PMATH 351 Real Analysis, Solutions to the Exercises for Chapter 1

1: (a) Find a bijective map $f:\mathbb{R}\to[0,1)$.

Solution: The map $h:\mathbb{R}\to(0,1)$ given by $h(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$ is a bijective map with inverse given by $h^{-1}(y)=\tan\big(\pi(y-\frac{1}{2})\big)$, and the map $g:(0,1)\to[0,1)$ given by $g\big(\frac{k}{k+1}\big)=\frac{k-1}{k}$ for $k\in\mathbb{Z}^+$ and by $g(x)=x$ when $x\neq\frac{k}{k+1}$ for any $k\in\mathbb{Z}^+$ is bijective with inverse given by $g^{-1}\big(\frac{k-1}{k}\big)=\frac{k}{k+1}$ for $k\in\mathbb{Z}^+$ and by $g^{-1}(y)=y$ for $y\neq\frac{k-1}{k}$ for any $k\in\mathbb{Z}^+$. The composite $f:\mathbb{R}\to[0,1)$ given by $f(x)=g(h(x))$ is bijective.

(b) Find an injective map $f:\mathbb{R}\to\mathbb{R}\setminus\mathbb{Q}$.

Solution: The map $h:\mathbb{R}\to(0,1)$ given by $h(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$ is a bijective map with inverse given by $h^{-1}(y)=\tan\big(\pi(y-\frac{1}{2})\big)$, and the map $g:(0,1)\to\mathbb{R}\setminus\mathbb{Q}$ given by

$$g(x)=\begin{cases} x & , \ x\notin\mathbb{Q} \\ x+\sqrt{2} & , \ x\in\mathbb{Q}\,. \end{cases}$$

is injective, so the composite $f:\mathbb{R}\to\mathbb{R}\setminus\mathbb{Q}$ given by $f(x)=g(h(x))$ is injective.

(c) Find a bijective map from $\mathbb{N}$ to the set of all finite subsets of $\mathbb{N}$.

Solution: Let A be the set of finite subsets of $\mathbb{N}$. We define a bijective map $F:\mathbb{N}\to A$ as follows. Given $n\in\mathbb{N}$ we can write n (uniquely) in its binary representation as $n=a_m a_{m-1}\cdots a_1 a_0$, so we have $n=\sum\limits_{i=0}^{m}a_i 2^i$ where each $a_i\in\{0,1\}$ with $a_m=1$ (unless $n=0$ in which case $m=a_m=0$). We then define

$$F(n)=F\Big(\sum_{k=0}^{m}a_k2^k\Big)=\big\{k\in\mathbb{N}\big|a_k=1\big\}.$$

(for example, when $n=19$, in binary notation $n=10011$ and so $F(n)=\{0,1,4\}$). The inverse map $G:A\to\mathbb{N}$ is given by

$$G(S)=\sum_{k=0}^{\infty}a_k2^k \text{ where } a_k=\begin{cases}1 \text{ if } k\in S,\\ 0 \text{ if } k\notin S.\end{cases}$$

In the above equation, S is a finite subset of $\mathbb{N}$, and the sum $\sum\limits_{k=0}^{\infty}a_k2^k$ finite because S is finite so that $a_k=1$ for only finitely many values of $k\in\mathbb{N}$.

2: Find the cardinality of each of the following sets without using cardinal arithmetic (that is, only using the material from Chapter 1 up until, and including, Theorem 1.24).

(a) The set of all functions $f : \mathbb{N} \to \mathbb{N}$.

Solution: Recall that $2^{\mathbb{N}}$ denotes the set of functions from $\mathbb{N}$ to $\{0,1\}$, and $\mathbb{N}^{\mathbb{N}}$ denotes the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Note that $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ (since every function from $\mathbb{N}$ to $\{0,1\}$ is also a function from $\mathbb{N}$ to $\mathbb{N}$) and so we have $|2^{\mathbb{N}}| \leq |\mathbb{N}^{\mathbb{N}}|$. Recall that each element $n \in \mathbb{N}$ can be written uniquely in the form $n = 2^{k-1}(2l+1)-1$ with $k, l \in \mathbb{N}$. Define $F : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$F(f)\big(2^k(2l-1)-1\big) = \begin{cases} 1 \text{ if } k = f(l), \\ 0 \text{ if } k \neq f(l). \end{cases}$$

