PMATH 351 Real Analysis, Exercises for Chapter 1: Cardinality

1: (a) Find a bijective map $f : \mathbb{R} \to [0, 1)$.
(b) Find an injective map $f : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$.
(c) Find a bijective map from $\mathbb{N}$ to the set of all finite subsets of $\mathbb{N}$.

2: Find the cardinality of each of the following sets without using cardinal arithmetic (that is, only using the material from Chapter 1 up until, and including, Theorem 1.24).
(a) The set of all functions $f : \mathbb{N} \to \mathbb{N}$.
(b) The set of all nondecreasing functions $f : \mathbb{N} \to \mathbb{N}$.
(c) The set of all nonincreasing functions $f : \mathbb{N} \to \mathbb{N}$.

3: Find the cardinality of each of the following sets.
(a) The set of all countably infinite subsets of $\mathbb{R}$.
(b) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$.
(c) The set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$.

4: (a) Show that every open set in $\mathbb{R}$ (using the standard topology) is equal to the union of finite or countably many disjoint open intervals.
(b) Find the cardinality of the set of all open sets in $\mathbb{R}$.

5: (a) Let $\mathbb{Q}^+ = \{x \in \mathbb{Q} | x > 0\}$ and let $\mathbb{Z}^+ = \{k \in \mathbb{Z} | k > 0\}$. Let $f : \mathbb{Q}^+ \to \mathbb{Z}^+$ be the injective map given by $f\left(\frac{k}{l}\right) = 2^{k-1}(2l - 1)$ for $k, l \in \mathbb{Z}^+$ with $\gcd(k, l) = 1$. Let $A = f(\mathbb{Q}^+)$. Let $a_0 = \min A$, $a_1 = \min A \setminus \{a_0\}$, $a_2 = \min A \setminus \{a_0, a_1\}$, and so on. Find a_{20} and find $\left|A \cap S_{100}\right|$ where for $m \in \mathbb{N}$ we write $S_m = \{0, 1, \cdots, m-1\}$.
(b) Let $A = B = \mathbb{N}$. Let $f : A \to B$ and $g : B \to A$ be the injective maps given by $f(k) = 2k$ and $g(k) = 3k$. Let $X_1 = A$ and $Y_1 = g(B)$, and for $k \geq 1$ let $X_{k+1} = g(f(X_k))$ and $Y_{k+1} = g(f(Y_k))$. Let $U = \bigcup_{k=1}^{\infty} (X_k \setminus Y_k)$. Find $\left|U \cap S_{100}\right|$ and find $\left|U \cap S_m\right|$ in the case that $m = 6^k$ with $k \in \mathbb{N}$.