PMATH 351 Real Analysis, Solutions to the Exercises for Chapter 3

1: (a) Define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=1-nx$ for $0\le x\le\frac{1}{n}$ and $f_n(x)=0$ for $\frac{1}{n}\le x\le 1$. Show that $f_n\to 0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2, but $f_n\not\to 0$ pointwise on $[0,1]$.

Solution: We have $f_n\to 0$ in $\big(\mathcal{C}[0,1],d_1\big)$ and $f_n\to 0$ in $\big(\mathcal{C}[0,1],d_2\big)$ by Part 5 of Theorem 3.2 because

$$d_1(f_n,0)=\int_0^1|f_n(x)|\,dx=\int_0^{1/n}1-nx\,dx=\Big[x-\tfrac{n}{2}\,x^2\Big]_0^{1/n}=\tfrac{1}{2n}\to 0\ ,\text{ and}$$
$$d_2(f_n,0)^2=\int_0^1 f_n(x)^2\,dx=\int_0^{1/n}1-2nx+n^2x^2\,dx=\Big[x-nx^2+\tfrac{n^2}{3}\,x^3\Big]_0^{1/n}=\tfrac{1}{3n}\to 0.$$

On the other hand, it is not the case that $f_n\to 0$ pointwise on $[0,1]$ because $\lim\limits_{n\to\infty}f_n(0)=\lim\limits_{n\to\infty}1=1$.

(b) Define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=n^2x-n^3x^2$ for $0\le x\le\frac{1}{n}$ and $f_n(x)=0$ for $\frac{1}{n}\le x\le 1$. Show that $f_n\to 0$ pointwise on $[0,1]$ but $f_n\not\to 0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2.

Solution: We claim that $f_n\to 0$ pointwise on $[0,1]$. When $x=0$ we have $f_n(x)=f_n(0)=0$ for all $n\in\mathbb{Z}^+$ so that $\lim\limits_{n\to\infty}f_n(x)=0$. Let $x\in(0,1]$. Choose $m\in\mathbb{Z}^+$ large enough so that $\frac{1}{m}<x$. Then for $n\ge m$ we have $\frac{1}{n}\le\frac{1}{m}<x$. so that $f_n(x)=0$. Since $f_n(x)=0$ for all $n\ge m$, we have $\lim\limits_{n\to\infty}f_n(x)=0$. Thus $f_n\to 0$ pointwise on $[0,1]$, as claimed.

On the other hand, we have $f_n\not\to 0$ in $\big(\mathcal{C}[0,1],d_1\big)$ and $f_n\not\to 0$ in $\big(\mathcal{C}[0,1],d_2\big)$ by Part 5 of Theorem 3.2 because

$$d_1(f_n,0)=\int_0^1|f_n(x)|\,dx=\int_0^{1/n}n^2x-n^3x^2\,dx=\Big[\tfrac{n^2}{2}x^2-\tfrac{n^3}{3}x^3\Big]_0^{1/n}=\tfrac{1}{6}\ ,\text{ and}$$
$$d_2(f_n,0)^2=\int_0^1 f_n(x)^2\,dx=\int_0^{1/n}n^4x^2-2n^5x^3+n^6x^4\,dx=\Big[\tfrac{n^4}{3}x^3-\tfrac{n^5}{2}x^4+\tfrac{n^6}{5}x^5\Big]_0^{1/n}=\tfrac{n}{30}\to\infty.$$

(c) Define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=\sqrt{n}\,x^n$. Show that $(f_n)_{n\ge1}$ converges in $\big(\mathcal{C}[0,1],d_1\big)$ but diverges in $\big(\mathcal{C}[0,1],d_2\big)$.

