1$ and so $1/|p(z)|\le 1$. Note that since p has no roots, $1/p$ is holomorphic in $\mathbf{C}$. In particular, $1/p$ is continuous in $\overline{D}(0,R)$ and so it attains its maximum value. Since $1/p$ is bounded in $\overline{D}(0,R)$ and $|1/p|\le 1$ outside $D(0,R)$, we know that $1/p$ is bounded in $\mathbf{C}$. By Liouville's Theorem, $1/p$ must be a constant. But this implies that p is constant, giving the desired contradiction.

Chapter 8. Power Series

8.1 Definition: A **sequence** of complex numbers is a function $f:\{k,k+1,k+2\ldots\}\to\mathbf{C}$ where $k\in\mathbf{Z}$. We usually write $f(n)$ as a_n and we denote the sequence f by $\{a_n\}_{n\geq k}$ or simply by $\{a_n\}$. For $a\in\mathbf{C}$, we say that the sequence $\{a_n\}$ **converges** to a, and we write

$$\lim_{n\to\infty} a_n = a \ \big(\text{ or we write } a_n\to a\big)$$

when for all $\epsilon>0$ there exists $N\in\mathbf{Z}$ such that $n\geq N \Rightarrow a_n\in D(a,\epsilon)$. If the sequence converges to some $a\in\mathbf{C}$, then we say it **converges**, otherwise we say it **diverges**. We say that the sequence $\{a_n\}$ **diverges** to ∞, and write

$$\lim_{n\to\infty} a_n = \infty \ \big(\text{or } a_n\to\infty\big)$$

when for all $R>0$ there exists $N\in\mathbf{Z}$ such that $n\geq N\Rightarrow a_n\notin\overline{D}(0,R)$.

8.2 Example: If $a_n=1/n$ then $a_n\to 0$. If $b_n=2+\left(\frac{1}{2}(1+i)\right)^n$ then $b_n\to 2$. If $c_n=(1+i)^n$ then $c_n\to\infty$. If $d_n=i^n$ then $\{d_n\}$ diverges.

8.3 Theorem: *Let $a_n=x_n+i\,y_n$ and let $a=x+i\,y\in\mathbf{C}$. Then $a_n\to a$ if and only if $x_n\to x$ and $y_n\to y$.*

Proof: Suppose first that $a_n\to a$. Note that $(x_n-x)=\mathrm{Re}\,(a_n-a)$ so $|x_n-x|\leq|a_n-a|$. So given $\epsilon>0$ we choose $N\in\mathbf{Z}$ so that $n\geq N\Rightarrow|a_n-a|<\epsilon$, and then for $n\geq N$ we have $|x_n-x|\leq|a_n-a|<\epsilon$. This shows that $x_n\to x$. Similarly, we can show that $y_n\to y$. Conversely, suppose that $x_n\to x$ and that $y_n\to y$. By the triangle inequality we have $|a_n-a|\leq|x_n-x|+|y_n-y|$. So given $\epsilon>0$ we choose $N\in\mathbf{Z}$ so that $n\geq N\Rightarrow\left(|x_n-x|<\frac{1}{2}\epsilon \text{ and } |y_n-y|<\frac{1}{2}\epsilon\right)$. Then for $n\geq N$ we will have $|a_n-a|\leq|x_n-x|+|y_n-y|<\epsilon$. This shows that $a_n\to a$.

8.4 Theorem: *Let $\{a_n\}$ and $\{b_n\}$ be sequences with $a_n\to a$ and $b_n\to b$ and let $c\in\mathbf{C}$. Then*
(a) $(c\,a_n)\to c\,a$
(b) $(a_n\pm b_n)\to a\pm b$
(c) $(a_nb_n)\to ab$
(d) $(a_n/b_n)\to a/b$, provided that $b\neq 0$ (and hence $b_n\neq 0$ for large n)
(e) $|a_n|\to|a|$
All parts except (c) and (d) hold for sequences in $\mathbf{R}^n$.

Proof: We shall only prove part (c) (the proofs of the other parts are similar). We write $a_n=x_n+i\,y_n$, $a=x+i\,y$, $b_n=u_n+i\,v_n$ and $b=u+i\,v$. We suppose that $a_n\to a$ and $b_n\to b$ so that by Theorem 8.3 we have $x_n\to x$, $y_n\to y$, $u_n\to u$ and $v_n\to v$. We have $a_nb_n=(x_n+i\,y_n)(u_n+i\,v_n)=(x_nu_n-y_nv_n)+i\,(x_nv_n+y_nu_n)$. From our knowlege of sequences of *real* numbers, we know that $(x_nu_n-y_nv_n)\to xu-yv$ and that $(x_nv_n+y_nu_n)\to xv-yu$. By Theorem 8.3 again, we see that

$$a_nb_n=(x_nu_n-y_nv_n)+i\,(x_nv_n+y_nu_n)\to(xu-yv)+i\,(xv+yu)=ab\,.$$

8.5 Definition: We write $\sum\limits_{n=k}^{\infty} a_n$ $\big($ or simply $\sum a_n\big)$ to denote the sequence $\{s_l\}$ where $s_l = \sum\limits_{n=k}^{l} a_n$. This kind of sequence is called a **series**, and the finite sums s_l are called the **partial sums**. We say the series $\sum a_n$ **converges** or **diverges** according to whether the sequence $\{s_l\}$ converges or diverges. We also write $\sum\limits_{n=k}^{\infty} a_n$ to denote the limit of $\{s_l\}$, if it exists, and we call the limit the **sum** of the series. If $s_l \to s$ then we write $\sum\limits_{n=k}^{\infty} a_n = s$. The series $\sum a_n$ is said to **converge absolutely** when the series $\sum |a_n|$ converges.

8.6 Theorem: *(Linearity) Suppose that $\sum a_n$ and $\sum b_n$ converge and let $c \in \mathbf{C}$. Then*

$$\sum_{n=k}^{\infty} c\, a_n = c \sum_{n=k}^{\infty} a_n \quad \textit{and} \quad \sum_{n=k}^{\infty} (a_n + b_n) = \sum_{n=k}^{\infty} a_n \; + \sum_{n=k}^{\infty} b_n \, .$$

Proof: This is immediate from Theorem 8.4.

8.7 Theorem: *(Convergence Tests) Let $\sum a_n$ be a series. Then*

(a) If $\sum a_n$ converges then $|a_n| \to 0$.

(b) If $\sum |a_n|$ converges then $\sum a_n$ converges and $\left| \sum\limits_{n=k}^{\infty} a_n \right| \leq \sum\limits_{n=k}^{\infty} |a_n|$.

(c) (The Ratio Test)

(i) If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$ *then $\sum |a_n|$ converges.*

(ii) If $\exists N \in \mathbf{Z}$ s.t. $n \geq N \Rightarrow \left| \dfrac{a_{n+1}}{a_n} \right| \geq 1$ then $|a_n| \not\to 0$ and so $\sum a_n$ diverges.

(d) (The Root Test)

(i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$ *then $\sum |a_n|$ converges.*

(ii) If $\exists N \in \mathbf{Z}$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} \geq 1$ then $|a_n| \not\to 0$ and so $\sum a_n$ diverges.

Proof: We shall only prove the ratio test here. Suppose first that $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = p < 1$. Choose r with $p < r < 1$. Choose N such that for $n \geq N$ we have $\left| \dfrac{a_{n+1}}{a_n} \right| \leq r$. Then we have $|a_{N+1}| \leq r|a_N|$, $a_{N+2} \leq r|a_{N+1}| \leq r^2|a_N|$, $|a_{N+3}| \leq r|a_{N+2}| \leq r^3|a_N|$ and so on, and so $|a_n| \leq r^{n-N}|a_N|$ for all $n \geq N$. Since $0 < r < 1$, the real-valius geometric series $\sum |a_N| r^{n-N}$ converges, and so $\sum |a_n|$ converges by the comparison test for series of positive real numbers.

On the other hand, if we suppose that there exists $N \in \mathbf{Z}$ such that for $n \geq N$ we have $a_n \neq 0$ and $\left| \dfrac{a_{n+1}}{a_n} \right| \geq 1$ then we have $|a_N| \leq |a_{N+1}| \leq |a_{N+2}| \leq \cdots$ and so $|a_n| \not\to 0$.

8.8 Example: The sum $\sum\limits_{n=0}^{\infty} \dfrac{1}{(n+i)^2}$ converges by part (b) since for $n \geq 2$ we have $|n+i| \geq n-1$ so $\left| \dfrac{1}{(n+i)^2} \right| \leq \dfrac{1}{(n-1)^2}$, and we know that $\sum \dfrac{1}{(n-1)^2}$ converges.

