PMATH 351 Real Analysis, Exercises for Chapter 3: Limits and Continuity

1: (a) Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = 1 - nx$ for $0 \le x \le \frac{1}{n}$ and $f_n(x) = 0$ for $\frac{1}{n} \le x \le 1$. Show that $f_n \to 0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2, but $f_n \not\to 0$ pointwise on $[0,1]$.

(b) Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = n^2 x - n^3 x^2$ for $0 \le x \le \frac{1}{n}$ and $f_n(x) = 0$ for $\frac{1}{n} \le x \le 1$. Show that $f_n \to 0$ pointwise on $[0,1]$ but $f_n \not\to 0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2.

(c) Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \sqrt{n}\, x^n$. Show that $(f_n)_{n\ge 1}$ converges in $\big(\mathcal{C}[0,1], d_1\big)$ but diverges in $\big(\mathcal{C}[0,1], d_2\big)$.

2: (a) For each $n \in \mathbb{Z}^+$, let $x_n = (x_{n,k})_{k\ge 1} \in \mathbb{R}^\infty$ be given by $x_n = \sum_{k=1}^{n} \frac{k+1}{k}\, e_k$, where e_k is the k^{th} standard basis vector in $\mathbb{R}^\infty$ $\big($so we have $x_{n,k} = \frac{k+1}{k}$ when $k \le n$ and $x_{n,k} = 0$ when $k > n\big)$. Find $\lim\limits_{n\to\infty} \big(\lim\limits_{k\to\infty} x_{n,k}\big)$ in $\mathbb{R}$, and find $\lim\limits_{k\to\infty} \big(\lim\limits_{n\to\infty} x_{n,k}\big)$ in $\mathbb{R}$, and determine whether the sequence $(x_n)_{n\ge 1}$ converges in (ℓ_∞, d_∞).

(b) Let $A \subseteq \mathbb{R}$ and let $\ell_1(A) = \big\{(a_n) \in \ell_1 \big| \text{each } a_n \in A\big\}$. Show that $\overline{\ell_1(A)} = \ell_1\big(\overline{A}\big)$ in (ℓ_1, d_1).

(c) Let c be the set of all convergent sequences of real numbers. Show that c is closed and that the interior of c is empty in $\big(\ell_\infty, d_\infty\big)$.

3: Let X and Y be metric spaces.

(a) Let A and B be closed sets in X with $X = A \cup B$, let $f : A \to Y$ and $g : B \to Y$ be continuous with $f(x) = g(x)$ for all $x \in A \cap B$, and define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x)\ , \text{ for } x \in A, \\ g(x)\ , \text{ for } x \in B\,. \end{cases}$$

Show that h is continuous.

(b) Let A be a dense subset of X and let $f, g : X \to Y$ be continuous maps with $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in X$.

(c) Show that a map $f : X \to Y$ is continuous if and only if for every $B \subseteq Y$ we have $f^{-1}(B^\circ) \subseteq f^{-1}(B)^\circ$.

(d) Show that a map $f : X \to Y$ is continuous if and only if for every $A \subseteq X$ we have $f\big(\overline{A}\big) \subseteq \overline{f(A)}$.

4: (a) Let $I : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the identity map given by $I(x) = x$ for all $x \in \mathbb{R}^\infty$. Determine whether I is continuous as a map $I : (\mathbb{R}^\infty, d_1) \to (\mathbb{R}^\infty, d_2)$ and whether I is continuous as a map $I : (\mathbb{R}^\infty, d_2) \to (\mathbb{R}^\infty, d_1)$.

(b) Determine whether the map $G : \big(\mathcal{C}[0,1], d_1\big) \to \big(\mathbb{R}, d_2\big)$ given by $G(f) = f(0)$ is continuous.

(c) Determine whether the map $H : \big(\mathcal{C}[0,1], d_2\big) \to (\mathbb{R}, d_2)$ given by $H(f) = \displaystyle\int_0^1 f(x)\,dx$ is continuous.

5: Define $F : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by $F(f)(x) = \displaystyle\int_0^x \frac{f(t)}{\sqrt{t}}\,dt$.

(a) Determine whether F is continuous as a map from $\big(\mathcal{C}[0,1], d_\infty\big)$ to $\big(\mathcal{C}[0,1], d_\infty\big)$.

(b) Determine whether F is continuous as a map from $\big(\mathcal{C}[0,1], d_1\big)$ to $\big(\mathcal{C}[0,1], d_1\big)$.

6: (a) Show that $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are isometric.

(b) Show that $(\mathbb{R}^3, d_2)$ is not isometric to either $(\mathbb{R}^3, d_1)$ or $(\mathbb{R}^3, d_\infty)$.

(c) Define $F : (\mathbb{R}^2, d_2) \to \big(\mathcal{C}[0, 2\pi], d_\infty\big)$ by $F(r\cos\alpha, r\sin\alpha)(t) = r\cos(t + \alpha)$, where $r, \alpha \in \mathbb{R}$ with $r \ge 0$. Show that F is an isometry from $\mathbb{R}^2$ to $F(\mathbb{R}^2)$.