Solution: Note that $f_n\to 0$ in $\big(\mathcal{C}[0,1],d_1\big)$ because

$$d_1(f_n,0)=\int_0^1|f_n(x)|\,dx=\int_0^1\sqrt{n}\,x^n\,dx=\Big[\tfrac{\sqrt{n}}{n+1}x^{n+1}\Big]_0^1=\tfrac{\sqrt{n}}{n+1}\to 0.$$

On the other hand, in $\big(\mathcal{C}[0,1],d_2\big)$, notice that for all $n\in\mathbb{Z}^+$ we have

$$\|f_n-f_{4n}\|_2=\left(\int_0^1\big(\sqrt{n}\,x^n-\sqrt{4n}\,x^{4n}\big)^2\,dx\right)^{1/2}=\left(\int_0^1\big(n\,x^{2n}-4n\,x^{5n}+4n\,x^{8n}\big)\,dx\right)^{1/2}$$
$$=\big(\tfrac{n}{2n+1}-\tfrac{4n}{5n+1}+\tfrac{4n}{8n+1}\big)^{1/2}\to\tfrac{1}{\sqrt5}\text{ as }n\to\infty.$$

It follows that the sequence (f_n) cannot converge because if we had $f_n\to g$ in $\mathcal{C}[0,1]$ then we could choose $m\in\mathbb{Z}^+$ so that when $n\ge m$ we have $\|f_n-g\|_2<\frac{1}{4\sqrt5}$ and then for $n\ge m$ we would have

$$\|f_n-f_{4n}\|_2\le\|f_n-g\|_2+\|g-f_{4n}\|_2<\tfrac{1}{4\sqrt5}+\tfrac{1}{4\sqrt5}=\tfrac{1}{2\sqrt5}$$

which contradicts the fact that $\|f_n-f_{4n}\|\to\frac{1}{\sqrt5}$.

2: (a) For each $n \in \mathbb{Z}^+$, let $x_n = (x_{n,k})_{k\geq 1} \in \mathbb{R}^\infty$ be given by $x_n = \sum\limits_{k=1}^{n} \frac{k+1}{k}\, e_k$, where e_k is the k^{th} standard basis vector in $\mathbb{R}^\infty$ (so we have $x_{n,k} = \frac{k+1}{k}$ when $k \leq n$ and $x_{n,k} = 0$ when $k > n$). Find $\lim\limits_{n\to\infty}\big(\lim\limits_{k\to\infty} x_{n,k}\big)$ in $\mathbb{R}$, and find $\lim\limits_{k\to\infty}\big(\lim\limits_{n\to\infty} x_{n,k}\big)$ in $\mathbb{R}$, and determine whether the sequence $(x_n)_{n\geq 1}$ converges in (ℓ_∞, d_∞).

Solution: Given $n \in \mathbb{Z}^+$, since $x_{n,k} = 0$ for all $k > n$, we have $\lim\limits_{k\to\infty} x_{n,k} = 0$, and so $\lim\limits_{n\to\infty}\big(\lim\limits_{k\to\infty} x_{n,k}\big) = 0$. Given $k \in \mathbb{Z}^+$, since $x_{n,k} = \frac{k+1}{k}$ for all $n \geq k$, we have $\lim\limits_{n\to\infty} x_{n,k} = \frac{k+1}{k}$, so $\lim\limits_{k\to\infty}\big(\lim\limits_{n\to\infty} x_{n,k}\big) = \lim\limits_{k\to\infty} \frac{k+1}{k} = 1$. We claim that (x_n) does not converge in (ℓ_∞, d_∞). Suppose, for a contradiction, that $x_n \to a$ in (ℓ_∞, d_∞). By Theorem 3.6, for all $k \in \mathbb{Z}^+$ we must have $a_k = \lim\limits_{n\to\infty} x_{n,k} = \frac{k+1}{k}$, and so $a = (a_k)_{k\geq 1} = \big(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots\big)$. For all $n \in \mathbb{Z}^+$ since $x_{n,k} = a_k = \frac{k+1}{k}$ for $k \leq n$ and $x_{n,k} = 0$ for $k > n$, we have $|x_{n,k} - a_k| = 0$ for $k \leq n$ and $|x_{n,k} - a_k| = \frac{k+1}{k}$ for $k > n$, and so $\|x_n - a\|_\infty = \sup\big\{\frac{k+1}{k}\big| k \geq n+1\big\} = \frac{n+2}{n+1} > 1$. Since $\|x_n - a\|_\infty > 1$ for all $n \in \mathbb{Z}^+$, it follows that $x_n \not\to a$ in (ℓ_∞, d_∞), so we have obtained the desired contradiction.