8.9 Definition: A **power series centred at** $a \in \mathbf{C}$ is a series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$, with $c_n \in \mathbf{C}$ where, by convention, we take $(z-a)^0 = 1$. A power series is a series for each value of $z \in \mathbf{C}$. It will converge for some values of z and diverge for others.

8.10 Example: The geometric series $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$ is a power series centred at $a = 0$. Its partial sums are given by $s_l = \sum_{n=0}^{l} z^n = \dfrac{1 - z^{l+1}}{1-z}$. For $|z| < 1$ we have $z^l \to 0$ as $l \to \infty$ and so $s_l \to \dfrac{1}{1-z}$ hence $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$. On the other hand, for $|z| \geq 1$ we have $|z^n| \geq 1$ for all n so $|z^n| \not\to 0$ and hence $\sum_{n=0}^{\infty} z^n$ diverges.

8.11 Theorem: *Let* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *be a power series.*

(a) There exists a number R *with* $0 \leq R \leq \infty$, *called the* **radius of convergence** *of the power series, such that*

(i) if $|z-a| < R$ *then* $\sum c_n(z-a)^n$ *converges absolutely.*

(ii) if $|z-a| > R$ *then* $|c_n(z-a)^n| \not\to 0$ *and so* $\sum c_n(z-a)^n$ *diverges.*

(b) The power series $\sum n\, c_n(z-a)^{n-1}$ *has the same radius of convergence* R.

(c) When $R > 0$ *then the function* f *defined by* $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ *for* $z \in D(a,R)$ *is holomorphic with* $f'(z) = \sum_{n=1}^{\infty} n\, c_n(z-a)^{n-1}$ *and* $\int f(z)\, dz = \sum_{n=0}^{\infty} \dfrac{1}{n+1} c_n(z-a)^{n+1}$.

(d) When $R > 0$, *the above function* $f(z)$ *has derivatives of all orders and the coefficients* c_n *are given by* $c_n = \dfrac{f^{(n)}(a)}{n!}$, *so we have* $f(z) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(z-a)^n$.

(e) When $R > 0$, *if* $\sum b_n(z-a)^n = \sum c_n(z-a)^n$ *for all* $z \in D(a,R)$ *then* $b_n = c_n$ *for all* n.

Proof: We shall give the proof in the case that $a = 0$.

To prove part (a), we shall show that if $\sum_{n=0}^{\infty} c_n w^n$ converges, where $w \in \mathbf{C}$ then $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely for all z with $|z| < |w|$. Fix $w \in \mathbf{C}$, suppose that $\sum c_n w^n$ converges, and let $z \in \mathbf{C}$ with $|z| < |w|$. Since $\sum c_n w^n$ converges, we know that $|c_n w^n| \to 0$ as $n \to \infty$ and so we can choose $M > 0$ so that $M \geq |c_n w^n|$ for all n. Then we have

$$|c_n z^n| = \left| c_n w^n \frac{z^n}{w^n} \right| = |c_n w^n| \left| \frac{z^n}{w^n} \right| \leq M \left| \frac{z}{w} \right|^n .$$

Since $\left|\frac{z}{w}\right| < 1$, the series $\sum M \left|\frac{z}{w}\right|$ converges and hence the series $\sum |c_n z^n|$ converges too by the comparison test (for series of positive real terms). The radius of convergence is $R = \sup\left\{ |w| \big| w \in \mathbf{C}, \sum c_n w^n \text{ converges} \right\}$. If $R = \infty$ then the series converges for all z.

Next we prove part (b). Let R be the radius of convergence of the series $\sum c_n z^n$ and let S be the radius of convergence of the series $\sum n\, c_n z^{n-1}$. First we show that $R \geq S$. If $S \neq 0$ then let z be any point with $|z| < S$. Then by part (a), the series $\sum |n\, c_n z^{n-1}|$ converges, and so $\sum |c_n z^{n-1}| = \sum \frac{1}{n} |n\, c_n z^{n-1}|$ also converges by comparison, and hence $\sum |c_n z^n| = |z| \sum |c_n z^{n-1}|$ also converges. This implies that $R \geq |z|$. Since z was arbitrary, we have $R \geq S$.

It is a bit harder to show that $R \leq S$. If $R \neq 0$ then let z be any point with $0 < |z| < R$. Choose $\rho > 0$ with $|z| < \rho < R$. We have $|n\, c_n z^{n-1}| = \frac{n}{|z|}\big(|z|/\rho\big)^n |c_n \rho^n|$. The series (of positive real terms) $\sum n\big(|z|/\rho\big)^n$ converges by the Ratio Test, so we know that $n\big(|z|/\rho\big)^n \to 0$ and hence we can choose $M > 0$ so that $M \geq n\big(|z|/\rho\big)^n$ for all n. Then we have $|n\, c_n z^{n-1}| \leq \frac{M}{|z|}\,|c_n \rho^n|$. Since $\rho < R$ we know that the series $\sum |c_n \rho^n|$ converges, so the series $\sum \frac{M}{|z|}|c_n \rho^n| = \frac{M}{|z|}\sum |c_n \rho^n|$ also converges, and hence the series $\sum |n\, c_n z^{n-1}|$ also converges by comparison. Thus $S \geq |z|$, and since z was arbitrary, $S \geq R$.

Now we prove part (c). Let $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ and let $g(z) = \sum\limits_{n=1}^{\infty} n\, c_n z^{n-1}$ for all $z \in D(0,R)$. We claim that $f'(z) = g(z)$. Given $z \in D(0,R)$ choose $r > 0$ with $|z| < r < R$. Then for $|w| < r$ we have

$$\begin{aligned}
\left|\frac{f(w)-f(z)}{w-z} - g(z)\right| &= \left|\frac{\sum\limits_{n=0}^{\infty} c_n w^n - \sum\limits_{n=0}^{\infty} c_n z^n}{w-z} - \sum_{n=0}^{\infty} n\, c_n z^{n-1}\right| \\
&= \left|\sum_{n=1}^{\infty} \frac{c_n(w^n - z^n)}{w-z} - \sum_{n=1}^{\infty} n\, c_n z^{n-1}\right| = \left|\sum_{n=2}^{\infty} c_n\Big(\frac{w^n - z^n}{w-z} - n\, z^{n-1}\Big)\right| \\
&= \left|\sum_{n=2}^{\infty} c_n\Big(w^{n-1} + w^{n-2}z + \cdots + w\, z^{n-2} + z^{n-1} - n\, z^{n-1}\Big)\right| \\
&= \left|\sum_{n=2}^{\infty} c_n(w-z)\Big(w^{n-2} + 2\, w^{n-3}z + 3\, w^{n-4}z^2 + \cdots + (n-1)z^{n-2}\Big)\right| \\
&\leq \sum_{n=2}^{\infty} |c_n||w-z|\Big(|w|^{n-2} + 2\,|w|^{n-3}|z| + \cdots + (n-1)|z|^{n-2}\Big) \\
&\leq \sum_{n=2}^{\infty} |c_n||w-z|\big(1 + 2 + \cdots + (n-1)\big) r^{n-2} \\
&= \frac{|w-z|}{2}\sum_{n=2}^{\infty} n(n-1)\,|c_n|\,r^{n-2}\,.
\end{aligned}$$

But notice that by part (b), the series $\sum c_n z^n$, $\sum n\, c_n z^{n-1}$ and $\sum n(n-1)\, c_n z^{n-2}$ all have the same radius of convergence R and so since $r < R$ we know that $\sum n(n-1)|c_n|\, r^{n-2}$ converges. Thus $|w-z|\,\frac{1}{2}\sum n(n-1)|c_n| r^{n-2} \to 0$ as $w \to z$. This proves that $f'(z) = g(z)$.

To complete the proof of part (c), note that by part (b) the power series $\sum \frac{1}{n+1} c_n z^{n+1}$ has the same radius of convergence R, and by our above proof that $f' = g$, the function h defined by $h(z) = \sum\limits_{n=0}^{\infty} \frac{1}{n+1} c_n z^{n+1}$ is holomorphic in $D(a,R)$ with $h' = f$.

Part (d) follows from part (c). Indeed, if $f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ then we have $f'(z) = c_1 + 2\, c_2 z + 3\, c_3 z^2 + 4\, c_4 z^3 + \cdots$, $f''(z) = 2\cdot 1\, c_2 + 3\cdot 2\, c_3 z + 4\cdot 3\, c_4 z^2 + 5\cdot 4\, c_5 z^3 + \cdots$ and $f'''(z) = 3\cdot 2\cdot 1\, c_3 + 4\cdot 3\cdot 2\, c_4 z + 5\cdot 4\cdot 3\, c_5 z^2 + \cdots$ and so on, and we have $f(0) = c_0$, $f'(0) = 1\, c_1$, $f''(0) = 2!\, c_2$, $f'''(0) = 3!\, c_3$ and so on. Using induction you can show that $f^{(n)}(0) = n!\, c_n$.

