PMATH 351 Real Analysis, Solutions to the Exercises for Chapter 7

1: (a) Show that for $A \subseteq \mathbf{R}$, if A is closed and of measure zero then A is nowhere dense.

Solution: Suppose that A is closed and of measure zero. We must show that $A^\circ = \emptyset$. Suppose, for a contradiction, that $A^\circ \neq \emptyset$. Choose a closed interval $[a,b] \subset A$, where $a < b$. We claim that $\lambda(A) \geq b-a > 0$ so that A is not of measure zero. Let $\mathcal{U}$ be a countable set of open intervals which covers A. Then $\mathcal{U}$ also covers $[a,b]$, which is compact. Choose a finite subcover $\mathcal{V} \subset \mathcal{U}$ of $[a,b]$. Choose an open interval $(a_1,b_1) \in \mathcal{V}$ with $a \in (a_1,b_1)$. If $b_1 > b$ then $[a,b] \subset (a_1,b_1)$. If $b \leq b_1$ then $b_1 \in [a,b]$ but $b_1 \notin (a_1,b_1)$, so we can choose an open interval $(a_2,b_2) \in \mathcal{V}$ with $b_1 \in (a_2,b_2)$. Recursively, we choose intervals $(a_1,b_1),(a_2,b_2),(a_3,b_3),\cdots$ so that $a \in (a_1,b_1)$, $b_1 \in (a_2,b_2)$, $b_2 \in (a_3,b_3)$ and so on. Since $\mathcal{V}$ is finite and covers $[a,b]$, eventually we obtain an interval (a_n,b_n) which contains b. Since $a_1 < a$ and $a_{i-1} < b_i$ for $2 \leq i \leq n$ and $b_n > b$, we have

$$\begin{aligned}\sum_{i=1}^n (b_i - a_i) &= (b_1-a_1)+(b_2-a_2)+(b_3-a_3)+\cdots+(b_{n-1}-a_{n-1})+(b_n-a_n)\\ &> (b_1-a)+(b_2-b_1)+(b_3-b_2)+\cdots+(b_{n-1}-b_{n-2})+(b-b_{n-1})\\ &= b-a\end{aligned}$$

Thus the sum of the lengths of all the intervals in $\mathcal{U}$ is greater than $b-a$. From the definition of outer measure, we have $\lambda(A) \geq b-a > 0$.

(b) Let X and Y be metric spaces and let $f : X \to Y$. For $\epsilon > 0$ let

$$D_\epsilon = \left\{ a \in X \,\middle|\, \forall\, \delta > 0\ \exists\, x,y \in B(a,\delta)\ \ d\big(f(x),f(y)\big) \geq \epsilon \right\}.$$

Show that the set of points in X at which f is continuous is of type $\mathcal{G}_\delta$ by showing that D_ϵ is closed in X for all $\epsilon > 0$ and that $\bigcup_{n=1}^\infty D_{1/n} = \left\{ a \in X \,\middle|\, f \text{ is not continuous at } a \right\}$.

Solution: Let $a \in X \setminus D_\epsilon$. Since $a \notin D_\epsilon$, we can choose $\delta > 0$ so that for all $x,y \in B(a,\delta)$ we have $d\big(f(x),f(y)\big) < \epsilon$. We claim that $B\big(a,\frac{\delta}{2}\big) \cap D_\epsilon = \emptyset$. Let $b \in B\big(a,\frac{\delta}{2}\big)$. Then for $x,y \in B\big(b,\frac{\delta}{2}\big)$ we have $x,y \in B(a,\delta)$ so $d\big(f(x),f(y)\big) < \epsilon$. Thus $b \notin D_\epsilon$. It follows that D_ϵ is closed.