(b) Let $A \subseteq \mathbb{R}$ and let $\ell_1(A) = \big\{(a_n) \in \ell_1 \big| \text{each } a_n \in A\big\}$. Show that $\overline{\ell_1(A)} = \ell_1\big(\overline{A}\big)$ in (ℓ_1, d_1).

Solution: Let $a = (a_k) \in \ell_1\big(\overline{A}\big)$. For each $n \in \mathbb{Z}^+$ construct a sequence $x_n = (x_{n,k}) \in \ell_1(A)$ (using the Axiom of Choice) by choosing $x_{n,k} \in B\big(a_k, \frac{1}{2^{n+k}}\big) \cap A$ (note that $B\big(a_k, \frac{1}{2^{n+k}}\big) \cap A$ is not empty since $a_k \in \overline{A}$). Then for all n, k we have $|x_{n,k} - a_k| \leq \frac{1}{2^{n+k}}$ and so for all n, $\|x_n - a\|_1 = \sum\limits_{k=1}^{\infty} |x_{n,k} - a_k| \leq \sum\limits_{k=1}^{\infty} \frac{1}{2^{n+k}} = \frac{1}{2^n}$, and so $x_n \to a$ in (l_1, d_1). Thus $a \in \overline{\ell_1(A)}$, so we have $\ell_1\big(\overline{A}\big) \subseteq \overline{\ell_1(A)}$.

Now let $a = (a_k) \in \overline{\ell_1(A)}$. We claim that each $a_k \in \overline{A}$. Let $r > 0$. Choose $b = (b_k) \in \ell_1(A)$ with $|a - b|_1 < r$, that is $\sum\limits_{k=0}^{\infty} |a_k - b_k| < r$. Then for each k we have $|a_k - b_k| \leq \sum\limits_{k=0}^{\infty} |a_k - b_k| < r$, and so $b_k \in B(a_k, r) \cap A$. Thus each $a_k \in \overline{A}$, as claimed, so $a \in \ell_1\big(\overline{A}\big)$, and hence $\overline{\ell_1(A)} \subseteq \ell_1\big(\overline{A}\big)$.

(c) Let c be the set of all convergent sequences of real numbers. Show that c is closed and that the interior of c is empty in $\big(\ell_\infty, d_\infty\big)$.

Solution: Let $(x_n)_{n\geq 1}$ be a sequence in c which converges in (ℓ_∞, d_∞), and let $a = \lim\limits_{n\to\infty} x_n$ in (ℓ_∞, d_∞). Note that $a \in \ell_\infty$ means that a is a bounded sequence of real numbers, say $a = (a_k)_{k\geq 1}$. By the Sequential Characterization of Closed Sets, it suffices to show that $a \in c$ or, in other words, that a converges in $\mathbb{R}$. For each $n \in \mathbb{Z}^+$, we have $x_n \in c$, which means that x_n is a convergent sequence of real numbers, say $x_n = (x_{n,k})_{k\geq 1}$. Let $\epsilon > 0$. Since $x_n \to a$ in ℓ_∞ we can choose, and fix, one value of $n \in \mathbb{Z}^+$ such that $\|x_n - a\|_\infty < \frac{\epsilon}{3}$, and then we have $|x_{n,k} - a_k| < \frac{\epsilon}{3}$ for all indices k. Since $(x_{n,k})_{k\geq 1}$ converges in $\mathbb{R}$, hence is Cauchy in $\mathbb{R}$, we can choose $m \in \mathbb{Z}^+$ such that for all $k, \ell \geq m$ we have $|x_{n,k} - x_{n,\ell}| < \frac{\epsilon}{3}$. Then for all $k, \ell \geq m$ we have

$$|a_k - a_\ell| \leq |a_k - x_{n,k}| + |x_{n,k} - x_{n,\ell}| + |x_{n,\ell} - a_\ell| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

Thus the sequence $a = (a_k)_{k\geq 1}$ is Cauchy in $\mathbb{R}$, so it converges in $\mathbb{R}$, that is $a \in c$ as required.