Finally, note that part (e) follows immediately from part (d).

8.12 Theorem: *(Taylor's Theorem) If $f(z)$ is holomorphic in $D(a,R)$ and $0<r<R\leq\infty$ then*

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \quad \text{where} \quad c_n=\frac{f^{(n)}(a)}{n!}=\frac{1}{2\pi i}\int_\sigma \frac{f(z)}{(z-a)^{n+1}}\,dz\,,$$

and where σ is the circle $\sigma(t)=a+r\,e^{i\,t}$ with $0\leq t\leq 2\pi$.

Proof: We give the proof in the case that $a=0$. Fix $z\in D(0,R)$ and choose r with $|z|<r<R$. Then by Cauchy's integral formula,

$$\begin{aligned}
f(z)&=\frac{1}{2\pi\,i}\int_\sigma \frac{f(w)}{w-z}\,dw\\
&=\frac{1}{2\pi\,i}\int_\sigma f(w)\,\frac{1}{w}\,\frac{1}{1-(z/w)}\,dw\\
&=\frac{1}{2\pi\,i}\int_\sigma f(w)\,\frac{1}{w}\left(1+\frac{z}{w}+\left(\frac{z}{w}\right)^2+\cdots+\left(\frac{z}{w}\right)^{N-1}+\frac{(z/w)^N}{1-(z/w)}\right)dw\\
&=\sum_{n=0}^{N-1}\frac{1}{2\pi\,i}\int_\sigma \frac{f(w)}{w^{n+1}}z^n\,dw\ +\frac{1}{2\pi\,i}\int_\sigma\frac{f(w)(z/w)^N}{w-z}\,dw\\
&=\sum_{n=0}^{N-1}\frac{f^{(n)}(0)}{n!}\,z^n\ +R_N\,,
\end{aligned}$$

where $R_N=\dfrac{1}{2\pi\,i}\displaystyle\int_\sigma\frac{f(w)(z/w)^N}{w-z}\,dw$. Setting $M=\max\limits_{w=\sigma(t)}|f(w)|$, the estimation theorem gives $|R_N|\leq\dfrac{1}{2\pi}\dfrac{M(|z|/r)^N}{(r-|z|)}\,2\pi r$. Since $|z|<r$, we have $R_N\to 0$ as $N\to\infty$

8.13 Example: The elementary complex functions have the same derivative formulas as their real counterparts, and so they have the same Taylor series centred at the origin (or centred at any real number). For all $z\in\mathbf{C}$ we have

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\qquad \cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$

$$\sinh z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\qquad \cosh z=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$$

For $|z|<1$ we have

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\qquad \frac{1}{1+z}=\sum_{n=0}^{\infty}(-1)^n\,z^n\qquad \frac{1}{1+z^2}=\sum_{n=0}^{\infty}(-1)^n\,z^{2n}$$

When $|z|<1$, the principal branch of logarithm and inverse tangent are given by

$$\log(1-z)=-\sum_{n=1}^{\infty}\frac{z^n}{n}\qquad \log(1+z)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}$$

$$\tan^{-1}(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1}$$

For $|z| < 1$ and for $a \in \mathbf{R}$, the principal branch of $(1+z)^a$ is given by

$$(1+z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n = 1 + a\,z + \frac{a(a-1)}{2!}\,z^2 + \frac{a(a-1)(a-2)}{3!}\,z^3 + \cdots$$

This last power series is called the **Binomial series**.

8.14 Note: We should point out two important differences between Taylor series of complex functions and Taylor series of real functions. The first difference is that holomorphic functions are always equal to their Taylor series. This is not the case for real $\mathcal{C}^\infty$ functions. The standard example is the real function $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$. This function is $\mathcal{C}^\infty$ at $x = 0$, but all its derivatives vanish so its Taylor series is equal to 0. The second difference we would like to mention is that a real function might be $\mathcal{C}^\infty$ in a large interval while its Taylor series might converge only in a small interval, but notice that if a function is holomorphic in an open disc, then its Taylor series will converge in the entire disc. An example which illustrates this difference is the real function $f(x) = 1/(1+x^2)$. This function is $\mathcal{C}^\infty$ for all x, but its Taylor series only converges for $|x| < 1$. The reason for this is that when we extend f to the complex numbers, so $f(z) = 1/(1+z^2)$, then we find that $f(z) = \dfrac{1}{(z-i)(z+i)}$ so that f is holomorphic in $\mathbf{C} \setminus \{\pm i\}$. The radius of convergence is equal to 1 because the disc $D(0,1)$ is the largest disc (centred at 0) which can be contained in the domain of $f(z)$.

8.15 Note: If f and g are both holomorphic at a then The product fg will also be holomorphic at a. The coefficients of the Taylor series of fg at a are given by $(fg)^{(n)}(a)/n!$, and so they can be computed, using the product rule, from the coefficients of the Taylor series for f and for g. One can show that the Taylor series at a for fg is obtained from the Taylor series at a of f and of g by multiplying the power series together as if they were polynomials. We have

$$\left(\sum_{n=0}^{\infty} a_n (z-a)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-a)^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right) z^n$$

Also, if f and g are holomorphic at a and $g(a) \neq 0$, then we can solve the equation $hg = f$ for h to obtain the Taylor series of $h = f/g$ centred at a from the Taylor series of f and of g. This is equivalent to calculating f/g using long division as if the power series were polynomials.

Also, if f is holomorphic at a and g is holomorphic at $b = f(a)$ then the composite $g \circ f$ is holomorphic at a and hence has a Taylor series centred at a. Using the chain rule, one can show that the Taylor series for $g \circ f$ at a can be computed by composing the Taylor series of g at b with that of f at a as if the power series were polynomials.

8.16 Example: Find the Taylor series at 0 for $f(z) = \dfrac{1}{(1-z)^2}$.

Solution: We give several solutions. But first we note that since $f(z)$ is holomorphic in $\mathbf{C} \setminus \{1\}$, we know that the Taylor series at 0 converges in $D(0,1)$.

For our first solution, we calculate the derivatives: $f(z) = (1-z)^{-2}$, $f'(z) = 2(1-z)^{-3}$, $f''(z) = 3!(1-z)^{-4}$, and so on. So $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$ and so on. Thus

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Our second solution uses the Binomial series:

$$\begin{aligned} f(z) &= (1-z)^{-2} = 1 + \frac{-2}{1!}(-z)^1 + \frac{(-2)(-3)}{2!}(-z)^2 + \frac{(-2)(-3)(-4)}{3!}(-z)^3 + \cdots \\ &= 1 + 2z + 3z^2 + \cdots \end{aligned}$$

Our third solution is to differentiate both sides of $\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ to obtain

$$f(z) = 0 + 1 + 2z + 3z^2 + \cdots$$

Our fourth solution is to mutiply the Taylor series for $\dfrac{1}{1-z}$ by itself as if it was a polynomial to obtain

$$\begin{aligned} f(z) &= (1 + z + z^2 + z^3 + \cdots)(1 + z + z^2 + z^3 + \cdots) \\ &= 1 + (1+1)z + (1+1+1)z^2 + (1+1+1+1)z^3 + \cdots \\ &= 1 + 2z + 3z^2 + 4z^3 + \cdots \end{aligned}$$

8.17 Example: Find the Taylor series for $f(z) = e^z/(1-z)$.

Solution: We have $f(z) = e^z \, \frac{1}{1-z} = \Big(\sum\limits_{n=0}^{\infty} \frac{1}{n!} z^n \Big) \Big(\sum\limits_{n=0}^{\infty} z^n \Big) = \sum\limits_{n=0}^{\infty} \big(\sum\limits_{i=0}^{n} \frac{1}{n!} \big) z^n$. We can write out the first few terms: $f(z) = 1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + \frac{65}{24}z^4 + \cdots$.

8.18 Example: Find the first few terms of the Taylor series about 0 for $f(z) = \tan z$.

Solution: We have $\tan z = \dfrac{\sin z}{\cos z}$. We can use long division:

$$\begin{array}{r|l} & z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots \\ \hline 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots & z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots \\ & z - \frac{1}{2}z^3 + \frac{1}{24}z^5 - \cdots \\ \hline & \frac{1}{3}z^3 - \frac{1}{30}z^5 + \cdots \\ & \frac{1}{3}z^3 - \frac{1}{6}z^5 + \cdots \\ \hline & \frac{2}{15}z^5 + \cdots \end{array}$$

We find that $f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$. We can also easily find the radius of convergence. Since $\cos z = 0 \iff z = \frac{\pi}{2} + \pi k$ for some $k \in \mathbf{Z}$, we know that $f(z)$ is holomorphic for $z \neq \frac{\pi}{2} + \pi k$, so the radius of convergence is $R = \frac{\pi}{2}$.

