PMATH 351 Real Analysis, Solutions to the Exercises for Chapter 4

1: (a) Let $c_0 = \big\{ a \in \ell_\infty(\mathbb{R}) \,\big|\, \lim\limits_{n\to\infty} a_n = 0 \big\}$. Show that (c_0, d_∞) is separable.

Solution: Recall (from the proof of Theorem 4.7) that $\mathbb{Q}^\infty$ is countable. We claim that $\mathbb{Q}^\infty$ is dense in c_0. Note that $\mathbb{Q}^\infty \subseteq c_0$ because for $a = (a_n)_{n\geq 1} \in \mathbb{Q}^\infty$, we can choose $m \in \mathbb{Z}^+$ such that $a_n = 0$ for all $n > m$, and hence $\lim\limits_{n\to\infty} a_n = 0$. Let $x \in c_0$ with $x \notin \mathbb{Q}^\infty$. We must show that x is a limit point of $\mathbb{Q}^\infty$. Let $r > 0$. Since $\lim\limits_{n\to\infty} x_n = 0$ we can choose $m \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$ with $n \geq m$ we have $|x_n| < \frac{r}{2}$. For each $n \in \mathbb{Z}^+$ with $1 \leq n \leq m$, choose $a_n \in \mathbb{Q}$ with $|x_n - a_n| < \frac{r}{2}$, then let $a = \sum\limits_{n=1}^{m} a_n e_n = (a_1, a_2, \cdots, a_n, 0, 0, \cdots) \in \mathbb{Q}^\infty$. Then we have $\|x - a\|_\infty = \sup\big\{ |x_1 - a_1|, |x_2 - a_2|, \cdots, |x_n - a_n|, |x_{n+1}|, |x_{n+2}|, \cdots \big\} \leq \frac{r}{2}$ so that $x \in B(a, r)$. And since $a \in \mathbb{Q}^\infty$ and $x \notin \mathbb{Q}^\infty$ so that $x \neq a$, we have $a \in B^*(x, r)$, and so x is a limit point of $\mathbb{Q}^\infty$, as required.

(b) Show that $(\ell_\infty(\mathbb{C}), d_\infty)$ is complete.

Solution: Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in ℓ_∞. For each $n \in \mathbb{Z}^+$, a_n is a bounded sequence of complex numbers, say $a_n = (a_{n,k})_{k\geq 1}$. Fix an index $k \in \mathbb{Z}^+$ and let $\epsilon > 0$. Since $(a_n)_{n\geq 1}$ is Cauchy in ℓ_∞, we can choose $N \in \mathbb{Z}^+$ such that $n, m \geq N \Longrightarrow \|a_n - a_m\|_\infty < \epsilon$. Then for $n, m \geq N$ we have $|a_{n,j} - a_{m,j}| < \epsilon$ for all $j \in \mathbb{Z}^+$ so, in particular, $|a_{n,k} - a_{m,k}| < \epsilon$. This shows that for each $k \in \mathbb{Z}^+$, the sequence $(a_{n,k})_{n\geq 1}$ is a Cauchy sequence in $\mathbb{C}$, so it converges. For each $k \in \mathbb{Z}^+$, let $b_k = \lim\limits_{n\to\infty} a_{n,k} \in \mathbb{C}$, and then let $b = (b_k)_{k\geq 1}$.

We claim that $b \in \ell_\infty$ (that is, the sequence $b = (b_k)_{k\geq 1}$ is bounded in $\mathbb{C}$). Since $(a_n)_{n\geq 1}$ is Cauchy in ℓ_∞, it is bounded in ℓ_∞, so we can choose $M \geq 0$ such that $\|a_n\|_\infty \leq M$ for all indices $n \in \mathbb{Z}^+$. Then for all $k, n \in \mathbb{Z}^+$ we have $|a_{n,k}| \leq \|a_n\|_\infty \leq M$ and hence, for all $k \in \mathbb{Z}^+$, $|b_k| = \big| \lim\limits_{n\to\infty} a_{n,k} \big| = \lim\limits_{n\to\infty} |a_{n,k}| \leq M$. Thus the sequence $(b_k)_{k\geq 1}$ is bounded in $\mathbb{C}$, that is $b \in \ell_\infty$, as claimed.