Let $a \in c$, say $a = (a_n)_{n\geq 1}$. Let $r > 0$. Let $x = (x_n)_{n\geq 1}$ be the sequence given by $x_n = a_n + \frac{r}{2}(-1)^n$. Then $x \in \ell_\infty$ $\big($since $|x_n| \leq |a_n| + \frac{r}{2}$ for all $n \in \mathbb{Z}^+$ so that $\|x\|_\infty \leq \|a\|_\infty + \frac{r}{2}\big)$ and $x \in B_\infty(a, r)$ $\big($since $|x_n - a_n| = \frac{r}{2}$ for all $n \in \mathbb{Z}^+$ so that $\|x - a\|_\infty = \frac{r}{2}\big)$ and $x \notin c$ $\big($since if we had $x \in c$ then we would also have $x - a \in c$, but $x - a$ is the sequence with terms $x_n - a_n = \frac{r}{2}(-1)^n$, which diverges$\big)$. Thus for all $r > 0$, the ball $B_\infty(a, r)$ is not contained in c, so $a \notin c^o$ Since $a \in c$ was arbitrary point, we have $c^o = \emptyset$.

3: Let X and Y be metric spaces.

(a) Let A and B be closed sets in X with $X = A \cup B$, let $f : A \to Y$ and $g : B \to Y$ be continuous with $f(x) = g(x)$ for all $x \in A \cap B$, and define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) \text{ , for } x \in A, \\ g(x) \text{ , for } x \in B\,. \end{cases}$$

Show that h is continuous.

Solution: Let $C \subseteq Y$ be closed. Then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)\,,$$

is closed (since it is the union of two closed sets).

(b) Let A be a dense subset of X and let $f, g : X \to Y$ be continuous maps with $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in X$.

Solution: Let $B = \{x \in X \big| f(x) = g(x)\}$. Note that $A \subseteq B$. We claim that B is closed. Let $a \in B^c$ so that $f(a) \neq g(a)$. Let $r = \frac{1}{2}\, d_Y\big(f(a), g(a)\big)$ so that $B_Y\big(f(a), r\big) \cap B_Y\big(g(a), r\big) = \emptyset$. Since f and g are continuous, the set $U = f^{-1}\big(B_Y\big(f(a), r\big)\big) \cap g^{-1}\big(B_Y\big(g(a), r\big)\big)$ is open, and we have $a \in U$. Choose $s > 0$ so that $B_X(a, s) \subseteq U$. Then for $x \in B_X(a, s)$, we have $f(x) \in B_Y\big(f(a), r\big)$ and $g(x) \in B_Y\big(g(a), r\big)$. Since $B_Y\big(f(a), r\big) \cap B_Y\big(g(a), r\big) = \emptyset$, we see that $f(x) \neq g(x)$, so $x \in B^c$. Thus B^c is open, so B is closed, as claimed. Since B is closed and $A \subseteq B$, we have $\overline{A} \subseteq B$. But A is dense in X, so $\overline{A} = X$, and so we have $X \subseteq B$. Thus $B = X$, as required.

(c) Show that a map $f : X \to Y$ is continuous if and only if for every $B \subseteq Y$ we have $f^{-1}(B^\circ) \subseteq f^{-1}(B)^\circ$.

Solution: Suppose that f is continuous. Since A° is open and f is continuous, $f^{-1}\big(A^\circ\big)$ is open. Since $A^\circ \subseteq A$, we have $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)$. Since $f^{-1}\big(A^\circ\big)$ is open and $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)$, we have $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)^\circ$.

Conversely, suppose that for every $A \subseteq Y$ we have $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)^\circ$. Let $U \subseteq Y$ be open. Then $U^\circ = U$, so $f^{-1}(U) = f^{-1}\big(U^\circ\big) \subseteq f^{-1}(U)^\circ$. Since $f^{-1}(U) \subseteq f^{-1}(U)^\circ$ and of course $f^{-1}(U)^\circ \subseteq f^{-1}(U)$, we have that $f^{-1}(U) = f^{-1}(U)^\circ$, so $f^{-1}(U)$ is open. Thus f is continuous.

(d) Show that a map $f : X \to Y$ is continuous if and only if for every $A \subseteq X$ we have $f\big(\overline{A}\big) \subseteq \overline{f(A)}$.