8.19 Example: Find the Taylor series centred at $2i$ for $f(z) = \frac{1}{z}$.

Solution: $f(z) = \frac{1}{z} = \frac{1}{z - 2i + 2i} = \frac{1}{2i} \frac{1}{1 + \frac{z-2i}{2i}} = -\frac{i}{2} \frac{1}{1 - \frac{i(z-2i)}{2}} = -\frac{i}{2} \sum_{n=0}^{\infty} \Big(\frac{i(z-2i)}{2}\Big)^n$
$= \sum_{n=0}^{\infty} -\Big(\frac{i}{2}\Big)^{n+1} (z-2i)^n$. The disc of convergence is $D(2i, 2)$.

8.20 Theorem: *(The Local Identity Theorem) Let f and g be holomorphic in the disc $D(a, r)$. Let $\{a_n\}$ be a sequence in $D^*(a, r)$ with $a_n \to a$. If $f(a_n) = g(a_n)$ for all n then $f(z) = g(z)$ for all $z \in D(a, r)$.*

Proof: Suppose that $f(a_n) = g(a_n)$ for all n. Let $h = f - g$. Then $h(a_n) = 0$ for all n. Since h is continuous at a, we have $h(a) = 0$. Since its holomorphic it is equal to its Taylor series $h(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$. We want to show that all the coefficients c_n are zero. Suppose not, and say m is the smallest integer such that $c_m \neq 0$. Let $k(z) = h(z)(z-a)^{-m}$ Then we have $k(z) = c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^{m+2} + \cdots$, so $k(z)$ is holomorphic in $D(a, r)$ and $k(a) = c_m \neq 0$. Since $k(z)$ is continuous with $k(a) \neq 0$, we can find $s > 0$ such that $k(z) \neq 0$ for all $z \in D(a, s)$. But since $(z-a)^m \neq 0$ in $D^*(a, s)$ and since $h(z) = k(z)(z-a)^m$, this would imply that $h(z) \neq 0$ in $D^*(a, s)$. This gives us a contradiction since we assumed that $h(a_n) = 0$ for all n.

8.21 Theorem: *(The Identity Theorem) Let $U \subseteq \mathbf{C}$ be a connected open set. Let $f, g : U \to \mathbf{C}$ be holomorphic in U. Let $A = \big\{z \in U \big| f(z) = g(z)\big\}$. Suppose that A has a limit point in U. Then $f(z) = g(z)$ for all $z \in U$.*

Proof: Let $h = f - g$ so that h is holomorphic in U and $A = h^{-1}(0) = \big\{z \in U \big| h(z) = 0\big\}$. We must show that $h(z) = 0$ for all $z \in U$, or equivalently that $A = U$. Let V be the set of all limit points of A which lie in U. Note that $V \neq \emptyset$ since A has a limit point in U. Note that $V \subseteq A$ since for $a \in U$, if $a \notin A$, that is if $h(a) \neq 0$, then since h is continuous, we can choose $r > 0$ so that $h(z) \neq 0$ for all $z \in D(a, r)$, and we see that a is not a limit point of A, that is $a \notin V$. Note that $U \setminus V$ is open since if $a \in U \setminus V$, that is if a is not a limit point of A, then we can choose $r > 0$ so that $D^*(a, r)$ is disjoint from A, and we see that each $z \in D(a, r)$ is not a limit point of A so we have $D(a, r) \subseteq U \setminus V$. Finally note that V is open because given $a \in V$ we can choose $r > 0$ so that $D(a, r) \subseteq U$ and then, by the Local Identity Theorem, we have $h(z) = 0$ for all $z \in D(a, r)$ so that $D(a, r) \subseteq V$. It follows that $V = U$ (otherwise the open sets V and $U \setminus V$ would separate U) and hence $A = U$ (since $V \subseteq A$).

8.22 Lemma: *Let f be holomorphic with $|f|$ constant in the disc $D(a, r)$. Then f is constant.*

Proof: Say $|f(z)| = c$ for all $z \in D(a, r)$. Let $u = \mathrm{Re}\,(f)$ and $v = \mathrm{Im}\,(f)$ so that $f = u + iv$. Then we have $u^2 + v^2 = c^2$. Differentiate with respect to x and to y to get the two equations $u\,u_x + v\,v_x = 0$ and $u\,u_y + v\,v_y = 0$. The Cauchy Riemann Equations then give $u\,u_x - v\,u_y = 0$ and $v\,u_x + u\,u_y = 0$. Eliminating u_y from these two equations gives $(u^2 + v^2)\,u_x = 0$, that is $c^2 u_x = 0$. Eliminating u_x gives $c^2 u_y = 0$. If $c = 0$ then we have $|f(z)| = c = 0$ so $f(z) = 0$ for all $z \in D(a, r)$. If $c \neq 0$ then we obtain $u_x = 0$ and $u_y = 0$ for all $x + iy \in D(a, r)$, and hence f is constant.

8.23 Theorem: *(The Local Maximum Modulus Theorem) Let f be holomorphic in the disc $D(a,r)$. Suppose that $\left|f(z)\right| \le \left|f(a)\right|$ for all $z \in D(a,r)$. Then f is constant.*

Proof: Fix s with $0 < s < r$. Let $\sigma(t) = a + s\,e^{i\,t}$ for $0 \le t \le 2\pi$. By Cauchy's Integral Formula, we have

$$f(a) = \frac{1}{2\pi\,i}\int_\sigma \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi\,i}\int_0^{2\pi} \frac{f(a+s\,e^{i\,t})\,i\,s\,e^{i\,t}}{s\,e^{i\,t}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(a+s\,e^{i\,t})\,dt\,.$$

Since $|f(z)| \le |f(a)|$ for all $z \in D(a,r)$, the Estimation Theorem gives

$$\left|f(a)\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f(a+s\,e^{i\,t})\right|dt \le \left|f(a)\right|$$

and so we must have $|f(a)| = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f(a+s\,e^{i\,t})\right|dt$. It follows that

$$\int_0^{2\pi} \Big(\left|f(a)\right| - \left|f(a+s\,e^{i\,t})\right|\Big)dt = 0\,.$$

Since the integrand is continuous and non-negative, it must be identically zero so we have $\left|f(a+s\,e^{i\,t})\right| = \left|f(a)\right|$ for all t, that is $|f(z)| = |f(a)|$ for all z with $|z-a| = s$. Since s was fixed but arbitrary, we have $|f(z)| = |f(a)|$ for all $z \in D(a,r)$. By the above lemma, it follows that f is constant.

8.24 Theorem: *(The Maximum Modulus Theorem) Let $U \subseteq \mathbf{C}$ be a bounded connected open set. Let $f : \overline{U} \to \mathbf{C}$ be holomorphic in U and continuous on $\overline{U}$. Then $|f|$ attains its maximum on ∂U.*

Proof: Since $|f|$ is continuous on $\overline{U}$ and $\overline{U}$ is compact, $|f|$ attains its maximum on $\overline{U}$. Since $\overline{U} = U \cup \partial U$, $|f|$ attains its maximum either on ∂U or on U. Consider the case that $|f|$ attains its maximum at a point $a \in U$. Choose $r > 0$ small enough so that $D(a,r) \subseteq U$. By the Local Maximum Modulus Theorem, f is constant in $D(a,r)$. By the Identity Theorem, f is constant in U. Since f is constant in U and continuous on $\overline{U}$, it is constant on $\overline{U}$. Since f is constant, it attains its maximum at all points, including points in ∂U.

Chapter 9. Laurent Series and Residues

9.1 Note: We have studied power series. We are also interested in series of the form

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=-\infty}^{-1} c_n(z-a)^n + \sum_{n=0}^{\infty} c_n(z-a)^n c = \sum_{n=1}^{\infty} c_{-n} w^n + \sum_{n=0}^{\infty} c_n(z-a)^n \,.$$

where we have written $w = 1/(z-a)$. If the first series has radius of convergence $1/R$ and the second has radius of convergence S, then the first converges when $|w| < 1/R$, that is when $|z-a| > R$, and the second converges for $|z-a| < S$. They both converge in the annulus $A = \left\{ z \in \mathbf{C} \middle| R < |z-a| < S \right\}$. The next theorem shows that every function which is holomorphic in an annulus can be expressed as a series of this form.