Suppose that f is discontinuous at $a \in X$. Choose $\epsilon > 0$ so that $\forall \delta > 0\ \exists x \in B(a,\delta)\ d\big(f(x),f(a)\big) \geq \epsilon$. Choose n so that $\frac{1}{n} \leq \epsilon$. Then $\forall \delta > 0\ \exists x \in B(a,\delta)\ d\big(f(x),f(a)\big) \geq \frac{1}{n}$. Hence (by taking $y = a$) we have $\forall \delta > 0\ \exists x,y \in B(a,\delta)\ d\big(f(x),f(y)\big) \geq \frac{1}{n}$. Thus $a \in D_{1/n}$.

Conversely, let $a \in \bigcup_{n=1}^\infty D_{1/n}$. Choose n so that $a \in D_{1/n}$. Let $\epsilon = \frac{1}{2n}$. Let $\delta > 0$. Since $a \in D_{1/n}$, we can choose $x,y \in B\big(a,\frac{\delta}{2}\big)$ so that $d\big(f(x),f(y)\big) \geq \frac{1}{n} = 2\epsilon$. Then either we have $d\big(f(x),f(a)\big) \geq \epsilon$ or we have $d\big(f(y),f(a) \geq \epsilon$, and so (by taking $z = x$ or $z = y$) we have $\exists z \in B(a,\delta)\ d\big(f(z),f(a)\big) \geq \epsilon$. We have shown that $\exists \epsilon > 0\ \forall \delta > 0\ \exists z \in B(a,\delta)\ f(z) \notin B\big(f(a),\epsilon\big)$, which means that f is discontinuous at a.

Thus $\bigcup_{n=1}^\infty D_{1/n} = \left\{ a \in X \,\middle|\, f \text{ is not continuous at } a \right\}$, as required, and by taking the complement we see that the set of points in X at which f is continuous is of type $\mathcal{G}_\delta$.

2: Let $\mathcal{G} = \mathcal{G}(\mathbf{R})$ be the set of open sets in $\mathbf{R}$, and let $\mathcal{F} = \mathcal{F}(\mathbf{R})$ be the set of closed sets in $\mathbf{R}$.

(a) Show that $\mathcal{F} \subseteq \mathcal{G}_\delta$ (or equivalently, by taking complements, that $\mathcal{G} \subseteq \mathcal{F}_\sigma$).

Solution: Let $\emptyset \neq A \in \mathcal{F}$. Since A is closed, for each $x \in \mathbf{R}$ the function $g_x : A \to [0, \infty)$ given by $g_x = |x - a|$ attains its minimum value. Define $f : \mathbf{R} \to [0, \infty)$ by $f(x) = \text{dist}(x, A) = \min\left\{|x - a|\,\middle|\,a \in A\right\}$ and note that $f(x) = 0 \iff x \in A$. Recall (or verify) that f is continuous, and so the set $\left\{x \in \mathbf{R}\,\middle|\,f(x) < \frac{1}{n}\right\} = f^{-1}\left(\frac{1}{n}, \infty\right)$ is open for each $n \in \mathbf{Z}^+$. Thus

$$A = \left\{x \in \mathbf{R}\,\middle|\,f(x) = 0\right\} = \bigcap_{n=1}^{\infty} \left\{x \in \mathbf{R}\,\middle|\,f(x) < \tfrac{1}{n}\right\} \in \mathcal{G}_\delta.$$

(b) Show that $\mathcal{F}_\sigma \neq \mathcal{G}_\delta$.