Solution: Suppose that f is continuous. Let $A \subseteq X$. Let $b \in f\big(\overline{A}\big)$, say $b = f(a)$ where $a \in \overline{A}$. We must show that $b \in \overline{f(A)}$. Let $r > 0$. Since $B_Y(b, r)$ is open and f is continuous, $f^{-1}\big(B_Y(b, r)\big)$ is open, so we can choose $s > 0$ so that $B_X(a, s) \subseteq f^{-1}\big(B_Y(b, r)\big)$. Since $a \in \overline{A}$, we have $B_X(a, s) \cap A \neq \emptyset$, so we can choose a point $c \in B_X(a, s) \cap A$. Since $c \in B_X(a, s) \subseteq f^{-1}\big(B_Y(b, r)\big)$ we have $f(c) \in B_Y(b, r)$, and since $c \in A$ we have $f(c) \in f(A)$, and so $f(c) \in B_Y(b, r) \cap f(A)$. Thus $B_Y(b, r) \cap f(A) \neq \emptyset$ and so $b \in \overline{f(A)}$, as required.

Conversely, suppose that for every $A \subseteq X$ we have $f\big(\overline{A}\big) \subseteq \overline{f(A)}$. Let $B \subseteq Y$ be closed. We claim that $f^{-1}(B)$ is closed. Let $A = f^{-1}(B)$. Note that $f(A) \subseteq B$. Let $x \in \overline{A}$. Then $f(x) \in f\big(\overline{A}\big) \subseteq \overline{f(A)} \subseteq \overline{B} = B$ and so $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$. Of course we also have $A \subseteq \overline{A}$, so $A = \overline{A}$, and so A is closed, as claimed. Thus f is continuous.

4: (a) Let $I : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the identity map given by $I(x) = x$ for all $x \in \mathbb{R}^\infty$. Determine whether I is continuous as a map $I : (\mathbb{R}^\infty, d_1) \to (\mathbb{R}^\infty, d_2)$ and whether I is continuous as a map $I : (\mathbb{R}^\infty, d_2) \to (\mathbb{R}^\infty, d_1)$.

Solution: We claim that I is continuous as a map $I : (\mathbb{R}^\infty, d_1) \to (\mathbb{R}^\infty, d_2)$. By Part 4 of Theorem 3.35, it suffices to show that the closed ball $\overline{B}_1(0,1) = \left\{x \in \mathbb{R}^\infty \,\middle|\, \|x\|_1 \le 1\right\}$ is bounded in $(\mathbb{R}^\infty, d_2)$. Let $x \in \mathbb{R}^\infty$ with $\|x\|_1 \le 1$, that is with $\sum\limits_{k=1}^{\infty} |x_k| \le 1$. Since $\sum\limits_{k=1}^{\infty} |x_k| \le 1$, we must have $|x_k| \le 1$ for all $k \in \mathbb{Z}^+$, and hence $|x_k|^2 \le |x_k|$ for all $k \in \mathbb{Z}^+$. Thus we have $\|x\|_2{}^2 = \sum\limits_{k=1}^{\infty} |x_k|^2 \le \sum\limits_{k=1}^{\infty} |x_k| = \|x\|_1 \le 1$, and hence $\|x\|_2 \le 1$. Thus the closed ball $\overline{B}_1(0,1)$ is bounded in $(\mathbb{R}^\infty, d_2)$, and so I is continuous as a map $I : (\mathbb{R}^\infty, d_1) \to (\mathbb{R}^\infty, d_2)$.

We claim that I is not continuous as a map $I : (\mathbb{R}^\infty, d_2) \to (\mathbb{R}^\infty, d_1)$. By Part 4 of Theorem 3.35, it suffices to show that the closed ball $\overline{B}_2(0,1) = \left\{x \in \mathbb{R}^\infty \,\middle|\, \|x\|_2 \le 1\right\}$ is unbounded in $(\mathbb{R}^\infty, d_1)$. Let $n \in \mathbb{Z}^+$. Let $x \in \mathbb{R}^\infty$ be given by $x = \sum\limits_{k=1}^{n^2} \frac{1}{n} e_k$ (so we have $x_k = \frac{1}{n}$ for $1 \le k \le n^2$ and $x_k = 0$ for $k > n^n$. Then $\|x\|_2{}^2 = \sum\limits_{k=1}^{\infty} |x_k|^2 = \sum\limits_{k=1}^{n^2} \frac{1}{n^2} = 1$ hence $\|x\|_2 \le 1$, so we have $x \in \overline{B}_2(0,1)$. And $\|x\|_1 = \sum\limits_{k=1}^{\infty} |x_k| = \sum\limits_{k=1}^{n^2} \frac{1}{n} = n$. Thus $\overline{B}_2(0,1)$ is not bounded in $(\mathbb{R}^\infty, d_1)$, and so I is not continuous as a map $I : (\mathbb{R}^\infty, d_2) \to (\mathbb{R}^\infty, d_1)$.