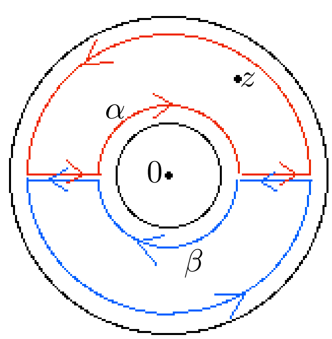
9.2 Theorem: *(Laurent's Theorem) Let $0 \leq R < \rho < S \leq \infty$ and let $a \in \mathbf{C}$. Suppose that f is holomorphic in the annulus $A = \left\{ z \in \mathbf{C} \middle| R < |z-a| < S \right\}$. Then for all $z \in A$,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad \text{where} \quad c_n = \frac{1}{2\pi i} \int_\sigma \frac{f(z)}{(z-a)^{n+1}}\, dz \,,$$

where σ is the circle $\sigma(t) = a + \rho\, e^{i\,t}$ with $0 \leq t \leq 2\pi$. In particular, we have

$$c_{-1} = \frac{1}{2\pi i} \int_\sigma f(z)\, dz \,.$$

Proof: To simplify notation, we take $a = 0$, so $A = \{z | R < |z| < S\}$. For $z \in A$ pick r and s so that $R < r < |z| < s < S$. Again to simplify notation, suppose that $\operatorname{Im}(z) > 0$. Let α be the loop in A which follows the semicircle counterclockwise from s to $-s$, then the line segment from $-s$ to $-r$, then the semicircle clockwise from $-r$ to r, and then the line segment from r to s. Let β be the loop which follows the line segment from s to r, then the semicircle clockwise from r to $-r$, then the line segment from $-r$ to $-s$, and then the semicircle counterclockwise from $-s$ to s.



Since $\eta(\alpha, z) = 1$ and $\eta(\beta, z) = 0$, Cauchy's theorem tells us that $\displaystyle\int_\alpha \frac{f(w)}{w-z}\, dw = 2\pi i\ f(z)$ and $\displaystyle\int_\beta \frac{f(w)}{w-z}\, dw = 0$. Also, since the integrals along the line segments cancel, we have

$\displaystyle\int_\alpha \frac{f(w)}{w-z}\,dw + \int_\beta \frac{f(w)}{w-z}\,dw = \int_{\sigma_s} \frac{f(w)}{w-z}\,dw - \int_{\sigma_r} \frac{f(w)}{w-z}\,dw$, where σ_r and σ_s are the circles $\sigma_r(t) = r\,e^{i\,t}$ and $\sigma_s(t) = s\,e^{i\,t}$ for $0 \le t \le 2\pi$. So we have

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\left(\int_{\sigma_s} f(w)\,\frac{1}{w-z}\,dw - \int_{\sigma_r} f(w)\,\frac{1}{w-z}\,dw\right) \\
&= \frac{1}{2\pi i}\left(\int_{\sigma_s} f(w)\,\frac{1}{w}\,\frac{1}{1-\frac{z}{w}}\,dw - \int_{\sigma_r} f(w)\,\frac{-1}{z}\,\frac{1}{1-\frac{w}{z}}\,dw\right) \\
&= \frac{1}{2\pi i}\left(\int_{\sigma_s} f(w)\sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}\,dw + \int_{\sigma_r} f(w)\sum_{m=0}^{\infty}\frac{w^m}{z^{m+1}}\,dw\right) \\
&= \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}\int_{\sigma_s}\frac{f(w)\,z^n}{w^{n+1}}\,dw + \sum_{m=0}^{\infty}\int_{\sigma_r}\frac{f(w)w^m}{z^{m+1}}\,dw\right) \\
&= \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}\Big(\int_{\sigma}\frac{f(w)}{w_{n+1}}\,dw\Big)\,z^n + \sum_{n=-\infty}^{-1}\Big(\int_{\sigma}\frac{f(w)}{w^{n+1}}\,dw\Big)\,z^n\right) \\
&= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\left(\int_{\sigma}\frac{f(w)}{w^{n+1}}\,dw\right)z^n
\end{aligned}
$$

In the second last equlity, we replaced m by $-n-1$, and we used the fact that each of the loops σ_s and σ_r is homotopic to σ in A. The interchange of summation and integration in the third equality should be justified. We can justify it as follows. For any positive integer N we have

$$
\begin{aligned}
\int_{\sigma_s} f(w)\,\frac{1}{w}\sum_{n=0}^{\infty}\big(\frac{z}{w}\big)^n\,dw &= \int_{\sigma_s} f(w)\,\frac{1}{w}\left(\sum_{n=0}^{N-1}\big(\frac{z}{w}\big)^n \;+ \frac{(z/w)^N}{1-(z/w)}\right)dw \\
&= \sum_{n=0}^{N-1}\int_{\sigma_s} f(w)\,\frac{1}{w}\,\frac{z^n}{w^n}\,dw \;+ \int_{\sigma_s} f(w)\,\frac{1}{w}\,\frac{(z/w)^N}{1-(z/w)}\,dw
\end{aligned}
$$

As $N \to \infty$ the first term tends to the infinite sum $\displaystyle\sum_{n=0}^{\infty}\int_{\sigma_s} f(w)\frac{z^n}{w^{n+1}}\,dw$ and the second term may be estimated using the Estimation Theorem:

$$
\left|\int_{\sigma_s} f(w)\,\frac{(z/w)^N}{w-z}\,dw\right| \le \max_{|w|=s}\big|f(w)\big|\,\frac{(|z|/s)^N}{(s-|z|)}\,2\pi s \to 0
$$

as $N \to \infty$ since $(|z|/s) < 1$.

9.3 Example: Let $f(z)=\dfrac{1}{z(z^2+4)}$. Note that f is holomorphic except at $z=0$ and $z=\pm 2i$. In particular, f is holomorphic in the annulus $A=\{z|0<|z|<2\}$ and in the annulus $B=\{z|2<|z|<\infty\}$ and also in the annulus $C=\{z|0<|z-2i|<2\}$. Find the Laurent series of $f(z)$ in A and in B and in C. Also, use the Laurent series to find the path integrals $\int_\alpha f$, $\int_\beta f$ and $\int_\gamma f$, where α, β and γ are the circles $\alpha(t)=e^{i\,t}$, $\beta(t)=3\,e^{i\,t}$ and $\gamma(t)=2i+e^{i\,t}$ for $0\le t\le 2\pi$.

Solution: We have $f(z)=\dfrac{1}{4z}\,\dfrac{1}{1+(z/2)^2}=\dfrac{1}{4z}\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,z^{2n-1}$. This is the Laurent series for f in A. Since the coefficient of z^{-1} in this series is $c_{-1}=\frac{1}{4}$, we have $\displaystyle\int_\alpha f=2\pi i\,c_{-1}=\frac{1}{2}\pi\,i$.

Also, we have $f(z)=\dfrac{1}{z^3}\,\dfrac{1}{1+(2/z)^2}=\dfrac{1}{z^3}\displaystyle\sum_{n=0}^{\infty}(-1)^n\big(\frac{2}{z}\big)^{2n}=\sum_{n=0}^{\infty}(-1)^n4^n\,z^{-2n-3}$. This is the Laurent series for f in B. Since the coefficient of z^{-1} is $c_{-1}=0$, we have $\displaystyle\int_\beta f=0$.

In the third annulus we write

$$\begin{aligned}
f(z)&=\frac{1}{z-2i}\,\frac{1}{z+2i}\,\frac{1}{z}=\frac{1}{z-2i}\,\frac{1}{(z-2i)+4i}\,\frac{1}{(z-2i)+2i}\\
&=\frac{1}{z-2i}\,\frac{1}{4i}\,\frac{1}{1+\frac{z-2i}{4i}}\,\frac{1}{2i}\,\frac{1}{1+\frac{z-2i}{2i}}\\
&=-\frac{1}{8}\,\frac{1}{z-2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2i}{4i}\right)^n\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2i}{2i}\right)^n\\
&=-\frac{1}{8}\,\frac{1}{z-2i}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^j}{(4i)^j}\,\frac{(-1)^{n-j}}{(2i)^{n-j}}\right)(z-2i)^n\\
&=-\frac{1}{8}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^n}{i^n2^{n+j}}\right)(z-2i)^{n-1}\\
&=-\frac{1}{8}\sum_{n=0}^{\infty}\frac{i^n}{2^n}\left(\sum_{j=0}^{n}\frac{1}{2^j}\right)(z-2i)^{n-1}\\
&=-\frac{1}{8}\sum_{n=0}^{\infty}\frac{i^n(2^{n+1}-1)}{2^{2n}}(z-2i)^{n-1}
\end{aligned}$$

This is the Laurent series in C. The coefficient of $(z-2i)^{-1}$ is $c_{-1}=-\frac{1}{8}$ so $\displaystyle\int_\gamma f=-\frac{1}{4}\pi i$.

9.4 Note: It should be remarked that all three of the path integrals in the above example are easy to compute using Cauchy's integral formula. In the following example, however, its easier to use the Laurent series to find the path integral.

