Chapter 7. The Baire Category Theorem

7.1 Definition: When I is the bounded open interval $I=(a,b)$, where $a,b\in\mathbb{R}$ with $a\leq b$, the diameter of I is $d(I)=b-a$. For a subset $A\subseteq\mathbb{R}$, we define the **Lebesgue outer measure** of A to be

$$\lambda(A)=\inf\left\{\sum_{k=1}^{\infty}d(I_k)\,\middle|\,\text{each } I_k \text{ is a bounded open interval in } \mathbb{R} \text{ and } A\subseteq\bigcup_{k=1}^{\infty}I_k\right\}$$

with $0\leq\lambda(A)\leq\infty$. We say that A has (Lebesgue) **measure zero** when $\lambda(A)=0$.

7.2 Note: Every finite or countable set $A\subseteq\mathbb{R}$ has measure zero. Indeed, if A is finite, say $A=\{a_1,a_2,\cdots,a_n\}$, then given $\epsilon>0$ then we can take $I_k=\left(a_k-\frac{\epsilon}{2n},a_k+\frac{\epsilon}{2n}\right)$ for $k\leq n$, and we can take $I_k=\emptyset$ for $k>n$, to get $A\subseteq\bigcup\limits_{k=1}^{\infty}I_k$ and $\sum\limits_{k=1}^{\infty}d(I_k)=\sum\limits_{k=1}^{n}\frac{\epsilon}{n}=\epsilon$. And if A is countably infinite, say $A=\{a_1,a_2,a_3,\cdots\}$, then we can take $I_k=\left(a-\frac{\epsilon}{2^{k+1}},a_k+\frac{\epsilon}{2^{k+1}}\right)$ for all $k\geq 1$ to get $A\subseteq\bigcup\limits_{k=1}^{\infty}I_k$ and $\sum\limits_{k=1}^{\infty}d(I_k)=\sum\limits_{k=1}^{\infty}\frac{\epsilon}{2^k}=\epsilon$. Perhaps surprisingly, it is not the case that every set of measure zero is at most countable.

7.3 Example: The (standard) **Cantor set** is the set $C\subseteq[0,1]$ constructed as follows. Let $C_0=[0,1]$. Let I_1 be the open middle third of C_0, that is let $I_1=\left(\frac{1}{3},\frac{2}{3}\right)$, and let $C_1=A_0\backslash U_1=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$. Let I_2 and I_3 be the open middle thirds of the two component intervals of C_1, that is let $I_2=\left(\frac{1}{9},\frac{2}{9}\right)$ and $I_3=\left(\frac{7}{9},\frac{8}{9}\right)$, and let $C_2=C_1\setminus(I_2\cup I_3)$. Having constructed the set C_n, which is the disjoint union of 2^n closed intervals each of length $\frac{1}{3^n}$, let $I_{2^n},I_{2^n+1},\cdots,I_{2^{n+1}-1}$ be the open middle thirds of these 2^n component intervals and let $C_{n+1}=C_n\setminus(I_{2^n},I_{2^n+1},\cdots,I_{2^{n+1}-1})$. Note that C_n is the set of all numbers $x\in[0,1]$ which can be written in base 3 such that the the first n digits of x are not equal to 1.

The Cantor set is the set

$$C=\bigcap_{n=0}^{\infty}C_n$$

or equivalently, C is the set of all numbers $x\in[0,1]$ which can be written in base 3 with none of the digits of x equal to 1.

Since $C=\bigcap\limits_{n=0}^{\infty}C_n$ with $C_0\supseteq C_1\supseteq C_2\supseteq\cdots$, it follows that $C\subseteq C_n$ for all $n\in\mathbb{N}$. Since C_n is the (disjoint) union of 2^n closed intervals each of size $\frac{1}{3^n}$, it follows that we can cover C_n (hence also C) by a union of 2^n open intervals each of size $\frac{2}{3^n}$, and so we have $\lambda(C)\leq 2^n\cdot\frac{2}{3^n}=\frac{2^{n+1}}{3^n}$. Since $\lambda(C)\leq\frac{2^{n+1}}{3^n}$ for all $n\in\mathbb{N}$ and $\frac{2^{n+1}}{3^n}\to 0$ as $n\to\infty$, it follows that $\lambda(C)=0$.

