PMATH 351 Real Analysis, Solutions to the Exercises for Chapter 5

1: (a) Prove Theorem 5.18: let X be a metric space (or topological space) and prove that the path components of X are path connected, and that every path connected subset of X is contained in one of the path components.

Solution: The path components of X are path connected since, for $a \in X$, if $x \in [a]$ and $y \in [a]$ then we have $x \sim a$ and $y \sim a$, and hence $x \sim y$ (since $\sim$ is an equivalence relation). We claim that every path connected subset of X is contained in one of the path components. Let $P \subseteq X$ be path connected. Let $p \in P$. Choose $a \in X$ such that $p \in [a] = \left\{x \in x \,\middle|\, x \sim a\right\}$. We claim that $P \subseteq [a]$. Let $x \in P$. Since $p \in [a]$ we have $p \sim a$ so $[p] = [a]$. Since $x \in P$ and $p \in P$ and P is path connected, we have $x \sim p$. Since $x \sim p$ we have $x \in [p] = [a]$. Thus $P \subseteq [a]$, as claimed.

(b) Let X be a metric space and let $A, B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$. Show that if A is connected then so is B.

Solution: Suppose that B is disconnected. Choose open sets U and V in X which separate B, that is with $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $U \cap V = \emptyset$ and $B \subseteq U \cup V$. We claim that U and V also separate A, and hence A is disconnected. We have $U \cap V = \emptyset$ and $A \subseteq B \subseteq U \cup V$ so it suffices to show that $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. Suppose, for a contradiction, that $U \cap A = \emptyset$. Then $A \subseteq U^c = X \setminus A$. Since U is open, it follows that U^c is closed. Since U^c is a closed set which contains A, it follows that $\overline{A} \subseteq U^c$ (since $\overline{A}$ is the intersection of all closed sets which contain A). Thus $B \subseteq \overline{A} \subseteq U^c$ and hence $U \cap B = \emptyset$, giving the desired contradiction. Thus $U \cap A \neq \emptyset$. Similarly $V \cap A \neq \emptyset$, and so U and V separate A, as claimed.

(c) Let X be a metric space. Show that the connected components of X are closed.

Solution: This follows from Part (b): if C is a component of X then C is connected, hence $\overline{C}$ is connected, hence $\overline{C}$ is contained in one of the components of X, hence $\overline{C} = C$.

2: For each of the following sets A in $\mathbb{R}^n$ (using its standard metric), determine whether A is complete, whether A is compact, and whether A is connected.

(a) $A = \left\{ x \in \mathbb{R}^3 \,\middle|\, \|x\| = \frac{n-1}{n^2} \text{ for some } n \in \mathbb{Z}^+ \right\}$.

Solution: Let $(a_n)_{n \geq 1}$ be the sequence given by $a_n = \frac{n-1}{n^2}$. Note that $a_1 = 0$, $a_2 = \frac{1}{4}$ and $a_3 = \frac{2}{9}$, and a_n is strictly decreasing for $n \geq 2$ because when $n \geq 2$ we have

$$a_n - a_{n+1} = \tfrac{n-1}{n^2} - \tfrac{n}{(n+1)^2} = \tfrac{(n-1)(n+1)^2 - n^3}{n^2(n+1)^2} = \tfrac{n^2-n-1}{n^2(n+1)^2} = \tfrac{n(n-1)-1}{n^2(n+1)^2} \geq \tfrac{1\cdot 2-1}{n^2(n+1)^2} = \tfrac{1}{n^2(n+1)^2} > 0,$$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n-1}{n^2} = 0$. Because $a_1 = 0$ and $a_2 = \frac{1}{4}$ and (a_n) is decreasing for $n \geq 2$, it follows that $\max\{a_n | n \in \mathbb{Z}^+\} = a_2 = \frac{1}{4}$. Thus A is bounded with $A \subseteq B(0,1)$, because when $x \in A$ we have $\|x\| = a_n$ for some n hence $\|x\| \leq a_2 = \frac{1}{4}$. We claim that A is closed. For $x \in \mathbb{R}^3$, we have

$$x \in A \iff \|x\| \in \{a_n \,|\, n \in \mathbb{Z}^+\} \iff \|x\| \in \{a_2, a_3, a_4, \cdots\} \cup \{0\}$$

and so

$$x \in A^c = \mathbb{R}^3 \setminus A \iff \|x\| \notin \{a_2, a_3, a_4, \cdots\} \cup \{0\} \iff \|x\| \in \bigcup_{n=1}^{\infty} I_n$$

where $I_1 = (a_2, \infty)$ and $I_n = (a_{n+1}, a_n)$ for $n \geq 2$. Thus

$$A^c = \bigcup_{n=1}^{\infty} U_n \quad \text{where} \quad U_n = \left\{ x \in \mathbb{R}^3 \,\middle|\, \|x\| \in I_n \right\}.$$