(b) Determine whether the map $G : \big(\mathcal{C}[0,1], d_1\big) \to \big(\mathbb{R}, d_2\big)$ given by $G(f) = f(0)$ is continuous.

Solution: We claim that G is not continuous. Since G is linear, it suffices to show that $G\big(\overline{B}_1(0,1)\big)$ is not bounded. For $n \ge 1$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2n - 2n^2 x \ , & 0 \le x \le \frac{1}{n} \\ \quad 0 & , \ \frac{1}{n} \le x \le 1 \, . \end{cases}$$

Then $\|f_n\|_1 = \displaystyle\int_0^1 |f_n(x)|\,dx = 1$, so $f_n \in \overline{B}_1(0,1)$, but $G(f_n) = f_n(0) = 2n$, so $G\big(\overline{B}_1(0,1)\big)$ is unbounded.

(c) Determine whether the map $H : \big(\mathcal{C}[0,1], d_2\big) \to (\mathbb{R}, d_2)$ given by $H(f) = \displaystyle\int_0^1 f(x)\,dx$ is continuous.

Solution: We claim that H is continuous. Since H is linear, it suffices to show that $H\big(\overline{B}_2(0,1)\big)$ is bounded. Let $f \in \overline{B}_2(0,1)$ so we have $\|f\|_2 \le 1$, that is $\displaystyle\int_0^1 f(x)^2 dx \le 1$. Then, since $|y| \le 1 + y^2$ for all $y \in \mathbb{R}$, we have

$$\big|H(f)\big| = \left|\int_0^1 f(x)\,dx\right| \le \int_0^1 |f(x)|\,dx \le \int_0^1 1 + f(x)^2 \ dx = 1 + \int_0^1 f(x)^2\,dx \le 2\,,$$

and so $H\big(\overline{B}_2(0,1)\big)$ is bounded, as required.

5: Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=\displaystyle\int_0^x \frac{f(t)}{\sqrt{t}}\,dt$.

(a) Determine whether F is continuous as a map from $\big(\mathcal{C}[0,1],d_\infty\big)$ to $\big(\mathcal{C}[0,1],d_\infty\big)$.

Solution: We claim that $F:\big(\mathcal{C}[0,1],d_\infty\big)\to\big(\mathcal{C}[0,1],d_\infty\big)$ is continuous. Let $f\in\mathcal{C}[0,1]$ with $\|f\|_\infty\le 1$. Then

$$\begin{aligned}\big\|F(f)\big\|_\infty &= \max_{x\in[0,1]}\big|F(f)(x)\big| = \max_{x\in[0,1]}\left|\int_0^x \frac{f(t)}{\sqrt{t}}\,dt\right| \le \max_{x\in[0,1]}\int_0^x \frac{|f(t)|}{\sqrt{t}}\,dt \\ &\le \max_{x\in[0,1]}\int_0^1 \frac{1}{\sqrt{t}}\,dt = \max_{x\in[0,1]}\Big[2\sqrt{t}\Big]_0^x = \max_{x\in[0,1]} 2\sqrt{x} = 2\,.\end{aligned}$$

Since F is linear and $F\big(\overline{B}(0,1)\big)$ is bounded, it follows that F is continuous.

(b) Determine whether F is continuous as a map from $\big(\mathcal{C}[0,1],d_1\big)$ to $\big(\mathcal{C}[0,1],d_1\big)$.