Finally, we claim that $a_n \to b$ in ℓ_∞. Let $\epsilon > 0$. Choose $N \in \mathbb{Z}^+$ so that $n, m \geq N \Longrightarrow \|a_n - a_m\|_\infty < \epsilon$. Then for $n, m \geq N$ we have $|a_{n,k} - a_{m,k}| < \epsilon$ for all indices $k \in \mathbb{Z}^+$. It follows that for all $n \geq N$ and for all $k \in \mathbb{Z}^+$ we have $|a_{n,k} - b_k| = \lim\limits_{m\to\infty} \big| a_{n,k} - a_{m,k} \big| \leq \epsilon$ and hence, for all $n \geq N$, we have $\|a_n - b\|_\infty \leq \epsilon$. This shows that $a_n \to b$ in ℓ_∞, as claimed.

2: (a) Show that $(\ell_2(\mathbb{R}), d_2)$ is separable.

Solution: Recall (from the proof of Theorem 4.7) that $\mathbb{Q}^\infty$ is countable. It is clear that $\mathbb{Q}^\infty \subseteq \ell_2(\mathbb{R})$. We claim $\mathbb{Q}^\infty$ is dense in $\big(\ell_2, d_2\big)$. Let $b = (b_k)_{k\geq 1} \in \ell_2$ with $b \notin \mathbb{Q}^\infty$. We need to show that b is a limit point of $\mathbb{Q}^\infty$. Let $r > 0$. Since $\sum\limits_{k=1}^{\infty} |b_k|^2 < \infty$, we can choose $n \in \mathbb{Z}^+$ so that $\sum\limits_{k=n+1}^{\infty} |b_k|^2 < \frac{r^2}{2}$. For each $k \in \mathbb{Z}^+$ with $1 \leq k \leq n$, choose $a_k \in \mathbb{Q}$ so that $|a_k - b_k| < \frac{r^2}{2^{k+1}}$. Let $a = \sum\limits_{k=1}^{n} a_k e_k = (a_1, a_2, \cdots, a_n, 0, 0, \cdots) \in \mathbb{Q}^\infty$. We have

$$\|a-b\|_2^2 = \sum_{k=1}^{\infty} |a_k - b_k|^2 = \sum_{k=1}^{n} |a_k - b_k|^2 + \sum_{k=n+1}^{\infty} |b_k|^2 < \Big(\sum_{k=1}^{n} \tfrac{r^2}{2^{k+1}}\Big) + \tfrac{r^2}{2} < \Big(\sum_{k=1}^{\infty} \tfrac{r^2}{2^{k+1}}\Big) + \tfrac{r^2}{2} = r^2$$

and hence $\|a-b\|_2 < r$ so that $b \in B(a,r)$. Since $a \in \mathbb{Q}^\infty$ and $b \notin \mathbb{Q}^\infty$, we have $b \in B_2^*(a,r)$ so that b is a limit point of $\mathbb{Q}^\infty$, as required.

(b) Show that $\big(\ell_2(\mathbb{C}), d_2\big)$ is complete.

Solution: Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $\big(\ell_2, d_2\big)$, say $x_n = (x_{n,k})_{k\geq 1}$. Note that for each fixed $k \in \mathbb{Z}^+$, the sequence $(x_{n,k})_{n\geq 1}$ is Cauchy; indeed given $\epsilon > 0$ we can choose $N \in \mathbb{Z}^+$ so that for all $n, m \in \mathbb{Z}^+$ we have $n, m \geq N \Longrightarrow \|x_n - x_m\|_2 < \epsilon$, and then for $n, m \geq N$ we have

$$|x_{n,k} - x_{m,k}| \leq \Big(\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2\Big)^{1/2} = \|x_n - x_m\|_2 < \epsilon\,.$$

Since $(x_{n,k})_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is complete, this sequence converges. Let

$$a = (a_k)_{k\geq 1}, \text{ where } a_k = \lim_{n\to\infty} x_{n,k}\,.$$

We claim that $a \in \ell_2$, that is $\sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$. For $K \in \mathbb{Z}^+$ we have

$$\sum_{k=1}^{K} |a_k|^2 = \sum_{k=1}^{K} \big|\lim_{n\to\infty} x_{n,k}\big|^2 = \sum_{k=1}^{K} \lim_{n\to\infty} {x_{n,k}}^2 = \lim_{n\to\infty} \sum_{k=1}^{K} {x_{n,k}}^2 \leq \lim_{n\to\infty} \sum_{k=1}^{\infty} {x_{n,k}}^2 = \lim_{n\to\infty} \|x_n\|_2^{\ 2}\,,$$

so it suffices to show that the sequence $\big(\|x_n\|_2\big)$ converges in $\mathbb{R}$. And since $\big|\|x_n\|_2 - \|x_m\|_2\big| \leq \|x_n - x_m\|_2$ (by the Triangle Inequality) we see that $\big(\|x_n\|_2\big)$ is Cauchy in $\mathbb{R}$, so it does converge.