Note that each set U_n is open in $\mathbb{R}^3$ because each set I_n is open in $\mathbb{R}$ (indeed I_n is an open interval) and the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = \|x\| = \left({x_1}^2 + {x_2}^2 + {x_3}^2\right)^{1/2}$ is continuous (indeed every elementary function is continuous), and we have $U_n = \left\{ x \in \mathbb{R}^3 \,\middle|\, f(x) \in I_n \right\} = f^{-1}(I_n)$. Since each set U_n is open, it follows that $A^c = \bigcup\limits_{n=1}^{\infty} U_n$ is open, and hence A is closed. Since $\mathbb{R}^3$ is complete and A is closed in $\mathbb{R}^3$, A is complete. Since A is closed and bounded in $\mathbb{R}^3$, A is compact. But A is not connected because, for $r \in (a_2, a_1) = \left(\frac{2}{9}, \frac{1}{4}\right)$, the open sets $U = \left\{ x \in \mathbb{R}^3 \,\middle|\, \|x\| < r \right\}$ and $V = \left\{ x \in \mathbb{R}^3 \,\middle|\, \|x\| > r \right\}$ separate A.

(b) Let $A = \left\{ (u,v,w,x,y,z) \in \mathbb{R}^6 \,\middle|\, \text{rank} \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} < 2 \right\}$.

Solution: Note that we have $\text{rank} \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} = 2$ if and only if some pair of columns is linearly independent if and only if one of the three 2×2 submatrices $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$, $\begin{pmatrix} u & w \\ x & z \end{pmatrix}$ and $\begin{pmatrix} v & w \\ y & z \end{pmatrix}$ is invertible if and only if one of the three determinants $uy - vx$, $uz - wx$ and $vz - wy$ is non-zero. Thus we have

$$\text{rank} \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} < 2 \iff \big(uy - vx = 0 \text{ and } uz - wx = 0 \text{ and } vz - wy = 0 \big)$$

and hence

$$A = f^{-1}\big(\{0\}\big) \cap g^{-1}\big(\{0\}\big) \cap h^{-1}\big(\{0\}\big)$$

where $f, g, h : \mathbb{R}^6 \to \mathbb{R}$ are given by

$$f(u,v,w,x,y,z) = uy - vx\,,$$
$$g(u,v,w,x,y,z) = uz - wx\,,$$
$$h(u,v,w,x,y,z) = vz - wy\,.$$

Since f, g and h are continuous (they are polynomials) and $\{0\}$ is closed in $\mathbb{R}$, it follows that the sets $f^{-1}\big(\{0\}\big)$, $g^{-1}\big(\{0\}\big)$ and $h^{-1}\big(\{0\}\big)$ are all closed, and hence the set A is closed. Since $\mathbb{R}^6$ is complete and A is closed in $\mathbb{R}^6$, it follows that A is complete. On the other hand, A is not bounded because for $e_1 = (1,0,0,0,0,0)$ we have $re_1 \in A$ for all $r \in \mathbb{R}$ and $\|re_1\| = |r|$. Since A is not bounded, it is not compact. Finally, note that A is path connected (hence connected) because for all $p = (u,v,w,x,y,z) \in A$, the map $\alpha : [0,1] \to A$ given by $\alpha(t) = tp$ is a path from 0 to p in A (α takes values in A because the matrices corresponding to tp all have rank 0 or 1) so we have $p \sim 0$, and hence for all $p, q \in A$ we have $p \sim 0$ and $q \sim 0$ hence $p \sim q$.

3: For each of the following sets A, determine whether A is complete and whether A is compact.

(a) $A = \left\{ a = (a_k)_{k\geq 1} \in \mathbb{R}^\infty \,\middle|\, \|a\|_\infty \leq 1 \right\} \subseteq \mathbb{R}^\infty \subseteq \ell_\infty(\mathbb{R})$, using the metric d_∞.

Solution: We claim that A is not closed in ℓ_∞ (using the metric d_∞). Let $(x_n)_{n\geq 1}$ be the sequence in $\mathbb{R}^\infty$ given by $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}\, e_k = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, 0, 0, \cdots\right)$. Note that for every $n \in \mathbb{Z}^+$ we have $\|x_n\|_\infty = 1$ so that $x_n \in A$, and so $(x_n)_{n\geq 1}$ is a sequence in A. Also note that $x_n \to a$ in ℓ_∞ where $a = (a_k)_{k\geq 1} \in \ell_\infty$ is given by $a_k = \frac{1}{k}$ for all $k \in \mathbb{Z}^+$, that is $a = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\right)$, indeed we have

$$\left\|a - x_n\right\|_\infty = \left\|\left(0, 0, \cdots, 0, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \tfrac{1}{n+3}, \cdots\right)\right\|_\infty = \tfrac{1}{n+1} \longrightarrow 0 \text{ as } n \to \infty.$$

But notice that $a \notin \mathbb{R}^\infty$, so $a \notin A$. Since $(x_n)_{\geq 1}$ is a sequence in A with $x_n \to \infty$ in ℓ_∞, but $a \notin A$, it follows that A is not closed in ℓ_∞. Since ℓ_∞ is complete and A is not closed in ℓ_∞, it follows that A is not complete. Since A is not closed in ℓ_∞, it follows that A is not compact in ℓ_∞ and hence A is not compact (in $\mathbb{R}^\infty$ or in itself).

