PMATH 351 Real Analysis, Solutions to the Exercises for Chapter 6

1: (a) Find an example of a function $f:\mathbb{R}\to\mathbb{R}$ such that $\big|f(y)-f(x)\big|<|y-x|$ for all $x,y\in\mathbb{R}$ with $x\neq y$, but f has no fixed point in $\mathbb{R}$.

Solution: Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x+g(x)$ where $g:\mathbb{R}\to\mathbb{R}$ is any differential function with $g(x)>0$ and $-1<g'(x)<0$ for all $x\in\mathbb{R}$ $\big($for example, $g(x)=\frac{1}{4}\big(\sqrt{x^2+1}-x\big)$ or $g(x)=\frac{1}{2}-\frac{1}{\pi}\tan^{-1}x\big)$. Given $x<y$, by the Mean Value Theorem we can choose c with $x\leq c\leq y$ such that $g(y)-g(x)=g'(c)(y-x)$ and then

$$\big|f(y)-f(x)\big|=\big|(y-x)+(g(y)-g(x))\big|=\big|(y-x)+g'(c)(y-x)\big|=(1+g'(c))(y-x)<(y-x)$$

since $-1<g'(c)<0$. But f has no fixed points because for all $x\in\mathbb{R}$ we have $f(x)=x+g(x)>x$, since $g(x)>0$.

(b) The polynomial $p(x)=x^3-3x+1$ has a unique root in $\big[0,\frac{1}{2}\big]$. Approximate this root using the Banach Fixed Point Theorem as follows: Let $f(x)=\frac{1}{3}(x^3+1)$. Show that $f:\big[0,\frac{1}{2}\big]\to\big[0,\frac{1}{2}\big]$ is a contraction map whose unique fixed point is the desired root of p. Approximate the root by using a calculator to find x_5 where $x_0=0$ and $x_{n+1}=f(x_n)$.

Solution: We have $f'(x)=x^2$. Since $f'(x)>0$ for $x>0$, it follows that f is (strictly) increasing for $x\geq 0$, and since $f(0)=\frac{1}{3}$ and $f\big(\frac{1}{2}\big)=\frac{3}{8}$ we have $f:\big[0,\frac{1}{2}\big]\to\big[\frac{1}{3},\frac{3}{8}\big]\subseteq\big[0,\frac{1}{2}\big]$. Let $x,y\in\big[0,\frac{1}{2}\big]$. By the Mean Value Theorem we can choose t between x and y so that $f(x)-f(y)=f'(t)(x-y)$. Since t is between x and y, we have $0\leq t\leq\frac{1}{2}$ hence $0\leq t^2\leq\frac{1}{4}$, that is $0\leq f'(t)\leq\frac{1}{4}$. Thus $\big|f(x)-f(y)\big|=\big|f'(t)\big|\,|x-y|\leq\frac{1}{4}\,|x-y|$ so that f is a contraction map with contraction constant $c=\frac{1}{4}$. Using a calculator, we find that $x_0=0$, $x_1\cong 0.333333$, $x_2\cong 0.345679$, $x_3\cong 0.347102$, $x_4\cong 0.347273$ and $x_5\cong 0.347294$.

We remark that Newton's Method for finding this root (which many students will have seen) amounts to finding the fixed point of the contraction map $g(x)=x-\frac{p(x)}{p'(x)}=\frac{2x^3-1}{3(x^2-1)}$, and the resulting sequence converges faster than the sequence we found above (because, letting a be the root that we are approximating, when we repeatedly apply f on smaller intervals the contraction constant approaches $f'(a)=a^2\cong 0.12$ but p is a factor of g' so when we repeatedly apply g the contraction constant approaches $g'(a)=0$).

We also remark that the exact value of the root that we are approximating is $a=2\sin\frac{\pi}{9}=2\sin(10^\circ)$.

2: (a) Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=\int_0^x f(t)\,dt$. Show that F is not a contraction map but $F^2=F\circ F$ is.