Solution: Recall that $\mathbf{Q} \in \mathcal{F}_\sigma$ $\left(\text{indeed if } \mathbf{Q} = \{a_1, a_2, \cdots\} \text{ then } \mathbf{Q} = \bigcup_{k=1}^{\infty} \{a_k\}\right)$ and it follows (by taking the complement) that $\mathbf{Q}^c \in \mathcal{G}_\delta$. We claim that $\mathbf{Q}^c \notin \mathcal{F}_\sigma$ (and hence, by taking complements, $\mathbf{Q} \notin \mathcal{G}_\delta$). Suppose, for a contradiction, that $\mathbf{Q}^c \in \mathcal{F}_\sigma$. Let $\mathbf{Q} = \bigcup_{k=1}^{\infty} A_k$ where each A_k is a closed set (which is contained in $\mathbf{Q}$) and let $\mathbf{Q}^c = \bigcup_{k=1}^{\infty} B_k$ where each B_k is a closed set (which is contained in $\mathbf{Q}^c$). Then $\mathbf{R} = \mathbf{Q} \cup \mathbf{Q}^c = \bigcup_{n=1}^{\infty} C_n$ where $C_{2k} = A_k$ and $C_{2k-1} = B_k$. For each $n \in \mathbf{Z}^+$, when n is even C_n is contained in $\mathbf{Q}$ and when n is odd C_n is contained in $\mathbf{Q}^c$ and, in either case, it follows that C has an empty interior. Thus $\mathbf{R}$ is a countable union of closed sets with empty interiors, and so $\mathbf{R}$ is first category. We know this is impossible, by the Baire Category Theorem, and so we have obtained the desired contradiction.

(c) Show that $\mathcal{G}_\delta \cup \mathcal{F}_\sigma \neq \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$.

Solution: Let $a, b \in \mathbf{R}$ with $a < b$. Since $\mathbf{Q} \in \mathcal{F}_\sigma$ and $(a, b) \in \mathcal{G} \subseteq \mathcal{F}_\sigma$, we have $\mathbf{Q} \cap (a, b) \in \mathcal{F}_\sigma$. Since $\mathbf{Q}^c \in \mathcal{G}_\delta$ and $(a, b) \in \mathcal{G} \subseteq \mathcal{G}_\delta$, we have $\mathbf{Q}^c \cap (a, b) \in \mathcal{G}_\delta$. If we had $\mathbf{Q}^c \cap (a, b) \in \mathcal{F}_\sigma$ then we could write each of the sets $\mathbf{Q} \cap (a, b)$ and $\mathbf{Q}^c \cap (a, b)$ as a countable union of closed sets, with each closed set necessarily having empty interior, and then the union $(a, b) = \big(\mathbf{Q} \cap (a, b)\big) \cup \big(\mathbf{Q}^c \cap (a, b)\big)$ would also be a countable union of closed sets with empty interior, and this is not possible by the Baire Category Theorem. Thus $\mathbf{Q}^c \cap (a, b) \notin \mathcal{F}_\sigma$. If we had $\mathbf{Q} \cap (a, b) \in \mathcal{G}_\delta$ then, by taking complements, we would have $\mathbf{Q}^c \cup (a, b)^c \in \mathcal{F}_\sigma$, but then we would also have $\mathbf{Q}^c \cap (a, b) = \big(\mathbf{Q}^c \cup (a, b)^c\big) \cap (a, b) \in \mathcal{F}_\sigma$, which is not the case. Thus $\mathbf{Q}^c \cap (a, b) \notin \mathcal{G}_\delta$. To summarize, we have

$$\mathbf{Q} \cap (a, b) \in \mathcal{F}_\sigma\ ,\ \mathbf{Q} \cap (a, b) \notin \mathcal{G}_\delta\ ,\ \mathbf{Q}^c \cap (a, b) \in \mathcal{G}_\delta\ ,\ \mathbf{Q}^c \cap (a, b) \notin \mathcal{F}_\sigma.$$

Let $A = \big(\mathbf{Q} \cap (-1, 0)\big) \cup \big(\mathbf{Q}^c \cap (0, 1)\big)$. Since $\mathbf{Q} \cap (-1, 0) \in \mathcal{F}_\sigma \subseteq \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$ and $\mathbf{Q}^c \cap (0, 1) \in \mathcal{G}_\delta \subseteq \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$ we have $A \in \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$. If we had $A \in \mathcal{G}_\delta$ then we would also have $A \cap (-1, 0) \in \mathcal{G}_\delta$, but $A \cap (-1, 0) = \mathbf{Q} \cap (-1, 0) \notin \mathcal{G}_\delta$. If we had $A \in \mathcal{F}_\sigma$ then we would also have $A \cap (0, 1) \in \mathcal{F}_\sigma$, but $A \cap (0, 1) = \mathbf{Q}^c \cap (0, 1) \notin \mathcal{F}_\sigma$. Since $A \notin \mathcal{G}_\delta$ and $A \notin \mathcal{F}_\sigma$ it follows that $A \notin \mathcal{G}_\delta \cup \mathcal{F}_\sigma$.