(b) $A = \left\{ a = (a_k)_{k\geq 1} \in \ell_1(\mathbb{R}) \,\middle|\, \sum\limits_{k=1}^{\infty} a_k = 0 \right\} \subseteq \ell_1(\mathbb{R})$, using the metric d_1.

Solution: We claim that A is closed. We provide two proofs. For the first proof, let $(x_n)_{n\geq 1}$ be a sequence in A $x_n \to a$ in ℓ_1. Write $x_n = (x_{n,k})_{k\geq 1}$ and $a = (a_k)_{k\geq 1}$. Let $\epsilon > 0$. Since $x_n \to a$ in ℓ_1, we can choose $\ell \in \mathbb{Z}^+$ such that $\|x_\ell - a\|_1 < \frac{\epsilon}{2}$, that is $\sum\limits_{k=1}^{\infty} |x_{\ell,k} - a_k| < \frac{\epsilon}{2}$. Since $x_\ell \in A$ we have $\sum\limits_{k=1}^{\infty} x_{\ell,k} = 0$ so we can choose $m \in \mathbb{Z}^+$ such that $k \geq m \Longrightarrow \left|\sum\limits_{k=1}^{m} x_{\ell,k}\right| < \frac{\epsilon}{2}$. Then for $k \geq m$ we have

$$\left|\sum_{k=1}^{m} a_k\right| = \left|\sum_{k=1}^{m} (a_k - x_{\ell,k} + x_{\ell,k})\right| \leq \sum_{k=1}^{m} |a_k - x_{\ell,k}| + \sum_{k=1}^{m} |x_{\ell,k}| \leq \sum_{k=1}^{\infty} |a_k - x_{\ell,k}| + \sum_{k=1}^{m} |x_{\ell,k}| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

This proves that $\sum\limits_{k=1}^{\infty} a_k = 0$ so that $a \in A$, and so A is closed, as claimed.

For the second proof, note that A is a subspace of ℓ_1. Define $S : \ell_1 \to \mathbb{R}$ by $S(a) = \sum\limits_{k=1}^{\infty} a_k$. Note that S is linear (and S is well-defined because absolute convergence implies convergence) with $\text{Null}(S) = S$. Note that S is continuous because when $a \in \ell_1$ with $\|a\| \leq 1$ we have $\left|S(a)\right| = \left|\sum\limits_{k=1}^{\infty} a_k\right| \leq \sum\limits_{k=1}^{\infty} |a_k| = \|a\|_1 \leq 1$. Also not that $A = S^{-1}\left(\{0\}\right)$. Since S is continuous and $\{0\}$ is closed in ℓ_1, it follows that $A = S^{-1}\left(\{0\}\right)$ is closed in ℓ_1.

Since ℓ_1 is complete (using the metric d_1) and A is closed in ℓ_1, it follows that A is complete. On the other hand, note that A is not bounded because for $a = \sum\limits_{k=1}^{2n} (-1)^k e_k = (-1, 1, -1, 1, \cdots, -1, 1, 0, 0, \cdots)$, we have $a \in A$ with $\|a\|_1 = 2n$, and hence A is not compact.

(c) $A = \left\{ f \in \mathcal{C}([0,1], \mathbb{R}) \,\middle|\, |f(x)| \leq \frac{1}{x} \text{ for all } x \in (0,1) \right\} \subseteq \mathcal{C}([0,1], \mathbb{R})$, using the metric d_∞.

Solution: We claim that A is closed in $\mathcal{C}[0,1]$ (using the supremum metric d_∞). Let $f \in A^c = \mathcal{C}[0,1] \setminus A$. Choose $a \in (0,1)$ such that $\left|f(a)\right| > \frac{1}{a}$. Let $r = \left|f(a)\right| - \frac{1}{a}$ and note that $r > 0$. We claim that $B(f,r) \subseteq A^c$. Let $g \in B(f,r)$, that is let $g \in \mathcal{C}[0,1]$ with $\|g - f\|_\infty \leq 1$. Since $|f(a)| \leq \left|f(a) - g(a)\right| + \left|g(a)\right|$, we have

$$|f(a)| - |g(a)| \leq \left|f(a) - g(a)\right| \leq \|f - g\|_\infty < r = \left|f(a)\right| - \tfrac{1}{a}$$

and hence $\left|g(a)\right| > \frac{1}{a}$, so that $g \in A^c$. Thus $B(f,r) \subseteq A^c$, showing that A^c is open, hence A is closed, as claimed. Since $\mathcal{C}[0,1]$ is complete and A is closed in $\mathcal{C}[0,1]$, it follows that A is complete.