Solution: Note that F and F^2 are linear maps on the normed linear space $\mathcal{C}[0,1]$. When f is the constant function $f(x)=1$ we have $F(f)(x)=x$ so that $\|F(f)\|_\infty=1=\|f\|_\infty$, and so F is not a contraction. When $f\in\mathcal{C}[0,1]$ and $F(f)=g$ we have

$$\begin{aligned}\big|g(x)\big| &= \left|\int_0^x f(t)\,dt\right| \le \int_0^x \big|f(t)\big|\,dt \le \int_0^x \|f\|_\infty dt = \|f\|_\infty x \\ \big|F(g)(x)\big| &= \left|\int_0^x g(t)\,dt\right| \le \int_0^x \big|g(t)\big|\,dt \le \int_0^x \|f\|_\infty\, t\,dt = \tfrac{1}{2}\|f\|_\infty x^2\end{aligned}$$

so that $\|F^2(f)\|_\infty=\|F(g)\|_\infty\le\frac{1}{2}\|f\|_\infty$, and so F^2 is a contraction map with contraction constant $c=\frac{1}{2}$.

(b) Use the Banach Fixed Point Theorem to show that there exists a unique function $f\in\mathcal{C}[0,1]$ such that $f(x)=x+\int_0^x t\,f(t)\,dt$ for all $x\in[0,1]$.

Solution: Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=x+\int_0^x t\,f(t)\,dt$. Note that F is a contraction map because for $f,g\in\mathcal{C}[0,1]$ we have

$$\begin{aligned}\Big|F(f)(x)-F(g)(x)\Big| &= \left|\left(x+\int_0^x t\,f(t)\,dt\right)-\left(x+\int_0^x t\,g(t)\,dt\right)\right| = \left|\int_0^x t\big(f(t)-g(t)\big)\,dt\right| \\ &\le \int_0^x \|f-g\|_\infty t\,dt = \tfrac{1}{2}\,\|f-g\|_\infty\, x^2\end{aligned}$$

so that $\|F(f)-F(g)\|_\infty\le\frac{1}{2}\|f-g\|_\infty$. By the Banach Fixed-Point Theorem, F has a unique fixed point $f\in\mathcal{C}[0,1]$, so there is a unique function $f\in\mathcal{C}[0,1]$ such that $f(x)=x+\int_0^x t\,f(t)\,dt$ for all $x\in[0,1]$.

3: Solve the differential equation $y' = 1 + x^2 y$ with $y(0) = 0$ in the interval $[-1,1]$ using the following method: Define $F : \mathcal{C}[-1,1] \to \mathcal{C}[-1,1]$ by $F(f)(x) = x + \int_0^x t^2 f(t)\,dt$. Show that F is a contraction map (using the supremum norm) whose unique fixed point is the desired solution. Express the solution as a power series by finding a formula for $f_n(x)$ where $f_0(x) = 0$ and $f_{n+1}(x) = F(f_n)(x)$.

Solution: For $f, g \in \mathcal{C}[-1,1]$ and $x \in [-1,1]$ we have

$$\big|F(f)(x) - F(g)(x)\big| = \left|x + \int_0^x t^2 f(t)\,dt - x - \int_0^x t^2 g(t)\,dt\right| = \left|\int_0^x t^2\big(f(t) - g(t)\big)\,dt\right|.$$

When $0 \le x \le 1$ we have

$$\left|\int_0^x t^2\big(f(t)-g(t)\big)\,dt\right| \le \int_0^x t^2\big|f(t)-g(t)\big|\,dt \le \int_0^x t^2\|f-g\|_\infty\,dt = \tfrac{1}{3}x^3\,\|f-g\|_\infty \le \tfrac{1}{3}\|f-g\|_\infty$$

and when $-1 \le x \le 0$ we have

$$\left|\int_0^x t^2\big(f(t)-g(t)\big)\,dt\right| \le \int_x^0 t^2\big|f(t)-g(t)\big|\,dt \le \int_x^0 t^2\|f-g\|_\infty\,dt = -\tfrac{1}{3}x^3\,\|f-g\|_\infty \le \tfrac{1}{3}\|f-g\|_\infty.$$

Since $\big|F(f)(x) - F(g)(x)\big| \le \frac{1}{3}\|f-g\|_\infty$ for all $x \in [-,1,1]$, it follows that $\|F(f) - F(g)\|_\infty \le \frac{1}{3}\|f-g\|_\infty$, and so F is a contraction map. Since F is a contraction map on $\mathcal{C}[-1,1]$, which is a complete metric space, it follows from the Banach Fixed Point Theorem that F has a unique fixed point f.