Finally, we claim that $x_n \to a$ in $\big(\ell_2, d_2\big)$. Let $\epsilon > 0$. Choose $N \in \mathbb{Z}^+$ so that for all $n, m \in \mathbb{Z}^+$ we have

$$n, m \geq N \Longrightarrow \big\|x_n - x_m\big\|_2 < \tfrac{\epsilon}{2}\ , \text{ that is } \sum_{k=1}^{\infty} (x_{n,k} - x_{m,k})^2 < \tfrac{\epsilon^2}{4}\,.$$

Let $n \in \mathbb{Z}^+$. Then for all $K \in \mathbb{Z}^+$ we have

$$\sum_{k=1}^{K} (x_{n,k} - a_k)^2 = \sum_{k=1}^{K} \big(x_{n,k} - \lim_{m\to\infty} x_{m,k}\big)^2 = \lim_{m\to\infty} \sum_{k=1}^{K} (x_{n,k} - x_{m,k})^2 \leq \lim_{m\to\infty} \sum_{k=1}^{\infty} (x_{n,k} - x_{m,k})^2 \leq \tfrac{\epsilon^2}{4}$$

and so

$$\|x_n - a\|_2 = \Big(\sum_{k=1}^{\infty} (x_{n,k} - x_{m,k})^2\Big)^{1/2} \leq \tfrac{\epsilon}{2} < \epsilon\,.$$

3: (a) Show that $\big(\mathcal{B}([0,1],\mathbb{C}), d_\infty\big)$ is not separable.

Solution: Let A be any dense subset of $\mathcal{B}[0,1]$. We must show that A is uncountable. For each $n\in\mathbb{Z}^+$, define $f_n:[0,1]\to\mathbb{R}$ by $f_n\big(\frac{1}{n}\big)=1$ and $f_n(x)=0$ for all $x\neq\frac{1}{n}$. Note that each $f_n\in\mathcal{B}[0,1]$ with $\|f_n\|_\infty=1$. Let $\{0,1\}^{\mathbb{Z}^+}$ denote the set of binary sequences $\alpha=(\alpha_1,\alpha_2,\cdots)$. For each binary sequence $\alpha\in\{0,1\}^{\mathbb{Z}^+}$, define $g_\alpha:[0,1]\to\mathbb{R}$ by $g_\alpha=\sum\limits_{n=1}^{\infty}\alpha_n f_n$ and note that for any two distinct binary sequences $\alpha\neq\beta$ we have $\|g_\alpha-g_\beta\|_\infty=1$. Since A is dense in $\mathcal{B}[0,1]$, for each binary sequence α we can choose $g_\alpha\in A$ such that $\|g_\alpha-f_\alpha\|_\infty<\frac{1}{2}$. Define $F:\{0,1\}^{\mathbb{Z}^+}\to A$ by $F(\alpha)=g_\alpha$ (we remark that the Axiom of Choice is used here). Note that F is injective because when $\alpha\neq\beta$ we have

$$1=\|g_\alpha-g_\beta\|_\infty\le\|g_\alpha-f_\alpha\|_\infty+\|f_\alpha-f_\beta\|_\infty+\|f_\beta-g_\beta\|_\infty<\tfrac{1}{2}+\|g_\alpha-g_\beta\|_\infty+\tfrac{1}{2}$$

so that $\|g_\alpha-g_\beta\|_\infty>0$. Since F is injective we have $|A|\ge\big|\{0,1\}^{\mathbb{Z}^+}\big|=2^{\aleph_0}$, and so A is uncountable.

(b) Show that $\big(\mathcal{C}([-1,1],\mathbb{R}),d_1\big)$ is not complete.

