PMATH 351 Real Analysis, Exercises for Chapter 2: Metric Spaces

1: Determine which of the following functions $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are metrics on $\mathbb{R}$.

(a) $d(x, y) = (x - y)^2$

(b) $d(x, y) = \sqrt{|x - y|}$

(c) $d(x, y) = |x^2 - y^2|$

(d) $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$

2: (a) Let $S = \{(x, y) \in \mathbb{R}^2 \big| y > x^2\}$. Prove, from the definition of an open set, that S is open in $\mathbb{R}^2$.

(b) Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$. Show that Range$(f)$ is not closed in $\mathbb{R}^2$.

3: Determine which of the following statements are true for every metric space (X, d) and every $A \subseteq X$.

(a) $\overline{B(a, r)} = \overline{B}(a, r)$ for every $a \in X$ and every $r > 0$.

(b) $\left(\overline{A}\right)^c = \left(A^c\right)^\circ$.

(c) If $A = A^\circ$ then $A = \left(\overline{A}\right)^\circ$.

(d) If $A = \overline{A}$ then $\partial(\partial A) = \partial A$.

4: (a) Let (X, d) be a metric space with X uncountable. Show that for every $a \in X$ there exists $r > 0$ such that $B(a, r)$ is uncountable.

(b) Let (X, d) be a metric space with the property that for every $a \in X$ there exists $r > 0$ such that $B(a, r)$ is countable. Determine whether X must be countable.

5: (a) Show that there is no inner product on $\mathbb{R}^2$ which induces the 1-norm $\| \ \|_1$.

(b) Let $T = \{U \subseteq \mathbb{R} \,\big|\, U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite}\}$. Show that T is a topology on $\mathbb{R}$ which is not induced by any metric on $\mathbb{R}$ (T is called the *cofinite topology* on $\mathbb{R}$).

6: Let A denote the set of all real-valued sequences $(a_n)_{n\geq 1}$ for which $|a_n| \leq \frac{1}{2^n}$ for all $n \in \mathbb{Z}^+$.

(a) Show that $A^\circ = \emptyset$ in (ℓ_1, d_1).

(b) Show that $\overline{A} = A$ in (ℓ_1, d_1).

7: (a) Show that ℓ_1 is neither open nor closed in the metric space (ℓ_∞, d_∞).

(b) Determine whether every set $U \subseteq \ell_1$ which is open in (ℓ_1, d_2) is also open in (ℓ_1, d_1).

(c) Determine whether every set $U \subseteq \ell_1$ which is open in (ℓ_1, d_1) is also open in (ℓ_1, d_2).

8: (a) Verify that we can define a metric on the space $M_{k\times\ell}(\mathbb{R})$ of real $k \times \ell$ matrices by $d(A, B) = \text{rank}(B - A)$.

(b) Verify that we can define a metric on the unit sphere $\mathbb{S}^2 = \{u \in \mathbb{R}^3 \,\big|\, \|u\| = 1\}$ by $d(u, v) = \cos^{-1}(u \cdot v)$ where $u \cdot v$ is the standard inner product (the dot product) in $\mathbb{R}^3$. Hint: you may wish to use properties of the cross product in $\mathbb{R}^3$.