On the other hand, since C is the set of all real numbers $x\in[0,1]$ which can be written in base 3 using only the digits 0 and 2, it follows that $|C|=2^{\aleph_0}$.

7.4 Remark: Note that the set C of numbers $x\in[0,1]$ which *can* be written in base 3 without using the digit 1, is not equal to the complement of the set B of numbers $x\in[0,1]$ which *can* be written in base 3 using the digit 1 (at least once). For example, the number $x=\frac{1}{3}$ can be written in base 3 as $x=0.1$ so we have $x\in B$, but it can also be written in base 3 as $x=0.0222\cdots$, so we also have $x\in C$.

7.5 Exercise: Show that the set of all real numbers $x\in[0,1]$, which can be written in base 5 without using the digit 2, has measure zero.

7.6 Definition: Let X be a metric space and let $A \subseteq X$. Recall that A is **dense** (in X) when for every nonempty open ball $B \subseteq X$ we have $B \cap A \neq \emptyset$, equivalently when $\overline{A} = X$. We say A is **nowhere dense** (in X) when for every nonempty open ball $B \subseteq X$ there exists a nonempty open ball $C \subseteq B$ with $C \cap A = \emptyset$, or equivalently when $\overline{A}^{o} = \emptyset$.

7.7 Exercise: Show that the Cantor set is nowhere dense in $[0, 1]$ (or in $\mathbb{R}$).

7.8 Note: When $A \subseteq B \subseteq X$, note that if A is dense in X then so is B and, on the other hand, if B is nowhere dense in X then so is A.

7.9 Note: When $A, B \subseteq X$ with $B = A^c = X \setminus A$, note that A is nowhere dense $\iff$ $\overline{A}^{0} = \emptyset \iff \overline{B^0} = X \iff$ the interior of B is dense.

7.10 Definition: Let $A \subseteq X$. We say that A is **first category** (or that A is **meagre**) when A is equal to a countable union of nowhere dense sets. We say that A is **second category** when it is not first category. We say that A **residual** when A^c is first category.

7.11 Note: Every countable set in $\mathbb{R}$ is first category since if $A = \{a_1, a_2, a_3, \cdots\}$ then we have $A = \bigcup\limits_{k=1}^{\infty} \{a_k\}$. In particular $\mathbb{Q}$ is first category and $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ is residual.

7.12 Note: If $A \subseteq X$ is first category then so is every subset of A.

7.13 Note: If $A_1, A_2, A_3, \cdots \subseteq X$ are are all first category then so is $\bigcup\limits_{k=1}^{\infty} A_k$.

7.14 Theorem: *(The Baire Category Theorem) Let X be a complete metric space.*
(1) Every first category set in X has an empty interior.
(2) Every residual set in X is dense.
(3) Every countable union of closed sets with empty interiors in X has an empty interior.
(4) Every countable intersection of dense open sets in X is dense.

Proof: Parts (1) and (2) are equivalent by taking complements, and Parts (3) and (4) are special cases of Parts (1) and (2), so it suffices to prove Part (1). We sketch a proof.

Let $A \subseteq X$ be first category, say $A = \bigcup\limits_{n=1}^{\infty} C_n$ where each C_n is nowhere dense. Suppose, for a contradiction, that A has nonempty interior, and choose an open ball $B_0 = B(a_0, r_0)$ with $0 < r_0 < 1$ such that $\overline{B}_0 \subseteq A$. Since each C_n is nowhere dense, we can chose a nested sequence of open balls $B_n = B(a_n, r_n)$ with $0 < r_n < \frac{1}{2^n}$ such that $\overline{B}_n \subseteq B_{n-1}$ and $\overline{B}_n \cap C_n = \emptyset$. Because $r_n \to 0$, it folows that the sequence $\{a_n\}$ is Cauchy. Because X is complete, it follows that $\{a_n\}$ converges in X, say $a = \lim\limits_{n\to\infty} a_n$. Note that $a \in \overline{B}_n$ for all n since $a_k \in \overline{B}_n$ for all $k \geq n$. Since $a \in \overline{B}_0$ and $\overline{B}_0 \subseteq A$ we have $a \in A$. But since $a \in \overline{B}_n$for al $n \geq 1$, and $\overline{B}_n \cap C_n = \emptyset$, we have $a \notin C_n$ for all $n \geq 1$ hence $a \notin \bigcup\limits_{n=1}^{\infty} C_n$, that is $a \notin A$.