On the other hand, A is not bounded because given $r > 0$ we can define $f : [0,1] \to \mathbb{R}$ by $f(x) = r$ for $0 \leq x \leq \frac{1}{r}$ and $f(x) = \frac{1}{x}$ for $\frac{1}{r} \leq x \leq 1$ and then we have $f \in A$ with $\|f\|_\infty = r$. Since A is not bounded, it is not compact.

4: The theorems about connectedness in the lecture notes are stated for metric spaces. They also apply, more generally, to topological spaces with minor alterations in the proofs, with a slight change in Definition 5.1. Let X be a topological space and let $P \subseteq X$. For sets $A, B \subseteq X$ (not necessarily open sets), we say that A and B **separate** P in X when

$$A \cap P \neq \emptyset\ ,\ B \cap P \neq \emptyset\ ,\ A \cap \overline{B} = B \cap \overline{A} = \emptyset\ ,\ P \subseteq A \cup B\,.$$

Recall that we say that P is **connected** when P is not equal to the union of two disjoint nonempty open subsets $U, V \subseteq P$ (or equivalently when the only subsets of P which are both open and closed in P are the sets $\emptyset$ and P), and otherwise we say that P is **disconnected**.

(a) Let X be a topological space and let $A \subseteq P \subseteq X$. Let $\overline{A} = \mathrm{Cl}_X(A)$ be the closure of A in X and let $\mathrm{Cl}_P(A)$ be the closure of A in P. Show that $\mathrm{Cl}_P(A) = \mathrm{Cl}_X(A) \cap P$.

Solution: Since $\overline{A}$ is closed in X it follows that $\overline{A} \cap P$ is closed in P. Since $A \subseteq \overline{A}$ and $A \subseteq P$ we have $A \subseteq \overline{A} \cap P$. Since $\overline{A} \cap P$ is closed in P and $A \subseteq \overline{A} \cap P$, it follows from the definition of $\mathrm{Cl}_P(A)$ that $\mathrm{Cl}_P(A) \subseteq \overline{A} \cap P$.

Let F be any closed set in P with $A \subseteq F$. Choose a closed set K in X such that $F = K \cap P$. Since K is closed in X and $A \subseteq K$ we have $\overline{A} \subseteq K$. Thus $\overline{A} \cap P \subseteq K \cap P = F$. Since $\overline{A} \cap P \subseteq F$ for every closed set F in P which contains A, it follows, from the definition of $\mathrm{Cl}_P(A)$, that $\overline{A} \cap P \subseteq \mathrm{Cl}_P(A)$.

(b) Let X be a topological space and let $P \subseteq X$. Show that P is disconnected if and only if there exist sets $A, B \subseteq X$ which separate P in X, as defined above.

Solution: Suppose that P is disconnected, say $P = U \cup V$ where U and V are nonempty disjoint open subsets of P. Since U is closed in P we have $U = \mathrm{Cl}_P(U) = \mathrm{Cl}_X(U) \cap P = \overline{U} \cap P$. Since $V \subseteq P$ we have $V = V \cap P$ so $V \cap \overline{U} = V \cap P \cap \overline{U} = V \cap U = \emptyset$. Similarly, we have $U \cap \overline{V} = \emptyset$, and so we can take $A = U$ and $B = V$ and then A and B separate P.

Suppose, conversely, that there exist sets, say $A, B \subseteq X$, which separate P in X. Let $U = A \cap P$ and $V = B \cap P$, and note that $U \neq \emptyset$, $V \neq \emptyset$, $P = U \cup V$ and we have $U \cap \overline{V} = \emptyset$ (because $U \cap \overline{V} \subseteq A \cap \overline{B} = \emptyset$) and $V \cap \overline{U} = \emptyset$. We claim that U and V are open in P (and hence P is the union of two nonempty disjoint open subsets). Since $U \cap \overline{V} = \emptyset$ we also have $U \cap \mathrm{Cl}_P(V) = \emptyset$. Since $P = U \cup V$ we also have $P = U \cup \mathrm{Cl}_P(V)$. Thus P is the disjoint union of U and $\mathrm{Cl}_P(V)$. Since $\mathrm{Cl}_P(V)$ is closed in P, its complement $U = P \setminus \mathrm{Cl}_P(V)$ is open in P. Similarly V is open in P, as claimed.

(c) Let X be a metric space and let $P \subseteq X$. Show that there exist sets $A, B \subseteq X$ which separate P, as defined above, if and only if there exist open sets $U, V \subseteq X$ which separate P as in Definition 5.1, that is

$$U \cap P \neq \emptyset\ ,\ V \cap P \neq \emptyset\ ,\ U \cap P = \emptyset\ ,\ P \subseteq U \cup V\,.$$

Solution: Suppose there exist open sets, say $U, V \subseteq X$, such that $U \cap P \neq \emptyset$, $V \cap P \neq \emptyset$, $U \cap V = \emptyset$, and $P \subseteq U \cup V$. Since $U \cap V = \emptyset$, we have $V \subseteq U^c$ (where $U^c = X \setminus V$) and hence also $\overline{V} \subseteq U^c$ (by the definition of $\overline{V}$, since U^c is closed in X). Similarly, we have $V \cap \overline{U} = \emptyset$, so we can take $A = U$ and $B = V$ and then A and B separate P, as defined above.

