PMATH 351 Real Analysis, Exercises for Chapter 4: Separability and Completeness

1: (a) Let $c_0 = \big\{ a \in \ell_\infty(\mathbb{R}) \,\big|\, \lim\limits_{n\to\infty} a_n = 0 \big\}$. Show that (c_0, d_∞) is separable.

(b) Show that $(\ell_\infty(\mathbb{C}), d_\infty)$ is complete.

2: (a) Show that $(\ell_2(\mathbb{R}), d_2)$ is separable.

(b) Show that $\big(\ell_2(\mathbb{C}), d_2\big)$ is complete.

3: (a) Show that $\big(\mathcal{B}([0,1], \mathbb{C}), d_\infty\big)$ is not separable.

(b) Show that $\big(\mathcal{C}([-1,1], \mathbb{R}), d_1\big)$ is not complete.

4: (Absolute convergence implies convergence) Let X be a normed linear space. For a sequence $(x_k)_{k\geq 1}$ in X, the n^{th} *partial sum* of $(x_k)_{k\geq 1}$ is the element $s_n = \sum\limits_{k=1}^{n} x_k \in X$, the *series* $\sum\limits_{k=1}^{\infty} x_k$ is, by definition, equal to the sequence of partial sums $(s_n)_{n\geq 1}$, we say the series $\sum\limits_{k=1}^{\infty} x_k$ *converges* in X when the sequence of partial sums $(s_n)_{n\geq 1}$ converges in X and then the *sum* of the series $\big($also denoted by $\sum\limits_{k=1}^{\infty} x_k\big)$ is defined to be the limit of the sequence of partial sums in X. Show that X is complete if and only if X has the property that for every sequence $(x_k)_{k\geq 1}$ in X, if $\sum\limits_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$ then $\sum\limits_{k=1}^{\infty} x_k$ converges in X.

5: Let X be a metric space.

(a) Show that X is complete if and only if every decreasing sequence of closed balls

$$\overline{B}(a_1, r_1) \supseteq \overline{B}(a_2, r_2) \supseteq \overline{B}(a_3, r_3) \supseteq \cdots$$

in X with $r_n \to 0$ has a non-empty intersection.

(b) Show that the requirement in part (a) that $r_n \to 0$ is necessary.