If $y = f(x)$ is a solution to the given differential equation, so we have $f'(x) = 1 + x^2 f(x)$ for all $x \in [-1,1]$ with $f(0) = 0$, then by changing the variable to t and integrating from 0 to x (using the Fundamental Theorem of Calculus) we obtain

$$f(x) = \int_0^x f'(t)\,dt = \int_0^x 1 + t^2 f(t)\,dt = x + \int_0^x t^2 f(t)\,dt = F(f)(x)$$

so that $f = F(f)$, which means that f is a fixed point of F. If, on the other hand, f is a fixed point of F, which means that $f(x) = F(f)(x) = x + \int_0^x t^2 f(t)\,dt$ for all $x \in [-1,1]$, then taking $x = 0$ gives $f(0) = 0$, and differentiating on both sides gives $f'(x) = 1 + x^2 f(x)$, so that $y = f(x)$ is a solution to the given differential equation. Thus the solutions to the given differential equation (if there are any) are equal to the fixed points of F. Since F has a unique fixed point f, the given differential equation has a unique solution $y = f(x)$.

Finally, let us find a formula for $f_n(x)$ where $f_0(x) = 0$ and $f_{n+1} = F(f_n)$. We have

$$\begin{aligned}
f_0(x) &= 0\\
f_1(x) &= x + \int_0^x t^2 \cdot 0\; dt = x\\
f_2(x) &= x + \int_0^x t^2 \cdot t\; dt = x + \tfrac{1}{4}x^4\\
f_3(x) &= x + \int_0^x t^2\big(x + \tfrac{1}{4}t^4\big)\,dt = x + \tfrac{1}{4}x^4 + \tfrac{1}{4\cdot 7}x^7\,.
\end{aligned}$$

Let $n \ge 3$ and suppose, inductively, that

$$f_n(x) = x + \tfrac{1}{4}x^4 + \tfrac{1}{4\cdot 7}x^7 + \tfrac{1}{4\cdot 7\cdot 10}x^{10} + \cdots + \tfrac{1}{4\cdot 7\cdot 10\cdot \ldots \cdot(3n-2)}x^{3n-2}\,.$$

Then

$$\begin{aligned}
f_{n+1}(x) &= x + \int_0^x t^2\big(t + \tfrac{1}{4}t^4 + \tfrac{1}{4\cdot 7}t^7 + \cdots + \tfrac{1}{4\cdot 7\cdot \ldots \cdot(3n-2)}t^{3n-2}\big)\,dt\\
&= x + \tfrac{1}{4}t^4 + \tfrac{1}{4\cdot 7}x^7 + \tfrac{1}{4\cdot 7\cdot 10}x^{10} + \cdots + \tfrac{1}{4\cdot 7\cdot \ldots \cdot(3n-2)(3n+1)}x^{3n+1}
\end{aligned}$$

as required. The proof of the Banach Fixed Point Theorem shows that the fixed point is $f(x) = \lim_{n\to\infty} f_n(x)$, so we have

$$f(x) = x + \tfrac{1}{4}x^4 + \tfrac{1}{4\cdot 7}x^7 + \tfrac{1}{4\cdot 7\cdot 10}x^{10} + \cdots = \sum_{n=1}^{\infty} \tfrac{1}{1\cdot 4\cdot 7\cdot \ldots \cdot(3n-2)}x^{3n-2} = \sum_{n=1}^{\infty} \tfrac{(-1)^n}{3^n n!\binom{-1/3}{n}}x^{3n-2}.$$

We remark that you can obtain the same solution more easily simply by substituting $y = a_0 + a_1 x + a_2 x^2 + \cdots$ with $a_0 = y(0) = 0$ into the differential equation (differentiating term by term) to obtain a recursion formula for the coefficients a_n. This assignment problem is not providing you with a better method for solving this particular differential equation, it is intended to illustrate Banach's Fixed Point Theorem and its proof.