(In the above equation, $f : \mathbb{N} \to \mathbb{N}$ and $F(f) : \mathbb{N} \to \{0,1\}$). We claim that F is injective. Let $f, g : \mathbb{N} \to \mathbb{N}$. Suppose that $F(f) = F(g)$. Then $F(f)(n) = F(g)(n)$ for all $n \in \mathbb{N}$. Given $k, l \in \mathbb{N}$, let $n = 2^k(2l-1)-1$. Then we have $k = f(l) \iff F(f)(n) = 1 \iff F(g)(n) = 1 \iff k = g(l)$. Thus $f(l) = g(l)$ for all $l \in \mathbb{N}$, and so $f = g$. Thus F is injective, as claimed, and so we have $|\mathbb{N}^{\mathbb{N}}| \leq |2^{\mathbb{N}}|$. By the Cantor-Schroeder-Bernstein Theorem, it follows that $|\mathbb{N}^{\mathbb{N}}| = |2^{\mathbb{N}}|$, that is $|\mathbb{N}^{\mathbb{N}}| = 2^{\aleph_0}$.

(b) The set of all non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$.

Solution: Let A be the set of non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. Define $F : 2^{\mathbb{N}} \to A$ by $F(f)(k) = k + f(k)$ where $f \in 2^{\mathbb{N}}$ (so $f : \mathbb{N} \to \{0,1\}$) and $k \in \mathbb{N}$. Note that the function $F(f)$ is non-decreasing since for all $k \in \mathbb{N}$ we have $F(f)(k+1) - F(f)(k) = k+1+f(k+1)-k-f(k) = 1+f(k+1)-f(k) \geq 1+0-1 = 0$ (since $f(k), f(k+1) \in \{0,1\}$). Also note that the function F is injective because for $f, g \in 2^{\mathbb{N}}$, if $F(f) = F(g)$ then for all $k \in \mathbb{N}$ we have $F(f)(k) = F(g)(k)$ so that $k + f(k) = k + g(k)$, and hence $f(k) = g(k)$. Since $F : 2^{\mathbb{N}} \to A$ is injective we have $\left|2^{\mathbb{N}}\right| \leq |A|$. On the other hand, since $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $|\mathbb{N}^{\mathbb{N}}| = |2^{\mathbb{N}}|$ as shown in Part (a), we have $|A| \leq |2^{\mathbb{N}}|$. Thus $|A| = |2^{\mathbb{N}}| = 2^{\aleph_0}$, by the Cantor-Schroeder-Bernstein Theorem.

(c) The set of all non-increasing functions $f : \mathbb{N} \to \mathbb{N}$.

Solution: Let B be the set of non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$. Define $F : \mathbb{N} \to B$ by letting $F(n)$ be the constant function $F(n)(k) = n$ for all $k \in \mathbb{N}$. Then F is clearly injective so we have $|\mathbb{N}| \leq |B|$. Define $G : B \to \mathbb{N}$ as follows. Given $f \in B$, let $m = \min\big(\text{Range}(f)\big)$ and $l = \min\{k \in \mathbb{N} | f(k) = m\}$ (these exist by the Well-Ordering Property of $\mathbb{N}$), and then define $G(f) = \prod_{k=0}^{l} {p_k}^{f(k)+1}$ where $p_0 = 2$ and where p_k is the k^{th} odd prime for $k \geq 1$. The function G is injective because the function f is uniquely determined by the numbers $l, f(0), f(1), \cdots f(l)$ and because positive integers have unique prime factorization. Thus $|B| \leq |\mathbb{N}|$. Since $|\mathbb{N}| \leq |B|$ and $|B| \leq |\mathbb{N}|$, we have $|B| = |\mathbb{N}| = \aleph_0$ by the Cantor-Schroeder Bernstein Theorem.

We remark that the map $H : B \to \mathbb{N}$ given by $H(f) = \prod_{k=0}^{l} {p_k}^{f(k)}$ is not injective. Can you see why?

3: Find the cardinality of each of the following sets.

(a) The set of all countably infinite subsets of $\mathbb{R}$.