Suppose there exist sets, say $A, B \subseteq X$, which separate P as defined above, that is $A \cap P \neq \emptyset$, $B \cap P \neq \emptyset$, $A \cap \overline{B} = B \cap \overline{A} = \emptyset$ and $P \subseteq A \cup B$. For each $a \in A$, since $A \cap \overline{B} = \emptyset$ so that $A \subseteq \overline{B}^c$, which is open, we can choose $r_a > 0$ so that $B(a, 2r_a) \subseteq \overline{B}^c$, that is $B(a, 2r_a) \cap \overline{B} = \emptyset$. Similarly, for each $b \in B$ we can choose $s_b > 0$ such that $B(b, 2s_b) \cap \overline{A} = \emptyset$. Let $U = \bigcup_{a \in A} B(a, r_a)$ and let $V = \bigcup_{b \in B} B(b, s_b)$. Then U and V are open in X with $A \subseteq U$ and $B \subseteq V$ so that $U \cap P \neq \emptyset$ and $V \cap P \neq \emptyset$ and $P \subseteq U \cup V$. We claim that $U \cap V = \emptyset$. Suppose, for a contradiction, that $U \cap V \neq \emptyset$, say $p \in U \cap V$. Since $p \in U = \bigcup_{a \in A} B(a, r_a)$, we can choose $a \in A$ such that $p \in B(a, r_a)$. Likewise, we can choose $b \in B$ such that $p \in B(b, s_b)$. Say $r_a \leq s_b$ (the case $s_b \leq r_a$ is similar). Then we have $d(a, b) \leq d(a, p) + d(p, b) < r_a + s_b \leq 2s_b$ and hence $a \in B(b, 2s_b)$. But this contradicts our choice of s_b.

6: (a) Prove Theorem 5.37: let X and Y be metric spaces and let $f: X \to Y$. Show that if X is compact and f is continuous then f is uniformly continuous.

Solution: Suppose that X is compact and f is continuous. Let $\epsilon > 0$. Given $a \in X$, since f is continuous at a we can chose $\delta_a > 0$ such that for all $x \in X$, if $d(x, a) < 2\delta_a$ then $d(f(x), f(a)) < \frac{\epsilon}{2}$. Note that the set $\{B(a, \delta_a) \mid a \in X\}$ is an open cover of X. Since X is compact we can choose a finite subcover, so we can choose finitely many points $a_1, a_2, \cdots, a_n \in X$ such that $X = \bigcup_{k=1}^{n} B(a_k, \delta_{a_k})$. Choose $\delta = \min\{\delta_{a_1}, \delta_{a_2}, \cdots, \delta_{a_n}\}$. Let $x, y \in X$ with $d(x, y) < \delta$. Choose an index k such that $x \in B(a_k, \delta_{a_k})$. Since $d(x, a_k) < \delta_{a_k}$ and $d(x, y) < \delta \leq \delta_{a_k}$, we have $d(y, a_k) < 2\delta_{a_k}$ (by the Triangle Inequality) and hence $d(f(y), f(a_k)) < \frac{\epsilon}{2}$. Since $d(x, a_k) < \delta \leq \delta_{a_k} < 2\delta_{a_k}$ we also have $d(f(x), f(a_k)) < \frac{\epsilon}{2}$. Since $d(f(x), f(a_k)) < \frac{\epsilon}{2}$ and $d(f(y), f(a_k)) < \frac{\epsilon}{2}$, it follows, again from the Triangle Inequality, that $d(f(x), f(y)) < \epsilon$.

(b) Let X be a metric space. Show that if X is compact then X must be separable.

Solution: Suppose that X is compact. By Theorem 5.39, X is totally bounded, so for each $n \in \mathbb{Z}^+$, we can cover X with finitely many open balls of radius $\frac{1}{n}$. For each $n \in \mathbb{Z}^+$, choose $a_{n,1}, a_{n,2}, \cdots, a_{n,\ell_n} \in X$ such that $X = \bigcup_{k=1}^{\ell_n} B\big(a_{n,k}, \frac{1}{n}\big)$. Let $A = \big\{a_{n,k} \big| n \in \mathbb{Z}^+, 1 \leq k \leq \ell_n\big\}$ and note that A is finite or countable. We claim that A is dense in X. It suffices to show that for all $x \in X$ and all $\epsilon > 0$ there exists $a \in A$ such that $d(x, a) < \epsilon$. Let $x \in X$ and let $\epsilon > 0$. Choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \epsilon$. Since $X = \bigcup_{k=1}^{\ell_n} B\big(a_{n,k}, \frac{1}{n}\big)$ we can choose k with $1 \leq k \leq \ell_n$ such that $x \in B\big(a_{n,k}, \frac{1}{n}\big)$. Then we have $a_{n,k} \in A$ and $d(x, a_{n,k}) < \frac{1}{n} < \epsilon$, as required.