Solution: For each $n\in\mathbb{Z}^+$, define $f_n:[-1,1]\to\mathbb{R}$ by $f_n(x)=x^{\frac{1}{2n-1}}$. Note that each f_n is continuous on $[-1,1]$, and the sequence $(f_n)_{n\ge1}$ is Cauchy in $\big(\mathcal{C}[-1,1],d_1\big)$ because for $m\ge n\ge N$ we have

$$\begin{aligned}\|f_n-f_m\|_1&=\int_{x=-1}^{1}\big|f_n(x)-f_m(x)\big|\,dx=2\int_{x=0}^{1}x^{\frac{1}{2m-1}}-x^{\frac{1}{2n-1}}\,dx\\&=2\Big[\tfrac{2m-1}{2m}\,x^{\frac{2m+1}{2m-1}}-\tfrac{2n-1}{2n}\,x^{\frac{2n+1}{2n-1}}\Big]_{x=0}^{1}=\tfrac{2m-1}{m}-\tfrac{2n-1}{n}=\tfrac{1}{n}-\tfrac{1}{m}\le\tfrac{1}{N}.\end{aligned}$$

Note that for each $x\in[-1,1]$ we have $\lim\limits_{n\to\infty}f_n(x)=g(x)$ in $\mathbb{R}$ where $g(x)=-1$ for $x<0$, $g(x)=1$ for $x>0$ and $g(0)=0$ $\big($so we have $f_n\to g$ pointwise on $[-1,1]\big)$ Suppose, for a contradiction, that $(f_n)_{n\ge1}$ converges in $\mathcal{C}[-1,1]$, and let $h=\lim\limits_{n\to\infty}f_n$ in $\mathcal{C}[-1,1]$. Note that the restriction of h to $[0,1]$ is continuous. Let $\epsilon>0$. Choose $n\in\mathbb{Z}^+$ such that $\|f_n-h\|_1<\frac{\epsilon}{2}$ and also $\frac{1}{2n}<\frac{\epsilon}{2}$. Then

$$\begin{aligned}\int_{x=0}^{1}\big|h(x)-1\big|\,dx&\le\int_{x=0}^{1}\big|h(x)-f_n(x)\big|+\big|f_n(x)-1\big|\,dx\le\int_{x=-1}^{1}\big|h(x)-f_n(x)\big|\,dx+\int_{x=0}^{1}\big|f_n(x)-1\big|\,dx\\&=\|h-f_n\|_1+\int_{x=0}^{1}1-x^{\frac{1}{2n-1}}\,dx=\|h-f_n\|_1+\frac{1}{2n}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Since $\displaystyle\int_{x=0}^{1}\big|h(x)-1\big|\,dx<\epsilon$ for every $\epsilon>0$, it follows that $\displaystyle\int_{x=0}^{1}\big|h(x)-1\big|\,dx=0$ and, since the function $h(x)-1$ is continuous on $[0,1]$, it follows that $h(x)-1=0$ for all $x\in[0,1]$. Thus we have $h(x)=1$ for all $x\in[0,1]$. A similar argument shows that $h(x)=-1$ for all $x\in[-1.,0]$. But this is not possible since we cannot have $h(0)=1$ and $h(0)=-1$.

4: (Absolute convergence implies convergence) Let X be a normed linear space. For a sequence $(x_k)_{k\geq 1}$ in X, the n^{th} *partial sum* of $(x_k)_{k\geq 1}$ is the element $s_n = \sum_{k=1}^{n} x_k \in X$, the *series* $\sum_{k=1}^{\infty} x_k$ is, by definition, equal to the sequence of partial sums $(s_n)_{n\geq 1}$, we say the series $\sum_{k=1}^{\infty} x_k$ *converges* in X when the sequence of partial sums $(s_n)_{n\geq 1}$ converges in X and then the *sum* of the series $\big($also denoted by $\sum_{k=1}^{\infty} x_k\big)$ is defined to be the limit of the sequence of partial sums in X. Show that X is complete if and only if X has the property that for every sequence $(x_k)_{k\geq 1}$ in X, if $\sum_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$ then $\sum_{k=1}^{\infty} x_k$ converges in X.