Solution: We claim that $F:\big(\mathcal{C}[0,1],d_1\big)\to\big(\mathcal{C}[0,1],d_1\big)$ is not continuous. For $n\in\mathbb{Z}^+$ define $f_n:[0,1]\to\mathbb{R}$ by

$$f_n(t)=\begin{cases} 2n-2n^2t\,, & 0\le t\le \frac{1}{n} \\ 0\,, & \frac{1}{n}\le t\le 1\,.\end{cases}$$

Note that f_n is continuous and $\big\|f_n\big\|_1=\displaystyle\int_0^1 f_n(t)\,dt=\Big[2n\,t-n^2t^2\Big]_0^{1/n}=1$. For $0\le x\le\frac{1}{n}$ we have

$$F(f_n)(x)=\int_0^x \frac{f_n(t)}{\sqrt{t}}\,dt=\int_0^x \frac{2n}{\sqrt{t}}-2n^2\sqrt{t}\;dt=\Big[4n\,t^{1/2}-\tfrac{4n^2}{3}\,t^{3/2}\Big]_0^x=4n\,x^{1/2}-\tfrac{4n^2}{3}\,x^{3/2}\,,$$

and in particular $F(f_n)\big(\frac{1}{n}\big)=4\sqrt{n}-\frac{4}{3}\sqrt{n}=\frac{8\sqrt{n}}{3}$. For $\frac{1}{n}\le x\le 1$ we have

$$F(f_n)(x)=\int_0^{1/n}\frac{f(t)}{\sqrt{t}}\,dt+\int_{1/n}^x 0\;dt=\int_0^{1/n}\frac{f(t)}{\sqrt{t}}\,dt=F(f_n)\big(\tfrac{1}{n}\big)=\tfrac{8\sqrt{n}}{3}\,.$$

Thus

$$\big\|F(f_n)\big\|_1=\int_0^1\big|F(f_n)(x)\big|\,dx\ge\int_{1/n}^1 \tfrac{8\sqrt{n}}{3}\,dx=\frac{8\sqrt{n}\,\big(1-\frac{1}{n}\big)}{3}\longrightarrow\infty \text{ as } n\to\infty\,.$$

Since F is linear and $F\big(\overline{B}(0,1)\big)$ is unbounded, it follows that F is not continuous.

6: (a) Show that $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are isometric.

Solution: An isometry from $(\mathbb{R}^2, d_1)$ to $(\mathbb{R}^2, d_\infty)$ should send the square $B_1(0,1)$ to the square $B_\infty(0,1)$. We define $f:\mathbb{R}^2\to\mathbb{R}^2$ to be the composite of the rotation about the origin by $45°$ with the scaling by $\sqrt{2}$, so f is given by $f(x,y)=(x-y,x+y)$. This map f is bijective; its inverse g is the composite of the rotation about the origin by $-45°$ with the scaling by $\frac{1}{\sqrt{2}}$, given by $g(x,y)=\left(\frac{x+y}{2},\frac{-x+y}{2}\right)$. For $u=(x,y)\in\mathbb{R}^2$,

$$\|f(u)\|_\infty=\big\|f(x,y)\big\|_\infty=\big\|(x-y,x+y)\big\|_\infty=\max\big\{|x-y|,|x+y|\big\}=|x|+|y|=\big\|(x,y)\big\|_1=\|u\|_1\,,$$

and so for $u,v\in\mathbb{R}^2$,

$$d_\infty\big(f(u),f(v)\big)=\big\|f(u)-f(v)\big\|_\infty=\big\|f(u-v)\big\|_\infty=\|u-v\|_1=d_1(u,v)\,.$$

Thus f preserves distance, so it is an isometry.

(b) Show that $(\mathbb{R}^3, d_2)$ is not isometric to either $(\mathbb{R}^3, d_1)$ or $(\mathbb{R}^3, d_\infty)$.

Solution: $(\mathbb{R}^3, d_2)$ cannot be isometric to either $(\mathbb{R}^3, d_1)$ or $(\mathbb{R}^3, d_\infty)$ because in $(\mathbb{R}^3, d_2)$ we can have at most 4 points x_i with $d_2(x_i,x_j)=1$ for all $i\neq j$, but in $(\mathbb{R}^3, d_1)$ we have the 6 points $\pm\big(\frac12,0,0\big)$, $\pm\big(0,\frac12,0\big)$, $\pm\big(0,0,\frac12\big)$, and in $(\mathbb{R}^3, d_\infty)$ we have the 8 points (e_1,e_2,e_3) with each $e_i\in\{0,1\}$.