7.15 Example: Recall that $\mathbb{Q}$ is first category and $\mathbb{Q}^c$ is residual. The Baire Category Theorem shows that $\mathbb{Q}^c$ cannot be first category because if $\mathbb{Q}$ and $\mathbb{Q}^c$ were both first category then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would also be first category, but this is not possible since $\mathbb{R}$ does not have empty interior.

7.16 Exercise: Let $f \in \mathcal{C}^\infty(\mathbb{R})$ and suppose that for all $x \in \mathbb{R}$ there exists $n_x \in \mathbb{Z}^+$ such that $f^{(n_x)}(x) = 0$. Show that there exists a nonempty open interval $(a,b) \subseteq \mathbb{R}$ such that the restriction of f to (a,b) is a polynomial.

7.17 Exercise: For each $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for all $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}^+$ such that $f_n(x) \in \mathbb{Q}$. Prove that there exists $n \in \mathbb{Z}^+$ such that f_n is constant in some nondegenerate interval.

7.18 Remark: Let $\mathcal{C}_1 = \big\{A \subseteq \mathbb{R} \big| \, A \text{ is finite or countable}\big\}$, $\mathcal{C}_2\big\{A \subseteq \mathbb{R} \big| \, \lambda(A) = 0\big\}$ and $\mathcal{C}_3 = \big\{A \subseteq \mathbb{R} \big| \, A \text{ is first category}\big\}$. Note that if $\mathcal{C} = \mathcal{C}_k$ for some $k \in \{1,2,3\}$, then $\mathcal{C}$ has the following properties:

(1) if $A \subseteq B$ and $B \in \mathcal{C}$ then $A \in \mathcal{C}$,

(2) if $A_1, A_2, A_3, \dots \in \mathcal{C}$ then $\bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{C}$, and

(3) if $A \in \mathcal{C}$ then $A^0 = \emptyset$.

Because of this, it seems reasonable to consider each set $\mathcal{C}_k$ to be, in some sense, "small". Perhaps surprisingly, the following theorem states that every set in $\mathbb{R}$ is the union of two such small sets.

7.19 Theorem: *Every subset of $\mathbb{R}$ is equal to the disjoint union of a set of measure zero and a set of first category.*

Proof: Let $\mathbb{Q} = \{a_1, a_2, a_3, \cdots\}$. For $k, \ell \in \mathbb{Z}^+$, let $I_{k,\ell} = \big(a_\ell - \frac{1}{2^{k+\ell}}\,,\, a_\ell + \frac{1}{2^{k+\ell}}\big)$ and for $k \in \mathbb{Z}^+$, let $U_k = \bigcup\limits_{\ell=1}^{\infty} I_{k,\ell}$. Note that each U_k is open with $\mathbb{Q} \subseteq U_k$, so each U_k is a dense open set. Also note that for each $k \in \mathbb{Z}^+$ we have $\lambda(U_k) \leq \sum\limits_{\ell=1}^{\infty} d(I_{k,\ell}) = \frac{1}{2^{k-1}}$. Let $B = \bigcap\limits_{k=1}^{\infty} U_k$ and note that B is residual, since it is a countable intersection of dense open sets. Since $B = \bigcap\limits_{k=1}^{\infty} U_k$ and $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$, we have $B \subseteq U_k$ for all k, hence $\lambda(B) \leq \lambda(U_k) \leq \frac{1}{2^{k-1}}$ for all $k \in \mathbb{Z}^+$, and it follows that $\lambda(B) = 0$. Thus $\mathbb{R}$ is the disjoint union of the set B, which has measure zero, and its complement B^c which is first category (since B is residual). Finally note that any set $A \subseteq \mathbb{R}$ is equal to the disjoint union $A = (A \cap B) \cup (A \cap B^c)$, and we have $\lambda(A \cap B) = 0$ and the set $A \cap B^c$ is first category.