(c) Let U be a non-trivial finite-dimensional vector space over $\mathbb{R}$. Show that there does not exist a norm on U which makes U compact, but there does exist a metric on U which makes U compact.

Solution: Let $\| \ \|$ be any norm on U. Choose $0 \neq u \in U$. Since $u \neq 0$ we have $\|u\| > 0$ and so $\lim_{t\to\infty} \|tu\| = \lim_{t\to\infty} t\|u\| = \infty$, and hence U is not bounded. Since U is not bounded, it cannot be compact. On the other hand, let us show that U can be given a metric to make it compact. Note that since U is finite dimensional we have $|U| = 2^{\aleph_0}$. Indeed if $\{u_1, \cdots, u_n\}$ is a basis for U then the map $F: \mathbb{R}^n \to U$ given by $F(t) = \sum_{k=1}^{n} t_k u_k$ is bijective, so we have $|U| = |\mathbb{R}^n| = (2^{\aleph_0})^n = 2^{\aleph_0 \cdot n}$ and, using various properties of cardinal arithmetic, note that $2^{\aleph_0} = 2^{\aleph_0 \cdot 1} \leq 2^{\aleph_0 \cdot n} \leq 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$ so that $2^{\aleph_0 \cdot n} = 2^{\aleph_0}$ by the Cantor-Schröder- Bernstein Theorem. Since $|U| = 2^{\aleph_0} = \big|[0, 1]\big|$, we can choose any bijective map $g: U \to [0, 1]$ and then give U the metric which makes g an isometry from U to the compact metric space $[0, 1]$ (using its standard topology). That is, we define the metric d on U by $d(x, y) = \big|g(x) - g(y)\big|$. It is easy to check that d is a metric and that g is an isometry. Since g is an isometry it is a homeomorphism, and since g^{-1} is continuous and $[0, 1]$ is compact, it follows that U is compact.

7: (a) Define a metric on $\mathbb{R}$ by $d(x,y)=\frac{|x-y|}{1+|x-y|}$ (you do not need to prove that d is a metric). Show that $(\mathbb{R},d)$ is bounded but not totally bounded (hence not compact).

Solution: Note that $(\mathbb{R},d)$ is bounded, indeed we have $B(0,1)=\mathbb{R}$ because for all $x\in\mathbb{R}$ we have

$$d(x,0)=\tfrac{|x|}{|x|+1}<\tfrac{|x|+1}{|x|+1}=1.$$

We claim that $(\mathbb{R},d)$ is not uniformly bounded, indeed we claim that, using this metric d, $\mathbb{R}$ cannot be covered by finitely many open balls of radius $\frac{1}{4}$. Let $A\subseteq X$ be any subset such that $X=\bigcup\limits_{a\in A}B\big(a,\frac{1}{4}\big)$. For each $n\in\mathbb{Z}^+$, choose $a_n\in A$ such that $n\in B\big(a_n,\frac{1}{4}\big)$. Note that

$$\begin{aligned} n\in B\big(a_n,\tfrac{1}{4}\big) &\Longrightarrow d(n,a_n)<\tfrac{1}{4}\Longrightarrow \tfrac{|n-a_n|}{|n-a_n|+1}<\tfrac{1}{4}\Longrightarrow 4|n-a_n|<|n-a_n|+1\\ &\Longrightarrow 3|n-a_n|<1\Longrightarrow |n-a_n|<\tfrac{1}{3}\Longrightarrow a_n\in\big(n-\tfrac{1}{3},n+\tfrac{1}{3}\big).\end{aligned}$$

Since the intervals $\big(n-\frac{1}{3},n+\frac{1}{3}\big)$ are disjoint, it follows that the map $f:\mathbb{Z}^+\to A$ given by $f(n)=a_n$ is injective, so we have $|A|\geq\aleph_0$ hence A is infinite.

(b) Let $X=2^{\mathbb{Z}^+}$ be the space of binary sequences $(x_k)_{k\geq1}$ with each $x_k\in\{0,1\}$. Define a metric on X by $d(x,y)=\sum\limits_{k=1}^{\infty}\frac{|x_k-y_k|}{2^k}$ (you do not need to prove that d is a metric). Show that X is complete and totally bounded (hence compact).