4: (a) Let $A = \left\{ \sum\limits_{k=1}^{n} f_k(x) g_k(y) \,\middle|\, n \in \mathbb{Z}^+, f_k, g_k \in \mathcal{C}[0,1] \right\}$. Show that A is dense in $\mathcal{C}\big([0,1]\times[0,1]\big)$.

Solution: It is easy to see that A is a subalgebra of $\mathcal{C}\big([0,1]\times[0,1]\big)$, and A vanishes nowhere because $1 \in A$, and A separates points because $x \in A$ and $y \in A$ (and for $x_1, x_2, y_1, y_2 \in [0,1]$, if $(x_1,y_1) \neq (x_2,y_2)$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$). Thus A is dense in $\mathcal{C}\big([0,1]\times[0,1]\big)$ by the Stone-Weierstrass Theorem.

(b) Let $A = \left\{ \sum\limits_{k=0}^{n} \big(a_k \sin(kx) + b_k \cos(kx)\big) \,\middle|\, 0 \leq n \in \mathbb{Z}, a_k, b_k \in \mathbb{R} \right\}$. Show that A is dense in $\mathcal{C}[0,\pi]$ but A is not dense in $\mathcal{C}[0,2\pi]$.

Solution: Note that A is a subalgebra of $\mathcal{C}[0,\pi]$ because

$$\begin{aligned}
\sin(kx)\sin(\ell x) &= \tfrac{1}{2}\Big(\cos\big((k-\ell)x\big) - \cos\big((k+\ell)x\big)\Big),\\
\sin(kx)\cos(\ell x) &= \tfrac{1}{2}\Big(\sin\big((k+\ell)x\big) + \sin\big((k-\ell)x\big)\Big),\\
\cos(kx)\sin(\ell x) &= \tfrac{1}{2}\Big(\sin\big((k+\ell)x\big) - \sin\big((k-\ell)x\big)\Big) \text{ and}\\
\cos(kx)\cos(\ell x) &= \tfrac{1}{2}\Big(\cos\big((k+\ell)x\big) + \cos\big((k-\ell)x\big)\Big),
\end{aligned}$$

and A vanishes nowhere because $1 \in A$, and A separates points because $\cos x \in A$ and $\cos x$ is strictly decreasing on $[0,2\pi]$. Thus A is dense in $\mathcal{C}\big([0,\pi]\big)$ by the Stone-Weierstrass Theorem.

The reason that A is not dense in $\mathcal{C}[0,2\pi]$ is that for every $f \in A$ we have $f(0) = f(2\pi)$. When $g \in \mathcal{C}[0,2\pi]$ with $g(0) \neq g(2\pi)$, for every $f \in A$ we have

$$\big|g(0) - g(2\pi)\big| \leq \big|g(0) - f(0) + f(2\pi) - g(2\pi)\big| \leq \big|g(0) - f(0)\big| + \big|f(2\pi) - g(2\pi)\big| \leq 2\|f-g\|_\infty$$

so that $\|f-g\|_\infty \geq \frac{1}{2}\big|g(0) - g(2\pi)\big|$.

5: (a) Let $f \in \mathcal{C}[0,1]$. Suppose that $\displaystyle\int_0^1 f(x)\,dx = 0$ and $\displaystyle\int_0^1 x^{12+3n} f(x)\,dx = 0$ for all $n \in \mathbb{Z}^+$. Use the Stone-Weierstrass Theorem to show that $f(x) = 0$ for all $x \in [0,1]$.