3: A function $f : [0,1] \to \mathbf{R}$ is called *nowhere monotonic* when it is not monotonic in any interval. Show that the set of all nowhere monotonic continuous functions $f : [0,1] \to \mathbf{R}$ is a residual set in $\big(\mathcal{C}[0,1], d_\infty\big)$.

Hint: for $N \in \mathbf{N}$ let

$$A_N = \Big\{ \pm f \in \mathcal{C}[0,1] \Big| \exists a \in [0,1] \ \forall x \in [0,1] \ \ |x-a| \le \tfrac{1}{N} \Longrightarrow \big(f(x)-f(a)\big)(x-a) \ge 0 \Big\}.$$

Solution: For $2 \le N \in \mathbf{N}$, let

$$A_N = \Big\{ f \in \mathcal{C}[0,1] \Big| \exists\, a \in \big[\tfrac{1}{N}, 1-\tfrac{1}{N}\big] \ \forall x \in [0,1] \ \ |x-a| < \tfrac{1}{N} \Longrightarrow \ \big(f(x)-f(a)\big)(x-a) \ge 0 \Big\}$$

and

$$B_N = \Big\{ f \in \mathcal{C}[0,1] \Big| \exists\, a \in \big[\tfrac{1}{N}, 1-\tfrac{1}{N}\big] \ \forall x \in [0,1] \ \ |x-a| < \tfrac{1}{N} \Longrightarrow \big(f(x)-f(a)\big)(x-a) \le 0 \Big\}$$

Let $f \in \mathcal{C}[0,1]$ and suppose that f is increasing in some interval I. Choose $a \in I$ and $1 \le N \in \mathbf{N}$ so that $\big(a-\frac{1}{N}, a+\frac{1}{N}\big) \subset I$. For $a \le x < a+\frac{1}{N}$ we have $(x-a) \ge 0$ and $\big(f(x)-f(a)\big) \ge 0$, and for $a-\frac{1}{N} < x \le a$ we have $(x,a) \le 0$ and $\big(f(x)-f(a)\big) \ge 0$, so for all $x \in [0,1]$ with $|x-a| < \frac{1}{N}$ we have $\big(f(x)-f(a)\big)(x-a) \ge 0$. Thus $f \in A_N$. Similarly, if $f \in \mathcal{C}[0,1]$ is decreasing in some interval I, then we have $f \in B_N$ for some N. Thus the set of *somewhere* monotonic functions is contained in $\bigcup_{N=1}^\infty (A_N \cup B_N)$, and so it suffices to show that each A_N and each B_N is nowhere dense.

We claim that each A_N is closed in $\big(\mathcal{C}[0,1], d_\infty\big)$. Fix N. Let $\langle f_n \rangle$ be a sequence in A_N which converges uniformly on $[0,1]$ to the function $f \in \mathcal{C}[0,1]$. We must show that $f \in A_N$. For each $n \in \mathbf{N}$, choose $a_n \in [0,1]$ so that $\forall x \in [0,1]$ $|x-a_n| < \frac{1}{N} \Longrightarrow \big(f_n(x)-f_n(a_n)\big)(x-a_n) \ge 0$. Choose a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ which converges in $[0,1]$, and let $a = \lim\limits_{k\to\infty} a_{n_k} \in [0,1]$. We claim that