Solution: Let $(x_n)_{n\geq1}$ be a Cauchy sequence in X. Each x_n is a binary sequence, say $x_n=(x_{n,k})_{k\geq1}$ with each $x_{n,k}\in\{0,1\}$. We claim that for each $\ell\in\mathbb{Z}^+$, the sequence $(x_{n,\ell})_{n\geq1}$ is eventually costant (hence convergent). Let $\ell\in\mathbb{Z}^+$. Since $(x_n)_{n\geq1}$ is Cauchy in X, we can choose $N\in\mathbb{Z}^+$ such that for all $n,m\geq N$ we have $d(x_n,x_m)<\frac{1}{2^\ell}$, that is $\sum\limits_{k=1}^{\infty}\frac{|x_{n,k}-x_{m,k}|}{2^k}<\frac{1}{2^\ell}$. Then for all $n\geq N$ we have $\frac{|x_{n,\ell}-x_{N,\ell}|}{2^\ell}\leq\sum\limits_{k=1}^{\infty}\frac{|x_{n,k}-x_{N,k}|}{2^k}<\frac{1}{2^\ell}$ so that $|x_{n,\ell}-x_{N,\ell}|<1$. Since $x_{n,\ell},x_{N,\ell}\in\{0,1\}$ it follows that $x_{n,\ell}=x_{N,\ell}$, thus the binary sequence $(x_{n,\ell})_{n\geq1}$ is evenually constant, as claimed. For each $\ell\in\mathbb{Z}^+$, let $b_\ell=\lim\limits_{nto\infty}a_{n,\ell}$ and note that each $b_\ell\in\{0,1\}$ since the sequence $(x_{n,\ell})_{n\geq1}$ is eventually constant.

We claim that $\lim\limits_{n\to\infty}x_n=b$ in X. Choose $\ell\in\mathbb{Z}^+$ such that $\frac{1}{2^\ell}<\epsilon$. Let $\epsilon>0$. since each sequence $(x_n,k)_{n\geq1}$ is eventually constant, we can choose $N\in\mathbb{Z}^+$ such that $x_{n,k}=x_{N,k}=b_k$ for all $n\geq N$ and all $1\leq k\leq\ell$. Then for all $n\geq N$ we have

$$d(x_n,b)=\sum_{k=1}^{\infty}\tfrac{|x_{n,k}-b_k|}{2^k}=\sum_{k=\ell+1}^{\infty}\tfrac{|x_{n,k}-b_k|}{2^k}\leq\sum_{k=\ell+1}^{\infty}\tfrac{1}{2^k}=\tfrac{1}{2^\ell}<\epsilon.$$

Thus $\lim\limits_{n\to\infty}x_n=b$ in X, as claimed, and so X is complete.

We claim that X is totally bounded. Let $\epsilon>0$. Choose $\ell\in\mathbb{Z}^+$ such that $\frac{1}{2^\ell}<\epsilon$. Let $A\subseteq X$ be the set of all binary sequences of the form $a=(a_1,a_2,\cdots,a_\ell,0,0,0,\cdots)$, and note that A is finite, indeed $|A|=2^\ell$. Given $x\in X$ we can choose $a\in A$ such that $(x_1,x_2,\cdots,x_\ell)=(a_1,a_2,\cdots,a_\ell)$ and then we have

$$d(x,a)=\sum_{k=1}^{\infty}\tfrac{|x_k-a_k|}{2^k}=\sum_{k=\ell+1}^{\infty}\tfrac{|x_k-a_k|}{2^k}\leq\sum_{k=\ell+1}^{\infty}\tfrac{1}{2^k}=\tfrac{1}{2^\ell}<\epsilon$$

so that $x\in B(a,\epsilon)$, This shows that $X=\bigcup\limits_{a\in A}B(a,\epsilon)$ and so X is totally bounded.

8: (a) Let $B = \overline{B}(0,1) = \{x \in \ell_2(\mathbb{R}) \,|\, ||x||_2 \leq 1\} \subseteq \ell_2(\mathbb{R})$. Show that B is not compact in $(\ell_2(\mathbb{R}), d_2)$.

Solution: Let $E = \{e_1, e_2, e_3, \cdots\}$, let $U_0 = \ell_2 \setminus E$, let $U_n = B(e_n, 1) = \{x \in \ell_2 \big| ||x - e_n||_2 < 1\}$ for $n \in \mathbb{Z}^+$. and let $\mathcal{U} = \{U_0, U_1, U_2, \cdots\}$. Note that E is closed (because for all $k \neq \ell$ we have $||e_k - e_\ell||_2 = \sqrt{2}$, so every Cauchy sequence in E is eventually constant) and so U_0 is open, and so $\mathcal{U}$ is an open cover of B. But $\mathcal{U}$ has no finite subcover, indeed $\mathcal{U}$ has no proper subcover, because the point $0 \in B$ only lies in the set U_0 and for each $k \in \mathbb{Z}^+$, the point $e_k \in B$ only lies in the set U_k (when $n \in \mathbb{Z}^+$ with $n \neq k$ we have $||e_k - e_n||_2 = \sqrt{2}$ so $e_k \notin B(e_n, 1) = U_n$).

(b) Let $r_k \geq 0$ for all $k \in \mathbb{Z}^+$, and let $S = \{x \in \ell_2(\mathbb{R}) \,|\, |x_k| \leq r_k \text{ for all } k \in \mathbb{Z}^+\} \subseteq \ell_2(\mathbb{R})$. Show that S is compact in $(\ell_2(\mathbb{R}), d_2)$ if and only if $\sum\limits_{k=1}^{\infty} {r_k}^2$ converges in $\mathbb{R}$.