7.20 Remark: At first glance, it might appear that the set B constructed in the above proof might simply be equal to $\mathbb{Q}$. But in fact, B must be uncountable, because if B was countable then B would be first category, but then B and B^c would both be first category, and hence $\mathbb{R} = B \cup B^c$ would also be first category. But $\mathbb{R}$ is not first category by the Baire Category Theorem.

7.21 Example: Most students will have seen that it is possible to construct a continuous function $f : [0,1] \to \mathbb{R}$ such that f is nowhere differentiable. Show that the set of nowhere differentiable functions is residual (hence dense) in $\mathcal{C}[0,1]$.

Solution: Let A be the complement of the set of nowhere differentiable functions in $\mathcal{C}[0,1]$, that is

$$A = \Big\{ f \in \mathcal{C}[0,1] \,\Big|\, f \text{ is differentiable at some point } a \in [0,1] \Big\}.$$

For each $k, \ell \in \mathbb{Z}^+$, let

$$A_{k,\ell} = \Big\{ f \in \mathcal{C}[0,1] \,\Big|\, \exists a \in [0,1]\ \forall x \in [0,1]\ \ 0 < |x-a| < \tfrac{1}{k} \implies \big|\tfrac{f(x)-f(a)}{x-a}\big| \le \ell \Big\}.$$

We shall show that $A = \bigcup\limits_{k,\ell} A_{k,\ell}$, and that each $A_{k,\ell}$ is closed in $\mathcal{C}[0,1]$ with an empty interior and so A is first category. Thus the set of nowhere differentiable functions is residual, and hence dense by the Baire Category Theorem.

We claim that $A = \bigcup\limits_{k,\ell} A_{k,\ell}$. Let $f \in A$. Choose $a \in [0,1]$ such that f is differentiable at a. Choose $\ell \in \mathbb{Z}^+$ such that $\big|f'(a)\big| \le \ell$. Choose $\delta > 0$ such that for all $x \in [0,1]$ we have $0 < |x-a| < \delta \implies \big|\frac{f(x)-f(a)}{x-a} - f'(a)\big| < \ell - |f'(a)|$. Choose $k \in \mathbb{Z}^+$ with $\frac{1}{k} \le \delta$. Then for all $x \in [0,1]$, if $0 < |x-a| < \frac{1}{k}$ then we have $\big|\frac{f(x)-f(a)}{x-a} - f'(a)\big| < \ell - |f'(a)|$ and hence

$$\big|\tfrac{f(x)-f(a)}{x-a}\big| \le \big|\tfrac{f(x)-f(a)}{x-a} - f'(a)\big| + |f'(a)| \le \big(\ell - |f'(a)|\big) + |f'(a)| = \ell$$

so that $f \in A_{k,\ell}$. Thus $A = \bigcup\limits_{k,\ell} A_{k,\ell}$, as claimed.

We claim that each set $A_{k,\ell}$ is closed in $\mathcal{C}[0,1]$. Let $(f_n)_{n\ge1}$ be a sequence in $A_{k,\ell}$ which converges in $\mathcal{C}[0,1]$, and let $g = \lim\limits_{n\to\infty} f_n$ in $\mathcal{C}[0,1]$. Then $f_n \to g$ uniformly in $[0,1]$, and we need to show that $g \in A_{k,\ell}$. For each $n \in \mathbb{Z}^+$, since $f_n \in A_{k,\ell}$ we can choose $a_n \in [0,1]$ such that for all $x \in [0,1]$ we have $0 < |x-a_n| < \frac{1}{k} \implies \big|\frac{f_n(x)-f(a_n)}{x-a_n}\big| \le \ell$. Since $[0,1]$ is compact, we can choose a convergent subsequence $(a_{n_k})_{k\ge1}$ of the sequence $(a_n)_{n\ge1}$ and let $a = \lim\limits_{k\to\infty} a_{n_k} \in [0,1]$. Note that the corresponding subsequence $(f_{n_k})_{k\ge1}$ of $(f_n)_{n\ge1}$ converges in $\mathcal{C}[0,1]$ with the same limit $g = \lim\limits_{k\to\infty} f_{n_k}$ in $\mathcal{C}[0,1]$. Note that when $0 < |x-a| < \frac{1}{k}$, since $a_{n_k} \to a$ it follows that we also have $0 < |x-a_{n_k}| < \frac{1}{k}$ for sufficiently large $k \in \mathbb{Z}^+$. Since $f_{n_k} \to g$ uniformly on $[0,1]$ and $a_{n_k} \to a$ in $[0,1]$, recall (or verify) that $\lim\limits_{k\to\infty} f_{n_k}(a_{n_k}) = g(a)$ and so, for all $x \in [0,1]$ with $0 < |x-a| < \frac{1}{k}$