$$\lim_{k\to\infty} f_{n_k}(a_{n_k}) = f(a).$$

Let $\epsilon > 0$. Since f is continuous at a we can choose $\delta > 0$ so that for all $y \in [0,1]$ we have $|y-a| < \delta \Longrightarrow \big|f(y)-f(a)\big| < \frac{\epsilon}{2}$, and then since $a_{n_k} \to a$ and since $f_{n_k} \to f$ uniformly on $[0,1]$ we can choose $K \in \mathbf{N}$ so that for $k \in \mathbf{N}$ with $k \ge K$ we have $\big|a_{n_k} - a\big| < \delta$ and we have $\big|f_{n_k}(y) - f(y)\big| < \frac{\epsilon}{2}$ for all $y \in [o,1]$. Then for $k \ge K$ we obtain

$$\big|f_{n_k}(a_{n_k}) - f(a)\big| \le \big|f_{n_k}(a_{n_k}) - f(a_{n_k})\big| + \big|f(a_{n_k}) - f(a)\big| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Now, let $x \in [0,1]$ with $|x-a| < \frac{1}{N}$. Note that for sufficiently large k we have $\big|x - a_{n_k}\big| < \frac{1}{N}$; indeed if we choose K so that $k \ge K \Longrightarrow \big|a_{n_k} - a\big| < \frac{1}{N} - |x-a|$, then for $k \ge K$ we have $\big|x-a_{n_k}\big| \le |x-a| + \big|a-a_{n_k}\big| < |x-a| + \frac{1}{N} - |x-a| = \frac{1}{N}$. Since we have $\big(f(x) - f(a_{n_k})\big)(x-a_{n_k}) \ge 0$ for all k sufficiently lage that $|x-a_{n_k}| < \frac{1}{N}$, it follows that

$$\big(f(x)-f(a)\big)(x-a) = \lim_{k\to\infty} \big(f(x) - f_{n_k}(a_{n_k})\big)(x-a_{n_k}) \ge 0.$$

Thus $f \in A_N$, so we have shown that A_N is closed. Similarly, each B_N is closed.

We claim that each A_N has empty interior. Fix N, let $f \in A_N$, and let $r > 0$. We shall construct $g \in B_\infty(f,r)$ with $g \notin A_N$. Since f is uniformly continuous on $[0,1]$, we can choose $\delta > 0$ so that for all $x, y \in [0,1]$ we have $\big|f(x)-f(y)\big| < \frac{r}{4}$. Choose ω large enough so that $\frac{2\pi}{\omega} < \min\big(\frac{1}{N}, \delta\big)$. Let

$$g(x) = f(x) + \tfrac{r}{2}\sin(\omega x).$$

Since $|g-f|_\infty = \frac{r}{2}$, we have $g \in B_\infty(f,r)$. Let $a \in [\frac{1}{N}, 1-\frac{1}{N}]$. If $\sin(\omega a) \ge 0$ then we choose $x \in \big(a, a+\frac{2\pi}{\omega}\big]$ so that $\sin(\omega x) = -1$, and then we have $(x-a) > 0$ and

$$g(x)-g(a) = \big(f(x)-f(a)\big) + \tfrac{r}{2}\big(\sin(\omega x) - \sin(\omega a)\big) \le \tfrac{r}{4} - \tfrac{r}{2} < 0.$$

If $\sin(\omega a) \le 0$ then we choose $x \in \big[a - \frac{2\pi}{\omega}, a\big)$ so that $\sin(\omega x) = 1$, and then we have $(x-a) < 0$ and

$$g(x)-g(a) = \big(f(x)-f(a)\big) + \tfrac{r}{2}\big(\sin(\omega x) - \sin(\omega a)\big) \ge -\tfrac{r}{4} + \tfrac{r}{2} > 0.$$

In either case we obtain $x \in [0,1]$ with $|x-a| < \frac{1}{N}$ such that $\big(g(x)-g(a)\big)(x-a) < 0$. Thus $g \notin A_N$.