Solution: Let C be the set of all countably infinite subsets of $\mathbb{R}$. Note that the map $f:\mathbb{R}\to C$ given by $f(r)=\{r,r+1,r+2,\cdots\}$ is injective so that we have $2^{\aleph_0}=|\mathbb{R}|\leq|C|$, and the map $g:\mathbb{R}^\omega\to C$ given by $g(x_1,x_2,x_3,\cdots)=\{x_1,x_2,x_3,\cdots\}$ is surjective so that, by Example 1.32, we have $2^{\aleph_0}=|\mathbb{R}^\omega|\geq|C|$. Thus $|C|=2^{\aleph_0}$ by the the Cantor-Schoeder-Bernstein Theorem.

(b) The set of continuous functions $f:\mathbb{R}\to\mathbb{R}$.

Solution: Let $\mathcal{C}(\mathbb{R})$ denote the set of continuous maps $f:\mathbb{R}\to\mathbb{R}$ and let $\mathcal{C}(\mathbb{Q})$ denote the set of continuous maps $f:\mathbb{Q}\to\mathbb{R}$. Since every continuous map $f:\mathbb{R}\to\mathbb{R}$ is determined by its restriction to $\mathbb{Q}$ (indeed given $a\notin\mathbb{Q}$, we can choose a sequence $\langle x_n\rangle$ in $\mathbb{Q}$ with $x_n\to a$ and then we must have $f(a)=\lim\limits_{n\to\infty}f(x_n)\big)$, the map $F:\mathcal{C}(\mathbb{R})\to\mathcal{C}(\mathbb{Q})$, which sends $f\in\mathcal{C}(\mathbb{R})$ to its restriction to $\mathbb{Q}$, is a bijection. Thus we have $\big|\mathcal{C}(\mathbb{R})\big|=\big|\mathcal{C}(\mathbb{Q})\big|$. Since $\mathcal{C}(\mathbb{Q})\subseteq\mathbb{R}^{\mathbb{Q}}$, we have

$$\big|\mathcal{C}(\mathbb{R})\big|=\big|\mathcal{C}(\mathbb{Q})\big|\leq\big|\mathbb{R}^{\mathbb{Q}}\big|=\big(2^{\aleph_0}\big)^{\aleph_0}=2^{\aleph_0\cdot\aleph_0}=2^{\aleph_0}\,.$$

Also the map $G:\mathbb{R}\to\mathcal{C}(\mathbb{R})$, which sends $a\in\mathbb{R}$ to the constant map f_a given by $f_a(x)=a$ for all $x\in\mathbb{R}$, is injective so

$$2^{\aleph_0}=|\mathbb{R}|\leq\big|\mathcal{C}(\mathbb{R})\big|\,.$$

By the Cantor-Schroeder-Bernstein Theorem, we have $\big|\mathcal{C}(\mathbb{R})\big|=2^{\aleph_0}$.

(c) The set of all bounded functions $f:\mathbb{R}\to\mathbb{R}$.

Solution: Note that $\mathcal{B}(\mathbb{R})\subseteq\mathbb{R}^{\mathbb{R}}$ so we have

$$\big|\mathcal{B}(\mathbb{R})\big|\leq\big|\mathbb{R}^{\mathbb{R}}\big|=(2^{\aleph_0})^{2^{\aleph_0}}=2^{\aleph_0\cdot2^{\aleph_0}}\leq2^{2^{\aleph_0}\cdot2^{\aleph_0}}=2^{2^{\aleph_0+\aleph_0}}=2^{2^{\aleph_0}}\,.$$

Also note that $2^{\mathbb{R}}\subseteq\mathcal{B}(\mathbb{R})$ so we have

$$2^{2^{\aleph_0}}=\big|2^{\mathbb{R}}\big|\leq\big|\mathcal{B}(\mathbb{R})\big|\,.$$

By the Cantor-Schröder-Bernstein Theorem, $\big|\mathcal{B}(\mathbb{R})\big|=2^{2^{\aleph_0}}$.

4: (a) Show that every open set in $\mathbb{R}$ (using the standard topology) is equal to the union of finite or countably many disjoint open intervals.