9.5 Definition: When f is holomorphic in an open set U which contains the punctured disc $D^*(a,R)$, and f is undefined at a, we say that f has an **isolated singularity** at a. We define the **multiplicity** of f at a, denoted by $m(f,a)$, as follows. Say the Laurent series of f in $D^*(a,R)$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \, .$$

If $c_n = 0$ for all $n \in \mathbf{Z}$ so that f is identically zero in $D^*(a,R)$, then we define $m(f,a) = \infty$. If for all $N \in \mathbf{Z}$ there exists $n < N$ with $c_n \neq 0$ then we define $m(f,a) = -\infty$. Otherwise, we define $m(f,a)$ be the smallest integer m such that $c_m \neq 0$.

Let $m = m(f,a)$. When $m = -\infty$ we say that f has an **essential singularity** at a. When $m < 0$ we say that f has a **pole** at a of **order** $|m|$. A pole of order 1 is also called a **simple pole**. When $m \geq 0$ we say that f has a **removable** singularity at a, and in this case we shall extend f so that it is holomorphic in the disc $D(a,r)$ by setting $f(a) = c_0$. If $m > 0$ then we say that f has a **zero** at a of **order** m. A zero of order 1 is also called a **simple zero**. In any of these cases, we define the **residue** of f at a to be $\mathrm{Res}\,(f,a) = c_{-1}$. If σ is the circle $\sigma(t) = a + r\,e^{i\,t}$ for $0 \leq t \leq 2\pi$ where $0 < r < R$ then we have

$$\mathrm{Res}\,(f,a) = c_{-1} = \frac{1}{2\pi i} \int_\sigma f(z)\, dz$$

9.6 Note: If f has a removable singularity at a, then of course we have $\lim\limits_{z \to a} f(z) = c_0$. If f has a pole at a then its not hard to show that $\lim\limits_{z \to a} f(z) = \infty$. If f has an essential singularity at a, then the limit $\lim\limits_{z \to a} f(z)$ does not exist, and in fact there is a (dificult) theorem called *Picard's Theorem* which states that for all $\epsilon > 0$ the image $f\big(D^*(a,\epsilon)\big)$ is either equal to $\mathbf{C}$ or to $\mathbf{C} \setminus \{p\}$ for some point p.

9.7 Definition: Let $A \subseteq \mathbf{C}$ (or $A \subseteq \mathbf{R}^n$). We say that an element $a \in A$ is an **isolated** point of A when there exists $r > 0$ such that $D^*(a,r) \subseteq \mathbf{C} \setminus A$. We say that A is **discrete** when every point in A is isolated.

9.8 Example: The set of zeros of the function $f(z) = \sin \frac{1}{z}$ is the set $A = \left\{ \frac{1}{k\pi} \middle| 0 \neq k \in \mathbf{Z} \right\}$, which is discrete. The set of zeros and singularities of $f(z) = \sin \frac{1}{z}$ is the set $A \cup \{0\}$, which is not discrete. The set of poles of the function $g(z) = 1/\sin \frac{1}{z}$ is the above discrete set A, and g also has an unisolated singularity at the point $\{0\}$, so the set of all singularities of $g(z)$ is the non-discrete set $A \cup \{0\}$.

9.9 Example: When f is holomorphic in an open set U, unless f is identically zero in some connected component of U, the set of zeros of f is discrete by the Identity Theorem.

9.10 Definition: We say that f is **meromorphic** in the open set U when there exists a discrete set $A \subseteq U$ such that $U \setminus A$ is open and $f : U \setminus A \to \mathbf{C}$ is holomorphic in $U \setminus A$ and has a pole at each point $a \in A$. We remark that such a map f can be extended to a holomorphic map $f : U \to \widehat{\mathbf{C}}$ by setting $f(a) = \infty$ for each $a \in A$.

9.11 Theorem: *(The Residue Theorem) Let $U \subseteq C$ be an open set, let $A \subseteq U$ be a discrete set such that $U \setminus A$ is open, and let $f : U \setminus A \to \mathbf{C}$ be holomorphic in $U \setminus A$. Let α be a loop in $U \setminus A$ which is homotopic in U to a constant loop. Then there are finitely many points $a \in A$ for which $\eta(\alpha, a) \neq 0$ and we have*

$$\int_\alpha f(z)\,dz = 2\pi i \sum_{a\in A} \eta(\alpha, a)\,\mathrm{Res}\,(f, a)\,.$$

Proof: Say $\alpha : [u, v] \to \mathbf{C}$. Let $F : [u, v] \times [0, 1] \to U$ be a homotopy from α to a constant loop κ in U. Since $[u, v] \times [0, 1]$ is compact and F is continuous, the image $\mathrm{Image}\,(F) = F\big([u, v] \times [0, 1]\big)$ is compact. For each $a \in \mathrm{Image}\,(F)$ let $U_a = D(a, r_a)$ where if $a \in A$ then $r_a > 0$ is such that $D^*(a, r_a) \subseteq U \setminus A$ and if $a \notin A$ then $r_a > 0$ is such that $D(a, r_a) \subseteq U \setminus A$. Since $\mathrm{Image}\,(F)$ is compact, the open cover $\mathcal{U} = \big\{U_a \big| a \in \mathrm{Image}\,(F)\big\}$ must have a finite subcover. Since each $a \in A$ is contained in a unique element of $\mathcal{U}$ (namely the set U_a), it follows that there are only finitely many elements in $A \cap \mathrm{Image}\,(F)$. But notice that when $a \notin \mathrm{Image}\,(F)$, the homotopy F takes values in $U \setminus \{a\}$ and so F is a homotopy from α to κ in $U \setminus \{a\}$ so that $\eta(\alpha, a) = \dfrac{1}{2\pi i}\displaystyle\int_\alpha \frac{dz}{z-a} = \int_\kappa \frac{dz}{z-a} = 0$. Thus there are only finitely many points $a \in A$ for which $\eta(\alpha, a) \neq 0$. Let B be the finite set

$$B = \big\{a \in A \big| \eta(\alpha, a) \neq 0\big\}.$$

Choose $R > 0$ so that for every $b \in B$ we have $D^*(b, R) \subseteq U \setminus A$. Inside each of these punctured discs, f is equal to the sum of its Laurent series. For fixed $b \in B$, write

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-b)^n = p_b(z) + h_b(z)\ ,\ \text{where}$$

$$p_b = \sum_{n=-\infty}^{-1} c_n (z-b)^n \quad \text{and} \quad h_b = \sum_{n=0}^{\infty} c_n (z-b)^n$$

(p_b is called the **principal part**, and h_b is called the **holomorphic part**, of f at b). Note that $\mathrm{Res}\,(f, b) = c_{-1} = \mathrm{Res}\,(p_b, b)$. Also note that h_b is holomorphic in the disc $D(b, R)$ (not just the punctured disc) and that p_b is holomorphic in all of $\mathbf{C} \setminus \{b\}$. Now we let

$$g(z) = f(z) - \sum_{b\in B} p_b(z)\,.$$

Although f was only holomorphic in $U \setminus A$, the map g is holomorphic in all of U, indeed in $D^*(b, R)$ we have $g(z) = f(z) - p_b(z) - \sum\limits_{a\neq b} p_a(z) = h_b(z) - \sum\limits_{a\neq b} p_a(z)$. Since α is homotopic to a constant loop in U, we have

$$\begin{aligned}0 = \int_\alpha g(z)\,dz &= \int_\alpha f(z) - \sum_{b\in B} p_b(z)\,dz = \int_\alpha f(z)\,dz - \sum_{b\in B}\int_\alpha p_b(z)\,dz \\ &= \int_\alpha f(z)\,dz - \sum_{b\in B} 2\pi i\ \eta(\alpha, b)\,\mathrm{Res}\,(f, b))\,.\end{aligned}$$

9.12 Note: We describe two situations in which it is easy to calculate $\text{Res}\,(f,a)$. Suppose first that $f(z)=\dfrac{g(z)}{(z-a)^{n+1}}$ where $g(z)$ is holomorphic in an open set U containing a. Let σ be a circle in U centred at a. Then Cauchy's Integral Formula gives

$$\text{Res}\,(f,a)=\frac{1}{2\pi i}\int_\sigma \frac{g(z)}{(z-a)^{n+1}}\,dz=\frac{g^{(n)}(a)}{n!}\,.$$

Suppose next that $f(z)=\dfrac{g(z)}{h(z)}$ where g and h are holomorphic in an open set U containing a with $h(a)=0$ and $h'(a)\neq 0$. Then we can write $h(z)=(z-a)k(z)$ with $k(a)=h'(a)\neq 0$. Note that $k(z)\neq 0$ in a small disc $D(a,r)$ and $g(z)/k(z)$ is holomorphic in this disc. We have $f(z)=\dfrac{g(z)/k(z)}{(z-a)}$ with $g(z)/k(z)$ holomorphic in $D(a,r)$ so if σ is a circle in $D(a,r)$ centred at a then

$$\text{Res}\,(f,a)=\frac{1}{2\pi i}\int_\sigma \frac{g(z)/h(z)}{z-a}\,dz=\frac{g(a)}{k(a)}=\frac{g(a)}{h'(a)}\,.$$

9.13 Example: Let α be a loop in $D(0,3)$ with $\eta(\alpha,0)=3$, $\eta(\alpha,\frac{\pi}{2})=-1$ and $\eta(\alpha,-\frac{\pi}{2})=1$, and let $f(z)=\dfrac{(z+1)\,e^z}{z\cos z}$. Find $\displaystyle\int_\alpha f(z)\,dz$.

