PMATH 351 Real Analysis, Exercises for Chapter 7: The Baire Category Theorem

1: (a) Show that for $A \subseteq \mathbf{R}$, if A is closed and of measure zero then A is nowhere dense.

(b) Let X and Y be metric spaces and let $f : X \to Y$. For $\epsilon > 0$ let

$$D_\epsilon = \left\{ a \in X \middle| \forall\, \delta > 0\ \exists\, x, y \in B(a,\delta)\ \ d\big(f(x), f(y)\big) \geq \epsilon \right\}.$$

Show that the set of points in X at which f is continuous is of type $\mathcal{G}_\delta$ by showing that D_ϵ is closed in X for all $\epsilon > 0$ and that $\displaystyle\bigcup_{n=1}^{\infty} D_{1/n} = \left\{ a \in X \,\middle|\, f \text{ is not continuous at } a \right\}$.

2: Let $\mathcal{G} = \mathcal{G}(\mathbf{R})$ be the set of open sets in $\mathbf{R}$, and let $\mathcal{F} = \mathcal{F}(\mathbf{R})$ be the set of closed sets in $\mathbf{R}$.

(a) Show that $\mathcal{F} \subseteq \mathcal{G}_\delta$ (or equivalently, by taking complements, that $\mathcal{G} \subseteq \mathcal{F}_\sigma$).

(b) Show that $\mathcal{F}_\sigma \neq \mathcal{G}_\delta$.

(c) Show that $\mathcal{G}_\delta \cup \mathcal{F}_\sigma \neq \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$.

3: A function $f : [0,1] \to \mathbf{R}$ is called *nowhere monotonic* when it is not monotonic in any interval. Show that the set of all nowhere monotonic continuous functions $f : [0,1] \to \mathbf{R}$ is a residual set in $\big(\mathcal{C}[0,1], d_\infty\big)$.

Hint: for $N \in \mathbf{N}$ let

$$A_N = \left\{ \pm f \in \mathcal{C}[0,1] \middle| \exists a \in [0,1]\ \forall\, x \in [0,1]\ \ |x - a| \leq \tfrac{1}{N} \implies \big(f(x) - f(a)\big)(x - a) \geq 0 \right\}.$$