Solution: For $a, b \in \mathbb{R}$, let $[a,b]$ denote the closed interval between a and b, that is

$$[a,b] = \{a + t(b-a) \,\big|\, 0 \leq t \leq 1\},$$

and note that $[a,b] = [b,a] = \big[\min\{a,b\}, \max\{a,b\}\big]$. Recall that the intervals in $\mathbb{R}$ are the sets with the intermediate value property: a subset $I \subseteq \mathbb{R}$ is an interval when it has the property that for every $a, b \in I$ we have $[a,b] \subseteq I$ (in other words, the intervals in $\mathbb{R}$ are equal to the convex subsets of $\mathbb{R}$). Let U be an open set in $\mathbb{R}$. Define a relation on U by stipulating that $a \sim b \iff [a,b] \subseteq U$. Note that this is an equivalence relation $\big($indeed we have $a \sim a$ because $[a,a] = \{a\}$, and if $a \sim b$ then $b \sim a$ because $[a,b] = [b,a]$, and if $a \sim b$ and $b \sim c$ then $a \sim c$ because $[a,c] \subseteq [a,b] \cup [b,c]\big)$. It follows that U is the disjoint union of the equivalence classes.

We claim that each equivalence class C is an interval. Let C be an equivalence class and let $a, b \in C$. Then we have $a \sim b$ and $C = \{x \in U \big| x \sim a\}$. Since $a \sim b$ we have $[a,b] \subseteq U$. For every $x \in [a,b]$ we have $[a,x] \subseteq [a,b] \subseteq U$ so that $x \sim a$ and hence $x \in C$. This shows that $[a,b] \subseteq C$, hence C is an interval.

We claim that each equivalence class C is open. Let C be an equivalence class and let $a \in C$. Then we have $C = \{x \in U \big| x \sim a\}$. Since U is open we can choose $r > 0$ such that $(a-r, a+r) \subseteq U$. For every $x \in (a-r, a+r)$ we have $[a,x] \subseteq (a-r, a+r) \subseteq U$ so that $x \sim a$ hence $x \in C$. This shows that for all $(a-r, a+r) \subseteq C$ and so C is open.

Finally, we claim that there are at most countably many equivalence classes C. We denote the set of equivalence classes by $U/\!\sim$. For each equivalence class $C \in U/\!\sim$, since C is a nonempty open interval we can choose a rational number $a_C \in C$. Because the equivalence classes are disjoint, the rational numbers a_C are distinct so the map $F : U/\!\sim \mathbb{Q}$ given by $F(C) = a_C$ is injective. Thus the set of equivalence classes $U/\!\sim$ is at most countable.

(b) Find the cardinality of the set of all open sets in $\mathbb{R}$.

Solution: Let S be the set of all open sets in $\mathbb{R}$. We claim that $|S| = 2^{\aleph_0}$. Since the map $F : \mathbb{R} \to S$ given by $F(a) = B(a,1) = (a-1, a+1)$ is injective, we have $|S| \geq |\mathbb{R}| = 2^{\aleph_0}$. It remains to show that $|S| \leq 2^{\aleph_0}$.

Let U be a nonempty open set. For each $a \in U$ we can choose $r_a > 0$ so that $(a - 3r_a, a + 3r_a) \subseteq U$ then we can choose $q_a \in \mathbb{Q}$ with $q_a \in \big(a - r, a + r\big)$ and we can choose $s_a \in \mathbb{Q}$ with $r < s < 2r$ and then we have $a \in (q_a - s_a, q_a + s_a) \subseteq (a - 3r_a, a + 3r_a) \subseteq U$. It follows that $U = \bigcup_{a \in U} (q_a - s_a, q_a + s_a)$. Thus every nonempty open set is a union of open intervals with rational centre and positive rational radius. Hence every open set (including the empty set) is a union of open intervals with rational centre and non-negative rational radius. Since there are only countably many such open intervals $\big($indeed $|\mathbb{Q} \times \mathbb{Q}^{\geq 0}| = \aleph_0\big)$ it follows that every open set in $\mathbb{R}$ is equal to a countable union of open intervals with rational centre and non-negative rational radius. It follows that the map $G : (\mathbb{Q} \times \mathbb{Q}^{\geq 0})^{\mathbb{Z}^+} \to S$ given by

$$G\big((q_1, s_1), (q_2, s_2), \cdots\big) = (q_1 - s_1, q_1 + s_1) \cup (q_2 - s_2, q_2 + s_2) \cup \cdots$$

is surjective. Thus we have

$$|S| \leq \left|(\mathbb{Q} \times \mathbb{Q}^{\geq 0})^{\mathbb{Z}^+}\right| = \aleph_0^{\aleph_0} \leq \big(2^{\aleph_0}\big)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