Solution: Suppose that X is complete. Let $(x_k)_{k\geq 1}$ be a sequence in X such that $\sum_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$. For each $n \in \mathbb{Z}^+$, let $t_n = \sum_{k=1}^{n} \|x_k\| \in \mathbb{R}$ and let $s_n = \sum_{k=1}^{n} x_k \in X$. Let $\epsilon > 0$. Since $\sum_{k=1}^{n} \|x_k\|$ converges in $\mathbb{R}$, the sequence $(t_n)_{n\geq 1}$ is Cauchy in $\mathbb{R}$, so we can choose $N \in \mathbb{Z}^+$ such that for $m > n \geq N$ we have $\sum_{k=n+1}^{m} \|x_k\| = |t_m - t_n| < \epsilon$. Then for $m > n \geq N$ we have $\|s_m - s_n\| = \big\|\sum_{k=n+1}^{m} x_k\big\| \leq \sum_{k=n+1}^{m} \|x_k\| < \epsilon$. This shows that the sequence $(s_n)_{n\geq 1}$ is Cauchy in X, and so it converges in X because X is complete.

Suppose, conversely, that X has the property that for every sequence $(y_k)_{k\geq 1}$ in X, if $\sum_{k=1}^{\infty} \|y_k\|$ converges in $\mathbb{R}$ then $\sum_{k=1}^{\infty} y_k$ converges in X. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Since $(x_n)_{n\geq 1}$ is Cauchy, we can choose $n_1 \in \mathbb{Z}^+$ such that $k, \ell \geq n_1 \implies \|x_k - x_\ell\| < \frac{1}{2}$, then we can choose $n_2 > n_1$ such that $k, \ell \geq n_2 \implies \|x_k - x_\ell\| < \frac{1}{2^2}$, then we can choose $n_3 > n_2$ so that $k, \ell \geq n_3 \implies \|x_k - x_\ell\| < \frac{1}{2^3}$ and so on, to obtain integers n_k with $1 \leq n_1 < n_2 < n_3 < \cdots$ such that $i, j \geq n_k \implies \|x_i - x_j\| < \frac{1}{2^k}$. For each $k \in \mathbb{Z}^+$, let $y_k = x_{n_{k+1}} - x_{n_k}$. Note that

$$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \sum_{k=1}^{\infty} \tfrac{1}{2^k} = 1.$$

Since $\sum_{k=1}^{\infty} \|y_k\|$ converges in $\mathbb{R}$, it follows that $\sum_{k=1}^{\infty} y_k$ converges in X. For each $\ell \in \mathbb{Z}^+$, let s_ℓ be the ℓ^{th} partial sum

$$s_\ell = \sum_{k=1}^{\ell} y_k = \sum_{k=1}^{\ell} (x_{n_{k+1}} - x_{n_k}) = x_{n_{\ell+1}} - x_{n_1}$$

and note that $x_{n_\ell} = s_{\ell-1} + x_{n_1}$ for $\ell \geq 2$. Since the series $\sum_{k=1}^{\infty} y_k$ converges in X, its sequence of partial sums $(s_\ell)_{\ell\geq 1}$ converges in X, and hence the sequence $\big(x_{n_\ell}\big)_{\ell\geq 1}$ converges in X. Since $(x_n)_{n\geq 1}$ is a Cauchy sequence, and the subsequence $(x_{n_\ell})_{\ell\geq 1}$ converges, it follows that $(x_n)_{n\geq 1}$ converges by Theorem 4.11.

5: Let X be a metric space.

(a) Show that X is complete if and only if every decreasing sequence of closed balls

$$\overline{B}(a_1,r_1) \supseteq \overline{B}(a_2,r_2) \supseteq \overline{B}(a_3,r_3) \supseteq \cdots$$

in X with $r_n \to 0$ has a non-empty intersection.

Solution: Suppose that X is complete. Let $\overline{B}(a_1,r_1) \supseteq \overline{B}(a_2,r_2) \supseteq \overline{B}(a_3,r_3) \supseteq \cdots$ be a decreasing sequence of balls in X with $r_n \to 0$. We claim that (a_n) is Cauchy. Let $\epsilon > 0$. Choose $N \in \mathbb{Z}^+$ so that $r_N \leq \frac{\epsilon}{2}$. For $n,m \in \mathbb{Z}^+$ with $n,m \geq N$ we have $a_n, a_m \in B(a_N, r_N)$ so that $d(a_n,a_m) \leq d(a_n,a_N)+d(a_N,a_m) < 2r_N \leq \epsilon$, and so (a_n) is Cauchy as claimed. Since X is complete, (a_n) converges in X. Let $a = \lim\limits_{n\to\infty} a_n$. Note that $a \in \bigcap\limits_{n=1}^{\infty} \overline{B}(a_n,r_n)$ since for each $N \in \mathbb{N}$, the sequence (a_n) lies in $\overline{B}(a_N,r_N)$ which is closed in X and hence complete, and so $a \lim\limits_{n\to\infty} a_n \in \overline{B}(a_N,r_N)$.