Solution: If $\sum\limits_{k=1}^{\infty} {r_k}^2 = \infty$. then S is unbounded because $s_n = \sum\limits_{k=1}^{n} r_k e_k \in S$ and $||s_n||_2 = \sum\limits_{k=1}^{n} {r_k}^2 \longrightarrow \infty$ as $n \to \infty$, and hence S is not compact. Suppose that $\sum\limits_{k=1}^{\infty} {r_k}^2 < \infty$. We claim that every sequence in S has a convergent subsequence whose limit lies in S. Let $\{x_n\}_{n \geq 1}$ be a sequence in S, say $x_n = \sum\limits_{k=1}^{\infty} x_{n,k} e_k$ with $|x_{n,k}| \leq r_k$ for all n, k. Since $x_{n,1} \in [-r_1, r_1]$ for all n, we can choose $m_1 < m_2 < m_3 < \cdots$ so that the sequence $\{x_{m_n,1}\}_{n \geq 1}$ converges in $\mathbb{R}$, say to $c_1 \in [-r_1, r_1]$. Denote the subsequence $\{x_{m_n}\}_{n \geq 1}$ of $\{x_n\}$ in ℓ_2 by $\{x_n^1\}$ so we have $x_n^1 = \sum\limits_{k=1}^{\infty} x_{n,k}^1 e_k$ with $x_{n,k}^1 = x_{m_n,k}$. Note that $x_{n,k}^1 \in [-r_k, r_k]$ for all n, k and $\lim\limits_{n \to \infty} x_{n,1}^1 = c_1 \in [-r_1, r_1]$. Since $x_{n,2}^1 \in [-r_2, r_2]$ for all n, we can re-choose $m_1 < m_2 < m_3 < \cdots$ so that the sequence $\{x_{m_n,2}^1\}_{n \geq 1}$ converges in $\mathbb{R}$, say to $c_2 \in [-r_2, r_2]$. Denote the subsequence $\{x_{m_n}^1\}_{n \geq 1}$ of $\{x_n^1\}$ in ℓ_2 by $\{x_n^2\}$ so we have $x_n^2 = \sum\limits_{k=1}^{\infty} x_{n,k}^2 e_k$ with $x_{n,k}^2 = x_{m_n,k}^1$. Note that $x_{n,k}^2 \in [-r_k, r_k]$ for all n, k and $\lim\limits_{n \to \infty} x_{n,1}^2 = c_1$ and $\lim\limits_{n \to \infty} x_{n,2}^2 = c_2$. Repeat this procedure to obtain successive subsequences $\{x_n^m\}_{n \geq 1}$ in ℓ_2 for each $m \in \mathbb{Z}^+$ given by $x_n^m = \sum\limits_{k=1}^{\infty} x_{n,k}^m e_k$ with $|x_{n,k}^m| \leq r_k$ for all m, n, k such that $\lim\limits_{n \to \infty} x_{n,k}^m = c_k \in [-r_k, r_k]$ in $\mathbb{R}$ for all $k \leq m$. Let $\{y_n\}$ be the diagonal sequence $y_n = x_n^n = \sum\limits_{k=1}^{\infty} x_{n,k}^n e_k$, and note that $\{y_n\}$ is a subsequence of the original sequence $\{x_n\}$. We claim that $y \to c$ in ℓ_2 where $c = \sum\limits_{k=1}^{\infty} c_k e_k$. Let $\epsilon > 0$. Since $\sum\limits_{k=1}^{\infty} {r_k}^2 < \infty$, we can choose $m \in \mathbb{Z}^+$ so that $\sum\limits_{k=m+1}^{\infty} {r_k}^2 < \frac{\epsilon^2}{8}$. Since $\lim\limits_{n \to \infty} x_{n,k}^m = c_k$ for all $k \leq m$, we can choose $N \in \mathbb{Z}^+$ with $N \geq m$ so that for all $n \geq N$ we have $\left|x_{n,k}^m - c_k\right| < \frac{\epsilon^2}{2m}$ for all $k \leq m$. Note that when $m' \geq m$, $\{x_n^{m'}\}$ is a subsequence of $\{x_n^m\}$ so for each $n \in \mathbb{Z}^+$ we have $x_n^{m'} = x_{n'}^m$ for some $n' \geq n$. In particular, when $n \geq N$ we have $y_n = x_n^n = x_{n'}^m$ for some $n' \geq n$, and so

$$\begin{aligned} ||y_n - c||_2^2 &= \sum_{k=1}^{\infty} |y_{n,k} - c_k|^2 = \sum_{k=1}^{m} |y_{n,k} - c_k|^2 + \sum_{k=m+1}^{\infty} |y_{n,k} - c_k|^2 \\ &\leq \sum_{k=1}^{m} |x_{n',k}^m - c_k| + \sum_{k=m+1}^{\infty} (2r_k)^2 \leq m \cdot \tfrac{\epsilon^2}{2m} + 4 \cdot \tfrac{\epsilon^2}{8} = \epsilon^2 \end{aligned}$$

hence $||y_n - c||_2 < \epsilon$. Since every sequence in S has a subsequence which converges to an element in S, it follows that S is compact (by Theorem 5.39).