5: (a) Let $\mathbb{Q}^+ = \{x \in \mathbb{Q} | x > 0\}$ and let $\mathbb{Z}^+ = \{k \in \mathbb{Z} | k > 0\}$. Let $f : \mathbb{Q}^+ \to \mathbb{Z}^+$ be the injective map given by $f\left(\frac{k}{l}\right) = 2^{k-1}(2l-1)$ for $k, l \in \mathbb{Z}^+$ with $\gcd(k,l) = 1$. Let $A = f(\mathbb{Q}^+)$. Let $a_0 = \min A$, $a_1 = \min A \setminus \{a_0\}$, $a_2 = \min A \setminus \{a_0, a_1\}$, and so on. Find a_{20} and find $\left|A \cap S_{100}\right|$ where as usual, for $m \in \mathbb{N}$ we write $S_m = \{0, 1, \cdots, m-1\}$.

Solution: The elements in $\mathbb{Z}^+ \setminus A = \mathbb{Z}^+ \setminus f(\mathbb{Q}^+)$ are the elements $2^{k-1}(2l-1)$ with $\gcd(k,l) \neq 1$, so

$$\mathbb{Z}^+ \setminus A = \left\{2^0(2l-1)\middle| \gcd(1,l) \neq 1\right\} \cup \left\{2^1(2l-1)\middle| \gcd(2,l) \neq 1\right\} \cup \left\{2^2(2l-1)\middle| \gcd(3,l) \neq 1\right\} \cup \cdots$$

We have $\gcd(1,l) = 1$ for all $l \in \mathbb{Z}^+$, and so $\left\{2^0(2l-1)\middle| \gcd(1,l) \neq 1\right\} = \emptyset$. We have $\gcd(2,l) \neq 1$ for $l \in \{2, 4, 6, 8, 10, \cdots\}$, and so

$$\left\{2^1(2l-1)\middle| \gcd(2,l) \neq 1\right\} = \{2 \cdot 3, 2 \cdot 7, 2 \cdot 11, 2 \cdot 15, 2 \cdot 19, \cdots\} = \{6, 14, 22, 30, 38, \cdots\}.$$

We have $\gcd(3,l) \neq 1$ for $l \in \{3, 6, 9, 12, \cdots\}$ and so

$$\left\{2^2(2l-1)\middle| \gcd(3,l) \neq 1\right\} = \{4 \cdot 5, 4 \cdot 11, 4 \cdot 17, 4 \cdot 23, \cdots\} = \{20, 44, 68, 92, \cdots\}.$$

Similarly,

$$\begin{aligned}
&\left\{2^3(2l-1)\middle| \gcd(4,l) \neq 1\right\} = \left\{8(2l-1)\middle| l = 2, 4, 6, 8, 10\right\} = \{24, 56, 58, 120, \cdots\} \\
&\left\{2^4(2l-1)\middle| \gcd(5,l) \neq 1\right\} = \left\{16(2l-1)\middle| l = 5, 10, 15, 20, \cdots\right\} = \{144, 304, \cdots\} \\
&\left\{2^5(2l-1)\middle| \gcd(6,l) \neq 1\right\} = \left\{32(2l-1)\middle| l = 2, 3, 6, 8, 9, 12, 14, 15, 18, \cdots\right\} = \{96, 160, 352, \cdots\}
\end{aligned}$$

For $k \geq 7$ it is clear that $\left\{2^{k-1}(2l-1)\middle| \gcd(k,l) \neq 1\right\} \cap S_{100} = \emptyset$, and so we have

$$\left(\mathbb{Z}^+ \setminus A\right) \cap S_{100} = \{6, 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94\} \cup \{20, 44, 68, 92\} \cup \{24, 56, 58\} \cup \{96\}.$$

Thus $\left|(\mathbb{Z}^+ \setminus A) \cap S_{100}\right| = 12 + 4 + 3 + 1 = 20$ and so $\left|A \cap S_{100}\right| = 99 - 20 = 79$. Also, the first few term in the sequence $\langle a_k \rangle_{k \geq 0}$ are as follows

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_k	1	2	3	4	5	7	8	9	10	11	12	13	15	16	17	18	19	21	23	25	26

and in particular, $a_{20} = 26$.