Solution: Notice that f is holomorphic in $\mathbf{C}$ except at $z=0$ and $z=\frac{\pi}{2}+k\pi$ for $k\in\mathbf{Z}$. In particular, f is holomorphic in $D(0,3)$ except at $z=0$ and $z=\pm\frac{\pi}{2}$. So by the Residue Theorem,

$$\int_\alpha f(z)\,dz=2\pi i\Big(3\,\text{Res}\,(f,0)-\text{Res}\,(f,\tfrac{\pi}{2})+\text{Res}\,(f,-\tfrac{\pi}{2})\Big)\,.$$

To find $\text{Res}\,(f,0)$, note that we can write $f(z)=g(z)\big/z$ where $g(z)=(z+1)\,e^z/\cos z$ (which is holomorphic at 0) so by the first part of the above note

$$\text{Res}\,(f,0)=g(0)=1\,.$$

To find $\text{Res}\big(f,\frac{\pi}{2}\big)$ and $\text{Res}\big(f,-\frac{\pi}{2}\big)$, note that we can write $f(z)=g(z)/h(z)$ where now $g(z)=(z+1)e^z/z$ (which is holomorphic at $\pm\frac{\pi}{2}$) and $h(z)=\cos z$ (which has a simple zero at $\pm\frac{\pi}{2}$). By the second part of the above note, we have

$$\text{Res}\,\big(f,\tfrac{\pi}{2}\big)=\frac{g\big(\frac{\pi}{2}\big)}{h'\big(\frac{\pi}{2}\big)}=\frac{\big(\frac{\pi}{2}+1\big)\,e^{\pi/2}\big/\frac{\pi}{2}}{-\sin\frac{\pi}{2}}=-\big(1+\tfrac{2}{\pi}\big)\,e^{\pi/2}\ ,\text{ and}$$

$$\text{Res}\,\big(f,-\tfrac{\pi}{2}\big)=\frac{g\big(-\frac{\pi}{2}\big)}{h'\big(-\frac{\pi}{2}\big)}=\frac{\big(-\frac{\pi}{2}+1\big)\ e^{-\pi/2}\big/\big(-\frac{\pi}{2}\big)}{-\sin\big(-\frac{\pi}{2}\big)}=\big(1-\tfrac{2}{\pi}\big)\,e^{-\pi/2}\,.$$

Thus we obtain

$$\int_\alpha f(z)\,dz=2\pi i\,\big(3+(1+\tfrac{2}{\pi})e^{\pi/2}+(1-\tfrac{2}{\pi})e^{-\pi/2}\big)\,.$$

9.14 Example: Find $\displaystyle\int_{|z|=4} \frac{dz}{z^4 \sinh z}$.

Solution: Let $f(z) = \dfrac{1}{z^4 \sinh z}$. Since $\sinh z = 0$ when $z = k\pi i, k \in \mathbf{Z}$ we see that f is holomorphic except at $z = k\pi i$. Note that the loop $|z| = 4$ surrounds the singularities at 0 and $\pm i\pi$, so we need to find the residue of f at these points. To find $\text{Res}\,(f, \pm i\pi)$, write $f = g/h$ with $g(z) = 1/z^4$ and $h(z) = \sinh z$. Then

$$\text{Res}\,(f, \pm i) = \frac{g(\pm i)}{h'(\pm i)} = \frac{1/\pi^4}{\cosh(\pm i\pi)} = \frac{1/\pi^4}{\cos(\pm\pi)} = -\tfrac{1}{\pi^4}.$$

To find $\text{Res}\,(f, 0)$, we find the first few terms of the Laurent series for f in the annulus $A = \left\{z \in \mathbf{C}\middle|0 < |z| < \pi\right\}$. We have $f(z) = \dfrac{1}{z^4}\dfrac{1}{\sinh z} = \dfrac{1}{z^4}\dfrac{1}{z(1 + \frac{1}{6}z^2 + \frac{1}{120}z^4 + \cdots)}$. We use long division:

$$\begin{array}{r|l}
 & 1 - \tfrac{1}{6}z^2 + \tfrac{7}{360}z^4 + \cdots \\
1 + \tfrac{1}{6}z^2 + \tfrac{1}{120}z^4 + \cdots & 1 + 0\,z^2 + 0\,z^4 + \cdots \\
 & 1 + \tfrac{1}{6}z^2 + \tfrac{1}{120}z^4 + \cdots \\
 & \quad -\tfrac{1}{6}z^2 - \tfrac{1}{120}z^4 + \cdots \\
 & \quad -\tfrac{1}{6}z^2 - \tfrac{1}{36}z^4 + \cdots \\
 & \qquad\qquad \tfrac{7}{360}z^5 + \cdots
\end{array}$$

We find that $f(z) = z^{-5} - \frac{1}{6}z^{-3} + \frac{7}{360}z^{-1} + \cdots$ so that

$$\text{Res}\,(f, 0) = c_{-1} = \tfrac{7}{360}.$$

Thus

$$\int_{|z|=4} f = 2\pi i\Big(\text{Res}\,(f, 0) + \text{Res}\,(f, i\pi) + \text{Res}\,(f, -i\pi)\Big) = 2\pi i\big(\tfrac{7}{360} - \tfrac{2}{\pi^4}\big).$$

9.15 Theorem: *(Zeros and Poles) Let f be meromorphic in U. Let A be the set of all zeros and poles of f in U. Let α be a loop in $U \setminus A$ which is homotopic in U to a constant loop. Then*

$$\int_\alpha \frac{f'(z)}{f(z)}\,dz = \sum_{a\in A} 2\pi i\ \eta(\alpha, a)\,m(f, a)\,.$$

Proof: Note that the function f'/f is holomorphic in $U \setminus A$. Let $a \in A$, let $m = m(f, a)$ and choose $R > 0$ so that $D^*(0, R) \subseteq U \setminus A$. Then for $z \in D^*(0, R)$ we have

$$f(z) = \sum_{n=m}^{\infty} c_n(z - a)^n = (z - a)^m g(z) \ \text{ where } \ g(z) = \sum_{n=0}^{\infty} c_{m+n}(z - a)^n.$$

Note that g is holomorphic in $D(0, R)$ with $g(a) = c_m \neq 0$. Choose r with $0 < r < R$ so that $g(z) \neq 0$ for all $z \in D(a, r)$. Then for all $z \in D(a, r)$, g'/g is holomorphic at z and

$$\frac{f'(z)}{f(z)} = \frac{m(z - a)^{m-1}g(z) + (z - a)^m g'(z)}{(z - a)^m g(z)} = \frac{m}{z - a} + \frac{g'(z)}{g(z)}.$$

Thus f'/f has a simple pole at a with $\text{Res}\,(f'/f, a) = m = m(f, a)$. This holds for every $a \in A$, so by the Residue Theorem,

$$\int_\alpha \frac{f'(z)}{f(z)}\,dz = 2\pi\, i \sum_{a\in A} \eta(\alpha, a)\,\text{Res}\,(f'/f, a) = 2\pi\, i \sum_{a\in A} \eta(\alpha, a)\,m(f, a).$$

9.16 Theorem: *(Rouché's Theorem) Let f and g be holomorphic in U. Let α be a loop in U. Suppose that $\big|f(z)-g(z)\big| < \big|f(z)\big|$ for all $z \in \text{Image}\,(\alpha)$. Then*

$$\sum_{a\in A} \eta(\alpha,a)\, m(f,a) = \sum_{b\in B} \eta(\alpha,a)\, m(g,b)$$

where A is the set of zeros of f in U and B is the set of zeros of g in U.