Solution: Let $A = \text{Span}\{1, x^{15}, x^{18}, x^{21}, \cdots\} = \left\{ p(x) = a_0 + \sum_{k=1}^{n} a_k x^{12+3k} \,\middle|\, n \in \mathbb{Z}^+, a_k \in \mathbb{R} \right\}$. Note that A is a subalgebra of $\mathcal{C}[0,1]$ (it is closed under scalar multiplication and under addition and multiplication of polynomials). Also note that A vanishes nowhere because $1 \in A$, and A separates points because $x^{15} \in A$ and x^{15} is strictly increasing on $[0,1]$, Thus A is dense in $\mathcal{C}[0,1]$ by the Stone-Weierstrass Theorem. Let $f \in \mathcal{C}[0,1]$ with $\int_0^1 f(x)\,dx = 0$ and $\int_0^1 x^{12+3n} f(x)\,dx = 0$ for all $n \in \mathbb{Z}^+$ Then we have $\int_0^1 pf = 0$ for every $p \in A$. Since A is dense in $\mathcal{C}[0,1]$ we can choose a sequence $(p_n)_{n\geq 1}$ in A with $p_n \to f$ in $\mathcal{C}[0,1]$. Since $p_n \to f$ uniformly on $[0,1]$, recall (or verify) that it follows that $p_n f \to f^2$ uniformly on $[0,1]$ and hence $\int_0^1 f^2 = \lim_{n\to\infty} \int_0^1 p_n f = \lim_{n\to\infty} 0 = 0$ (by Uniform Convergence and Integration). Since f is continuous on $[0,1]$ and $\int_0^1 f^2 = 0$, it follows that $f = 0$.

(b) Show that there does exist $0 \neq f \in \mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^2 x^{2n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbb{Z}$ but there does not exist $0 \neq f \in \mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^2 x^{3n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbb{Z}$.

Solution: If f is any continuous function whose restriction to $[-1,1]$ is odd and whose restriction to $[1,2]$ is zero (such as the function given by $f(x) = \sin(\pi x)$ for $-1 \leq x \leq 1$ and $f(x) = 0$ for $1 \leq x \leq 2$) then we have $\displaystyle\int_{-1}^2 x^{2n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbb{Z}$.

Let $A = \left\{ \sum_{k=0}^{n} c_k x^{3k} \,\middle|\, 0 \leq n \in \mathbb{Z}, c_k \in \mathbb{R} \right\}$. Note that A is a subalgebra of $\mathcal{C}[-1,2]$ and A vanishes nowhere because $1 \in A$, and A separates points because $x^3 \in A$ and x^3 is strictly increasing on $[-1,2]$, and so A is dense in $\mathcal{C}[-1,2]$ by the Stone-Weierstrass Theorem. Let $f \in \mathcal{C}[-1,2]$ with $\displaystyle\int_{-1}^2 x^{3n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbb{Z}$ and note that $\int_{-1}^2 pf = 0$ for every $p \in A$. Since A is dense in $\mathcal{C}[-1,2]$ we can choose a sequence $(p_n)_{n\geq 1}$ in A with $p_n \to f$ in $\mathcal{C}[-1,2]$. Then $p_n \to f$ uniformly on $[-1,2]$, so $p_n f \to f^2$ uniformly on $[-1,2]$, and hence $\int_{-1}^2 f^2 = \lim_{n\to\infty} \int_{-1}^2 p_n f = \lim_{n\to\infty} 0 = 0$. Since f is continuous on $[-1,2]$ and $\int_{-1}^2 f^2 = 0$, it follows that $f = 0$.

6: (a) For $n \in \mathbb{Z}^+$, define $f_n : [0, 2\pi] \to \mathbb{R}$ by $f_n(x) = (\sin x)^n$. Determine whether the set $A = \{f_n \,|\, n \in \mathbb{Z}^+\}$ is equicontinuous.

Solution: We claim that A is not equicontinuous. We need to show that there exists $\epsilon > 0$ such that for all $\delta > 0$ there exist $x, y \in [0, 2\pi]$ and there exists $n \in \mathbb{Z}^+$ such that $|x - y| < \delta$ and $|f_n(x) - f_n(y)| \geq \epsilon$. Choose $\epsilon = \frac{1}{2}$. Let $\delta > 0$. Choose $x = \frac{\pi}{2}$ and choose $y \in \left(0, \frac{\pi}{2}\right)$ with $|x - y| < \delta$. Note that $\sin x = 1$ and $0 < \sin y < 1$. Since $\lim\limits_{n\to\infty} (\sin y)^n = 0$ we can choose $n \in \mathbb{Z}^+$ such that $0 < (\sin y)^n \geq \frac{1}{2}$. Then we have $|f_n(x) - f_n(y)| = 1 - (\sin y)^n \geq \frac{1}{2}$, as required.