$$\Big|\frac{g(x)-g(a)}{x-a}\Big| = \lim_{k\to\infty} \Big|\frac{f_{n_k}(x) - f_{n_k}(a_{n_k})}{x - a_{n_k}}\Big| \le \ell.$$

This proves that $g \in A_{k,\ell}$ and so $A_{k,\ell}$ is closed in $\mathcal{C}[0,1]$, as claimed.

We claim that each set $A_{k,\ell}$ has empty interior in $\mathcal{C}[0,1]$. Let $f \in A_{k,\ell}$. We need to show that for all $r > 0$ there there is a function $g \in B(f,r)$ with $g \notin A_{k,\ell}$. Our strategy is to first find a piecewise linear function p with $\|p-f\|_\infty < \frac{r}{2}$ and then to add a rapidly oscillating sine function to obtain a function $g = p + \frac{r}{2}\sin(wx)$ with $g \notin A_{k,\ell}$ and with $\|g-f\|_\infty < r$. Let $r > 0$. Since f is uniformly continuous on $[0,1]$ we can choose $\delta > 0$ such that $|x-y| < \delta \implies \big|f(x)-f(y)\big| < \frac{r}{4}$. we can choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \delta$. Let $x_i = \frac{i}{n}$ for $0 \le i \le n$ and let $p \in \mathcal{C}[0,1]$ be the piecewise linear function whose graph has vertices at $\big(x_i, f(x_i)\big)$ for $0 \le i \le n$. Then for all i and for all $x \in [x_{i-1}, x_i]$, we have

$$\begin{aligned}\big|f(x)-p(x)\big| &\le \big|f(x)-f(x_i)\big| + \big|f(x_i)-p(x)\big| = \big|f(x)-f(x_i)\big| + \big|p(x_i)-p(x)\big| \\ &\le \big|f(x)-f(x_i)\big| + \big|p(x_i)-p(x_{i-1})\big| < \tfrac{r}{4} + \tfrac{r}{4} = \tfrac{r}{2}\end{aligned}$$

and hence $\|f-p\|_\infty < \frac{r}{2}$. Let $m = \max\limits_{t \ne x_i} \big|p'(t)\big| = \max\limits_{1 \le i \le n} n\big|f(x_i) - f(x_{i-1})\big|$. Choose $\omega \in \mathbb{R}$ such that $\frac{2\pi}{\omega} < \frac{1}{k}$ and $\frac{2\pi}{w} < \frac{r}{2(\ell+m)}$, and consider the function $g = p + \frac{r}{2}\sin(\omega x)$. Note that $\|g-f\|_\infty \le \|g-p\|_\infty + \|p-f\|_\infty < \frac{r}{2} + \frac{r}{2} = r$, so it remains only to show that $g \notin A_{k,\ell}$. Let $a \in [0,1]$. By our choice of ω we can choose $x \in [0,1]$ with $0 < |x-a| < \frac{1}{k}$ such that $|x-a| < \frac{r}{2(\ell+m)}$ and such that $\sin(\omega x) = \pm 1$ with $\sin(\omega x) = 1 \iff \sin(\omega a) \le 0$ so that $\big|\sin(\omega x) - \sin(\omega a)\big| \ge 1$. Then we have