Proof: For each $s \in [0,1]$, define $k_s : U \to \mathbf{C}$ by $k_s(z) = f(z) + s\big(g(z) - f(z)\big)$. For all $z \in \text{Image}\,(\alpha)$, since $|g(z) - f(z)| < |f(z)$ we have $|f(z)| > 0$ and we have $g(z) \in D\big(f(z), |f(z)|\big)$. Since $D\big(f(z), |f(z)|\big)$ is convex, it follows that the line segment from $f(z)$ to $g(z)$ is contained in $D\big(f(z), |f(z)|\big)$ and hence that $k_s(z) \neq 0$ for all $s \in [0,1]$ and all $z \in \text{Image}\,(\alpha)$. Define $\ell : [0,1] \to \mathbf{Z}$ by

$$\ell(s) = \frac{1}{2\pi\, i}\int_\alpha \frac{{k_s}'(z)}{k_s(z)}\,dz = \frac{1}{2\pi\, i}\int_\alpha \frac{f'(z) + s\big(g'(z) - f'(z)\big)}{f(z) + \big(g(z) - f(z)\big)}\,dz.$$

Note that $\ell(s)$ does take values in $\mathbf{Z}$ by the Zeros and Poles Theorem. We shall show below that $\ell(z)$ is continuous. Since ℓ is continuous and takes values in $\mathbf{Z}$, it is constant in $[0,1]$ and so, by the Zeros and Poles Theorem,

$$\sum_{a\in A} \eta(\alpha,a)\, m(f,a) = \ell(0) = \ell(1) = \sum_{b\in B} \eta(\alpha,b)\, m(f,b),$$

as required. To show that ℓ is continuous, let $h = g - f$ and note that for $s_1, s_2 \in [0,1]$ we have

$$\begin{aligned}
\big|\ell(s_1) - \ell(s_2)\big| &= \left|\frac{1}{2\pi\, i}\int_\alpha \frac{f' + s_1 h'}{f + s_1 h} - \frac{f' + s_2 h'}{f + s_2 h}\right| \\
&= \left|\frac{1}{2\pi\, i}\int_\alpha \frac{(f' + s_1 h')(f + s_2 h) - (f' + s_2 h)(f + s_1 h)}{(f + s_1 h)(f + s_2 h)}\right| \\
&= \left|\frac{1}{2\pi\, i}\int_\alpha \frac{(s_1 - s_2)(h' f - f' h)}{(f + s_1 h)(f + s_2 h)}\right| \\
&\le \frac{1}{2\pi}\cdot\frac{|s_1 - s_2|\, M}{m^2}\cdot \text{Length}(\alpha) \longrightarrow 0 \text{ as } s_1 \to s_2,
\end{aligned}$$

where $M = \max\limits_{z=\alpha(t)} \big|h'(z)f(z) - f'(z)h(z)\big|$ and $m = \min s \in [0,1], z = \alpha(t)\big|f(z) + sh(z)\big|$.

9.17 Example: Let $f(z) = z^5 + 15z + 1$. Show that f has exactly 4 zeros (counted with multiplicity) inside the annulus $A = \big\{z \in \mathbf{C}\big|\frac{3}{2} < z < 2\big\}$.

Solution: Let $g(z) = 15z$ and $h(z) = z^5$. When $|z| = \frac{3}{2}$ we have

$$|f(z) - g(z)| = |z^5 + 1| \le |z|^5 + 1 = \tfrac{243}{32} + 1 < \tfrac{45}{2} = 15|z| = |g(z)|$$

so, by Rouché s Theorem, f has the same number of zeros inside the circle $|z| = \frac{3}{2}$ as the function $g(z) = 15z$, namely 1 zero. When $|z| = 2$, we have

$$|f(z) - h(z)| = |15z + 1| \le 15|z| + 1 = 31 < 32 = |z|^5 = |h(z)|$$

so, by Rouché s Theorem, f has the same number of zeros inside the circle $|z| = 2$ as the function $h(z) = z^5$, namely 5 zeros.

9.18 Lemma: *Let f be holomorphic in the open set U and suppose that f has a zero of multiplicity m at $a \in U$. Choose $R > 0$ so that $D(a, R) \subseteq U$ and $f(z) \neq 0$ in $D^*(a, R)$, and let $0 < r < R$. Let $\delta = \min\limits_{|z-a|=r} |f(z)|$ and note that $\delta > 0$. Then for all $w \in D(0, \delta)$, the function $g(z) = f(z) - w$ has exactly m zeros (counted with multiplicity) in $D(a, r)$.*

Proof: Let $w \in D(0, \delta)$ and let $g(z) = f(z) - w$. Then for all z with $|z - a| = r$ we have

$$|f(z) - g(z)| = |w| < \delta \leq |f(z)|$$

so, by Rouché s Theorem, g has the same number of zeros as f in $D(a, r)$.

9.19 Theorem: *(The Conformal Mapping Theorem) Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$ be holomorphic and injective. Then $f'(z) \neq 0$ for all $z \in U$, so f is conformal.*

Proof: Suppose, for a contradiction, that $a \in U$ and $f'(a) = 0$. Let $g(z) = f(z) - f(a)$ and note that $g'(z) = f'(z)$ so we have $g(a) = 0$ and $g'(a) = 0$ so that g has a zero of multiplicity $m \geq 2$ at a. Choose $R > 0$ so that $D(a, R) \subseteq U$ and $g'(z) \neq 0$ in $D^*(a, R)$. Let $0 < r < R$ and let $\delta = \min\limits_{|z-a|=r} |g(z)| > 0$. Choose $w \in \mathbf{C}$ with $0 < |w| < \delta$, let $h(z) = g(z) - w = f(z) - f(a) - w$, and note that $h'(z) = g'(z) = f'(z)$. By the above lemma, h has $m \geq 2$ zeros inside the circle $|z - a| = r$. Since $h(a) = -w \neq 0$ and $h'(z) = g'(z) \neq 0$ in $D^*(a, R)$, it follows that h has exactly m distinct zeros, each of multiplicity 1, in $D^*(a, r)$. Choose $u, v \in D^*(a, r)$ with $u \neq v$ such that $h(u) = h(v) = 0$. Then

$$f(u) = h(u) + f(a) + w = f(a) + w = h(v) + f(a) + w = f(v)$$

which contradicts the fact that f is injective.

9.20 Theorem: *(The Open Mapping Theorem) Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$ be holomorphic and non-constant. Then $f(U)$ is open.*

Proof: Let $b \in f(U)$, and choose $a \in U$ so that $f(a) = b$. Let $g(z) = f(z) - b$ and note that $g(a) = 0$. Choose $R > 0$ so that $D(a, R) \subseteq U$ and $g(z) \neq 0$ in $D^*(a, R)$. Let $0 < r < R$ and let $\delta = \min\limits_{|z-a|=r} |g(z)|$. We claim that $D(b, \delta) \subseteq f(U)$. Let $w \in D(b, \delta)$. Let $v = w - b$ and note that $|v| < \delta$. By the above lemma, the function $h(z) = g(z) - v$ has at least one zero in $D(a, r)$. Choose $u \in D(a, r)$ so that $h(u) = 0$. Then $f(u) = g(u) + b = h(u) + v + b = v + b = w$.

9.21 Theorem: *(The Inverse Function Theorem) Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$ be holomorphic with $f'(a) \neq 0$ where $a \in U$. Then there exists $r > 0$ such that the restriction of f to $D(a,r)$ is invertible, and its inverse $g = f^{-1}$ is holomorphic with*

$$g'(w) = \frac{1}{f'\big(g(w)\big)}$$

for all $w \in f\big(D(a,r)\big)$

Proof: Let $h(z) = f(z) - f(a)$. Since $h(a) = 0$ and $h'(a) = f'(a) \neq 0$, h has a simple zero at a. Choose $R > 0$ so that $D(a,R) \subseteq U$ and $h(z) \neq 0$ in $D^*(a,R)$. Let $0 < s < R$ and let $\delta = \min\limits_{|z-a|=s} |h(z)|$. We claim that for every $w \in D(f(a),\delta)$ there exists a unique $u \in D(a,s)$ such that $f(u) = w$. Let $w \in D\big(f(a),\delta\big)$. Let $v = w - f(a)$ and note that $|v| < \delta$. By the above lemma, the function $k(z) = h(z) - v$ has exactly 1 simple zero in $D(a,s)$. Thus there is a unique $u \in D(a,s)$ such that $0 = k(u) = h(u) - v = \big(f(u) - f(a)\big) - \big(w - f(a)\big) = f(u) - w$ hence there is a unique $u \in D(a,s)$ such that $f(u) = w$, as claimed. It follows that if we choose $r > 0$ so that $D(a,r) \subseteq f^{-1}\big(D(f(a),\delta)\big)$ then the restriction of f to $D(a,r)$ is invertible. Let $g = f^{-1}$ be the inverse of the restriction of f to $D(a,r)$.

Let $z \in D(a,r)$ and let $w = f(z)$ so that $z = g(w)$. By the Conformal Mapping Theorem, we know that $f'\big(g(w)\big) f'(z) \neq 0$. By the Open Mapping Theorem, we know that g is continuous. For $v \in f\big(D(a,r)\big)$, as $v \to w$ with $v \neq w$, we have $g(v) \to g(w)$ with $g(v) \neq g(w)$ since g is continuous and injective, and so

$$\frac{g(v) - g(w)}{v - w} = \frac{g(v) - g(w)}{f\big(g(v)\big) - f\big(g(w)\big)} \longrightarrow \frac{1}{f'\big(g(w)\big)}.$$