(c) Define $F:(\mathbb{R}^2,d_2)\to\big(\mathcal{C}[0,2\pi],d_\infty\big)$ by $F(r\cos\alpha,r\sin\alpha)(t)=r\cos(t+\alpha)$, where $r,\alpha\in\mathbb{R}$ with $r\geq 0$. Show that F is an isometry from $\mathbb{R}^2$ to $F(\mathbb{R}^2)$.

Solution: We claim that F is injective. Suppose that $(r\cos\alpha,r\sin\alpha)=F(s\cos\beta,s\sin\beta)$ in $\mathcal{C}[0,2\pi]$, where $r,s\geq 0$ and $\alpha,\beta\in\mathbb{R}$. Then we have $r\cos(t+\alpha)=s\cos(t+\beta)=g(t)$ for all $t\in[0,2\pi]$. It follows that we have $r=s=\max\big\{|g(t)|\big|0\leq t\leq 2\pi\big\}=\|g\|_\infty$, and we have $\cos(t+\alpha)=\cos(t+\beta)$ for all $t\in[0,2\pi]$ so that $\alpha=\beta \bmod 2\pi$. Thus we have $(r\cos\alpha,r\sin\alpha)=(s\cos\beta,s\sin\beta)$ in $\mathbb{R}^2$, and so F is injective as claimed. Next, we claim that F preserves distance. By the Law of Cosines, the distance between $u=(r\cos\alpha,r\sin\alpha)$ and $v=(s\cos\beta,s\sin\beta)$ in $\mathbb{R}^2$ is given by

$$d(u,v)=\sqrt{r^2+s^2-2rs\cos(\beta-\alpha)}\,.$$

On the other hand, the distance between $F(u)$ and $F(v)$ is

$$d_\infty\big(F(u),F(v)\big)=\big\|F(u)-F(v)\big\|_\infty=\max_{t\in[0,2\pi]}\big|r\cos(t+\alpha)-s\cos(t+\beta)\big|\,.$$

For $A,B,t\in\mathbb{R}$ we have

$$\begin{aligned}A\cos t+B\sin t&=\sqrt{A^2+B^2}\big(\tfrac{A}{\sqrt{A^2+B^2}}\cos t+\tfrac{B}{\sqrt{A^2+B^2}}\sin t\big)\\&=\sqrt{A^2+B^2}\,\cos(t+\phi)\,,\end{aligned}$$

where ϕ is the angle with $\cos\phi=\frac{A}{\sqrt{A^2+B^2}}$ and $\sin\phi=\frac{B}{\sqrt{A^2+B^2}}$, and so

$$\begin{aligned}r\cos(t+\alpha)-s\cos(t+\beta)&=r\cos t\cos\alpha-r\sin t\sin\alpha-s\cos t\cos\beta+s\sin t\sin\beta\\&=(r\cos\alpha-s\cos\beta)\cos t+(s\sin\beta-r\sin\alpha)\sin t\\&=\sqrt{A^2+B^2}\cos(t+\phi)\,,\end{aligned}$$

where $A=r\cos\alpha-s\cos\beta$ and $B=s\sin\beta-r\sin\alpha$ and ϕ is as above. Thus we have

$$\begin{aligned}d_\infty\big(F(u),F(v)\big)&=\max_{t\in[0,2\pi]}\big|\sqrt{A^2+B^2}\,\cos(t+\phi)\big|\\&=\sqrt{A^2+B^2}\\&=\sqrt{(r\cos\alpha-s\cos\beta)^2+(s\sin\beta-r\sin\alpha)^2}\\&=\sqrt{r^2\cos^2\alpha-2rs\cos\alpha\cos\beta+s^2\cos^2\beta+s^2\sin^2\beta-2rs\sin\alpha\sin\beta+r^2\sin^2\alpha}\\&=\sqrt{r^2+s^2-2rs\cos(\beta-\alpha)}\\&=d(u,v)\,.\end{aligned}$$