(b) Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. For each $y \in [0, 1]$ define $f_y : [0, 1] \to \mathbb{R}$ by $f_y(x) = f(x, y)$. Show that the set $A = \{f_y \,|\, y \in [0, 1]\}$ is compact in $\mathcal{C}[0, 1]$.

Solution: Define $F : [0, 1] \to \mathcal{C}[0, 1]$ by $F(y) = f_y$ and note that $A = \text{Range}(F)$. We claim that F is continuous. Let $\epsilon > 0$. Since f is uniformly continuous on $[0, 1] \times [0, 1]$, we can choose $\delta > 0$ such that for all $x_1, y_1, x_2, y_2 \in [0, 1]$, if $\|(x_1, y_1) - (x_2, y_2)\|_2 < \delta$ then $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$. Let $y_1, y_2 \in [0, 1]$ with $|y_1 - y_2| < \delta$. For all $x \in [0, 1]$ we have $\|(x, y_1) - (x, y_2)\|_2 = |y_1 - y_2| < \delta$, so $|f(x, y_1) - f(x, y_2)| < \epsilon$, that is $|f_{y_1}(x) - f_{y_2}(x)| < \epsilon$. Since $|f_{y_1}(x) - f_{y_2}(x)| < \epsilon$ for all $x \in [0, 1]$ it follows that $\|f_{y_1} - f_{y_2}\|_\infty \leq \epsilon$, that is $\|F(y_1) - F(y_2)\|_\infty \leq \epsilon$. Thus F is continuous, as claimed. Since $F : [0, 1] \to \mathcal{C}[0, 1]$ is continuous and $[0, 1]$ is compact, it follows that $A = \text{Range}(F)$ is compact.

(c) Show that the closed unit ball $\overline{B}(0, 1) = \{f \in \mathcal{C}[0, 1] \,|\, \|f\|_\infty \leq 1\}$ cannot be covered by a countable set of compact sets in $\mathcal{C}[0, 1]$.

Solution: We claim that every compact set in $\mathcal{C}[0, 1]$ is nowhere dense. Let K be a compact set in $\mathcal{C}[0, 1]$. Since K is closed, we need to show that $K^o = \emptyset$, so we need to show that for every $g \in K$ and every $\epsilon > 0$ there exists $f \in B(g, \epsilon)$ with $f \notin K$. Let $g \in K$ and let $\epsilon > 0$. Since K is compact, it is equicontinuous, so we can choose δ with $0 < \delta < 1$ such that for all $x, y \in [0, 1]$ and for every $f \in K$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \frac{\epsilon}{4}$. Let $f = g + h$ where $h : [0, 1] \to \mathbb{R}$ is defined by $h(x) = \frac{\epsilon}{\delta} x$ for $0 \leq x \leq \frac{\delta}{2}$ and $h(x) = \frac{\epsilon}{2}$ for $x \geq \frac{\delta}{2}$. Note that $h(0) = 0$, $h\left(\frac{\delta}{2}\right) = \frac{\epsilon}{2}$ and $\|h\|_\infty = \frac{\epsilon}{2}$. We have $\|f - g\|_\infty = \|h\|_\infty = \frac{\epsilon}{2}$, so that $f \in B(g, \epsilon) \subseteq \mathcal{C}[0, 1]$. When $x = 0$ and $y = \frac{\delta}{2}$ we have $|y - x| = \frac{\delta}{2} < \delta$ and we have

$$\tfrac{\epsilon}{2} = |h(y) - h(x)| = |f(y) - g(y) - f(x) + g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)| \leq |f(y) - f(x)| + \tfrac{\epsilon}{4}$$

so that $|f(y) - f(x)| \geq \frac{\epsilon}{4}$, hence $f \notin K$. Thus $K^o = \emptyset$, as claimed.

Since every compact set in $\mathcal{C}[0, 1]$ is nowhere dense, it follows from the Baire Category Theorem that every countable union of compact sets has empty interior, and so the closed unit ball cannot be covered by a countable union of compact sets.