(b) Let $A = B = \mathbb{N}$. Let $f : A \to B$ and $g : B \to A$ be the injective maps given by $f(k) = 2k$ and $g(k) = 3k$. Let $X_1 = A$ and $Y_1 = g(B)$, and for $k \geq 1$ let $X_{k+1} = g(f(X_k))$ and $Y_{k+1} = g(f(Y_k))$. Let $U = \bigcup\limits_{k=1}^{\infty} (X_k \setminus Y_k)$. Find $\left|U \cap S_{100}\right|$ and find $\left|U \cap S_m\right|$ in the case that $m = 6^k$ with $k \in \mathbb{N}$.

Solution: For $m \in \mathbb{N}$ we write $m\mathbb{N} = \{mk | k \in \mathbb{N}\} = \{0, m, 2m, 3m, \cdots\}$. We have $X_1 = \mathbb{N}$ and $Y_1 = 3\mathbb{N}$. Note that $g(f(k)) = g(2k) = 6k$ for all $k \in \mathbb{N}$, and so we have $X_2 = 6\mathbb{N}$, $X_3 = 36\mathbb{N}$, and in general $X_n = 6^{n-1}\mathbb{N}$, and we have $Y_2 = 18\mathbb{N}$, $Y_3 = 108\mathbb{N}$ and in general $Y_n = 3 \cdot 6^{n-1}\mathbb{N}$.

$$\begin{aligned}
U \cap S_{100} &= (X_1 \setminus Y_1) \cap S_{100} \cup (X_2 \setminus Y_2) \cap S_{100} \cup (X_3 \setminus Y_3) \cap S_{100} \cup \cdots \\
&= (X_1 \cap S_{100}) \setminus (Y_1 \cap S_{100}) \cup (X_2 \cap S_{100}) \setminus (Y_2 \cap S_{100}) \cup (X_3 \cap S_{100}) \setminus (Y_3 \cap S_{100}) \cup \cdots \\
&= \{0, 1, 2, \cdots, 99\} \setminus \{0, 3, 6, 9, \cdots, 96, 99\} \cup \{0, 6, 12, \cdots, 96\} \setminus \{0, 18, 36, 54, 72, 90\} \cup \{0, 36, 72\} \setminus \{0\}
\end{aligned}$$

and so $\left|U \cap S_{100}\right| = (100 - 34) + (17 - 6) + (3 - 1) = 79$.

Let $m = 6^k$. For $j > k$ we have $X_j \cap S_m = Y_j \cap S_m = \{0\}$ and for $1 \leq j \leq k$ we have

$$\begin{aligned}
|X_j \cap S_m| &= |6^{j-1}\mathbb{N} \cap S_m| = \tfrac{1}{6^{j-1}}\, 6^k = 6^{k-j+1} \text{ , and} \\
|Y_j \cap S_m| &= |3 \cdot 6^{j-1}\mathbb{N} \cap S_m| = \tfrac{1}{3 \cdot 6^{j-1}}\, 6^k = \tfrac{1}{3}\, 6^{k-j+1}
\end{aligned}$$

so we have

$$U \cap S_m = \bigcup_{j=1}^{\infty} (X_j \setminus Y_j) \cap S_m = \bigcup_{j=1}^{\infty} (X_j \setminus S_m) \setminus (Y_j \cap S_m) = \bigcup_{j=1}^{k} (X_j \setminus S_m) \setminus (Y_j \cap S_m) \text{ , and}$$

$$\begin{aligned}
\left|U \cap S_m\right| &= \sum_{j=1}^{k} \big(|X_j \cap S_m| - |Y_j \cap S_m|\big) = \sum_{j=1}^{k} \left(6^{k-j+1} - \tfrac{1}{3}\, 6^{k-j+1}\right) = \tfrac{2}{3} \sum_{j=1}^{k} 6^{k-j+1} \\
&= \tfrac{2}{3}(6^k + 6^{k-1} + \cdots + 6^2 + 6) = \tfrac{2}{3} \cdot \tfrac{6}{5}(6^k - 1) = \tfrac{4}{5}(6^k - 1).
\end{aligned}$$