Conversely, suppose that every decreasing sequence of balls $\overline{B}(a_1,r_1) \supseteq \overline{B}(a_2,r_2) \supseteq \overline{B}(a_3,r_3) \supseteq \cdots$ with $r_n \to 0$ has non-empty intersection. Let (a_n) be a Cauchy sequence in X. Choose $n_0 \in \mathbb{Z}^+$ so that for all $n,m \in \mathbb{Z}^+$ we have $n,m \geq n_0 \Longrightarrow d(a_n,a_m) < \frac{1}{2}$. Having chosen $n_0 < n_1 < \cdots < n_{k-1}$, choose $n_k > n_{k-1}$ so that for all $n,m \in \mathbb{Z}^+$ we have $n,m \geq n_k \Longrightarrow d(a_n,a_m) < \frac{1}{2^{k+1}}$. Note that $\overline{B}\big(a_{n_k}, \frac{1}{2^k}\big) \subseteq \overline{B}\big(a_{n_{k-1}}, \frac{1}{2^{k-1}}\big)$ since

$$d\big(x,a_{n_k}\big) \leq \tfrac{1}{2^k} \Longrightarrow d\big(x,a_{n_{k-1}}\big) \leq d\big(x,a_{n_{k-1}}\big) + d\big(a_{n_{k1}},a_{n_{k-1}}\big) < \tfrac{1}{2^k} + \tfrac{1}{2^k} = \tfrac{1}{2^{k-1}}\,.$$

Since this decreasing sequence of closed balls has non-empty intersection, we can choose $a \in \bigcap\limits_{n=1}^{\infty} \overline{B}\big(a_{n_k}, \frac{1}{2^k}\big)$. Note that $a_{n_k} \to a$ in X since given $\epsilon > 0$ we can choose $K \in \mathbb{Z}^+$ so that $\frac{1}{2^{k-1}} < \epsilon$ and then for $k \geq K$ we have $d\big(a_{n_k}, a_{n_K}\big) < \frac{1}{2^{K+1}}$ by the choice of n_K, and we have $a \in \overline{B}\big(a_{n_K}, \frac{1}{2^K}\big)$ so that $d\big(a, a_{n_K}\big) \leq \frac{1}{2^K}$, and so $d\big(a_{n_k}, a\big) \leq d\big(a_{n_k}, a_{n_K}\big) + d\big(a_{n_K}, a\big) < \frac{1}{2^{K+1}} + \frac{1}{2^K} < \frac{1}{2^{K-1}} < \epsilon$. Finally note that since (a_n) is Cauchy and has a convergent subsequence, (a_n) converges (by Theorem 4.11).

(b) Show that the requirement in part (a) that $r_n \to 0$ is necessary.

Solution: Let $X = \big\{\frac{1}{2^n}\big| n \in \mathbb{Z}^+\big\}$. Define $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{, if } x = y \\ 1 + |x-y| & \text{, if } x \neq y\,. \end{cases}$$

Then d is clearly positive definite and symmetric, and by considering that cases $x = y = z$, $x = y \neq z$, $x = z \neq y$, $y = z \neq x$ and x,y,z all distinct, we see that d satisfies the triangle equality, so d is a metric on X. Under this metric, X is complete since if a sequence in X is Cauchy, then it must be eventually constant, so it converges. But if we take $a_n = \frac{1}{2^n}$ and $r_n = 1 + \frac{1}{2^n}$, then we have $\overline{B}(a_n,r_n) = \big\{\frac{1}{2^k}\big| k \geq n-1\big\}$, so $\overline{B}(a_1,r_1) \supseteq \overline{B}(a_2,r_2) \supseteq \overline{B}(a_3,r_3) \supseteq \cdots$ but $\bigcap_{n=1}^{\infty} \overline{B}(a_n,r_n) = \emptyset$.