$$\begin{aligned}\tfrac{r}{2}\,\big|\sin(\omega x)-\sin(\omega a)\big| &= \big|\big(g(x)-g(a)\big) - \big(p(x)-p(a)\big)\big| \le \big|g(x)-g(a)\big| + \big|p(x)-p(a)\big| \\ \big|g(x)-g(a)\big| &\ge \tfrac{r}{2}\big|\sin(\omega x)-\sin(\omega a)\big| - \big|p(x)-p(a)\big| \ge \tfrac{r}{2} - \big|p(x)-p(a)\big| \\ \left|\tfrac{g(x)-g(a)}{x-a}\right| &\ge \frac{r}{2|x-a|} - \left|\frac{p(x)-p(a)}{x-a}\right| \ge \frac{r}{2\cdot\frac{2(\ell+m)}{r}} - m = \ell\end{aligned}$$

so that $g \notin A_{k,\ell}$, as required.

7.22 Notation: Let X be a set. For any set $\mathcal{C}$ of subsets of X we write

$$\mathcal{C}_\sigma = \left\{ \bigcup_{k=1}^\infty A_k \,\middle|\, \text{each } A_k \in \mathcal{C} \right\} \text{ and } \mathcal{C}_\delta = \left\{ \bigcap_{k=1}^\infty A_k \,\middle|\, \text{each } A_k \in \mathcal{C} \right\}.$$

Note that $\mathcal{C}_{\sigma\sigma} = \mathcal{C}_\sigma$ and $\mathcal{C}_{\delta\delta} = \mathcal{C}_\delta$.

7.23 Definition: Let X be a set. A **σ-algebra** in X is a set $\mathcal{C}$ of subsets of X such that
(1) $\emptyset \in \mathcal{C}$,
(2) if $A \in \mathcal{C}$ then $A^c = X \setminus A \in \mathcal{C}$, and
(3) if $A_1, A_2, A_3, \dots \in \mathcal{C}$ then $\bigcup\limits_{k=1}^\infty A_k \in \mathcal{C}$.

Note that when $\mathcal{C}$ is a σ-algebra in X we have $\mathcal{C}_\sigma = \mathcal{C}$ and $\mathcal{C}_\delta = \mathcal{C}$.

7.24 Notation: In a metric space (or topological space) X, we let $\mathcal{G}$ denote the set of all open sets in X and we let $\mathcal{F}$ denote the set of all closed subsets of X. Note that $\mathcal{G}_\sigma = \mathcal{G}$ and $\mathcal{F}_\delta = \mathcal{F}$.

7.25 Example: For any set X, the set $\{\emptyset, X\}$ and the set $\mathcal{P}(X)$ of all subsets of X are σ-algebras in X,

7.26 Note: Note that given any set $\mathcal{C}$ of subsets of a set X there exists a unique smallest σ-algebra in X which contains $\mathcal{C}$, namely the intersection of all σ-algebras in X which contain $\mathcal{C}$.

7.27 Definition: In a metric space (or topological space) X, the **Borel** σ-algebra $\mathcal{B}$ is the smallest σ-algebra in X which contains $\mathcal{G}$ (hence also $\mathcal{F}$). The elements of $\mathcal{B}$ are called **Borel** sets. Note that $\mathcal{B}$ contains all of the sets $\mathcal{G}, \mathcal{G}_\delta, \mathcal{G}_{\delta\sigma}, \mathcal{G}_{\sigma\delta\sigma}, \cdots$ and all of the sets $\mathcal{F}, \mathcal{F}_\sigma, \mathcal{F}_{\sigma\delta}, \mathcal{F}_{\sigma\delta\sigma}, \cdots$.

7.28 Exercise: Using the Baire Category Theorem, show that in $\mathbb{R}$ we have $\mathcal{F} \subseteq \mathcal{G}_\delta$ (equivalently $\mathcal{G} \subseteq \mathcal{F}_\sigma$), $\mathcal{F}_\sigma \ne \mathcal{G}_\delta$, and $\mathcal{G}_\delta \cup \mathcal{F}_\sigma \subsetneqq \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$.