Chapter 6. Some Applications

Contraction Maps and Picard's Theorem

6.1 Note: In this chapter, unless otherwise stated, we work in the field $\mathbb{F}=\mathbb{R}$.

6.2 Definition: Let X be a metric space. A map $f:X\to X$ is called a **contraction map** on X when there exists a constant $c\in[0,1)$ such that for all $x,y\in X$ we have

$$d\big(f(x),f(y)\big)\leq c\,d(x,y).$$

Such a constant c is called a **contraction constant** for f. Note that every contraction map is uniformly continuous.

6.3 Definition: For a map $f:X\to X$ (where X is any set), a point $a\in X$ such that $f(a)=a$ is called a **fixed point** of f.

6.4 Theorem: *(The Banach Fixed-Point Theorem) Every contraction map on a complete metric space has a unique fixed point.*

Proof: Let X be a complete metric space and let $f:X\to X$ be a contraction map on X with contraction constant $c\in[0,1)$. Let $x_0\in X$ be any point. Let $x_1=f(x_0)$ and $x_2=f(x_1)=f^2(x_0)$ and so on, so that for $n\geq 1$ we have $x_n=f(x_{n-1})=f^n(x_0)$. Note that the sequence $(x_n)_{n\geq 0}$ is Cauchy because for $n<m$ we have

$$\begin{aligned}
d(x_n,x_m)&=d\big(f^n(x_0),f^n(x_{m-n})\big)\leq c^n\,d\big(x_0,x_{m-n}\big)\\
&\leq c^n\big(d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{m-n-1},x_{m-n})\big)\\
&\leq c^n d(x_0,x_1)\big(1+c+c^2+\cdots+c^{m-n-1}\big)\\
&\leq c^n d(x_0,x_1)\,\tfrac{1}{1-c}\ \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Since X is complete, the sequence $(x_n)_{n\geq 0}$ converges, so we can let $a=\lim\limits_{n\to\infty}x_n$. Note that $f(a)=a$ since f is continuous at a so $f(a)=f\big(\lim\limits_{n\to\infty}x_n\big)=\lim\limits_{n\to\infty}f(x_n)=\lim\limits_{n\to\infty}x_{n-1}=a$. Finally note that for $a,b\in X$, if $f(a)=a$ and $f(b)=b$ then since

$$d(a,b)=d\big(f(a),f(b)\big)\leq c\,d(a,b)$$

with $0\leq c<1$, it follows that $d(a,b)=0$ so that $a=b$.

6.5 Example: Define $f:[2,\infty)\to[2,\infty)$ by $f(x)=x+\frac{1}{x}$. Note that $f'(x)=1-\frac{1}{x^2}$ so that $\frac{3}{4}\leq f'(x)<1$ for all $x\in[2,\infty)$. By the Mean Value Theorem, given $x,y\in[2,\infty)$ we can choose c between x and y such that $f(x)-f(y)=f'(c)(x-y)$, and then we have $\big|f(x)-f(y)\big|=\big|f'(c)\big|\,|x-y|<|x-y|$. Thus f has the property that $\big|f(x)-f(y)\big|<|x-y|$ for all $x,y\in[2,\infty)$, but it is not a contraction map, and f has no fixed point because $f(x)=x+\frac{1}{x}>x$ for all $x\in[2,\infty)$.

6.6 Example: Define $f:\left[0,\frac{\pi}{3}\right]\to\left[0,\frac{\pi}{3}\right]$ by $f(x)=\cos x$ $\big($note that $\cos(0)=1$ and $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$ and $\cos x$ is decreasing, so we have $f\big(\left[0,\frac{\pi}{3}\right]\big)=\left[\frac{1}{2},1\right]\subseteq\left[0,\frac{\pi}{3}\right]\big)$. Since $\big|f'(x)\big|=\sin x$ which is increasing on $\left[0,\frac{\pi}{3}\right]$, we have $0\leq\big|f'(x)\big|\leq\frac{\sqrt{3}}{2}$ for all $x\in\left[0,\frac{\pi}{3}\right]$. By the Mean Value Theorem (as above) we have $\big|f(x)-f(y)\big|\leq\frac{\sqrt{3}}{2}|x-y|$ for all $x,y\in\left[0,\frac{\pi}{3}\right]$ so that f is a contraction map with contraction constant $c=\frac{\sqrt{3}}{2}$. By the Banach Fixed-Point Theorem, f has a unique point, that is there is a unique $a\in\left[0,\frac{\pi}{3}\right]$ such that $\cos a=a$. The proof of the theorem shows that we can find a as follows: choose any $x_0\in\left[0,\frac{\pi}{3}\right]$ and let $x_n=f(x_{n-1})=\cos(x_{n-1})$ for $n\geq 1$, and then $x_n\to a$.

6.7 Definition: Let $A \subseteq \mathbb{R}^2$ and let $f : A \to \mathbb{R}$. We say that f satisfies a **Lipschitz condition** on A when there exists a constant $\ell \geq 0$ such that for all $x, y_1, y_2 \in \mathbb{R}$ for which $(x, y_1) \in A$ and $(x, y_2) \in A$, we have

$$\left|f(x, y_2) - f(x, y_1)\right| \leq \ell\, |y_2 - y_1|.$$

Such a constant ℓ is called a **Lipschitz constant** for f.

6.8 Theorem: *(Picard) Let U be an open set in $\mathbb{R}^2$, let $(a, b) \in U$, and let $F : U \to \mathbb{R}$ satisfy a Lipschitz condition on U. Then there exists $\delta > 0$ such that the differential equation $\frac{dy}{dx} = F(x, y)$ has a unique solution $y = f(x)$ with $f(a) = b$, defined for all $x \in [a-\delta, a+\delta]$.*

Proof: We sketch a proof. First note that $y = f(x)$ is a solution to the differential equation $\frac{dy}{dx} = F(x, y)$ with $f(a) = b$ if and only if $f(x)$ satisfies the integral equation

$$f(x) = b + \int_a^x F\big(t, f(t)\big)\, dt$$

for all $x \in [a-\delta, a+\delta]$. Let ℓ be a Lipschitz constant for F. Choose $r > 0$ such that $\overline{B}\big((a, b), r\big) \subseteq U$ and let $k = \max\limits_{(x,y)\in\overline{B}((a,b),r)} \left|F(x, y)\right|$. Choose δ with $0 < \delta < \frac{1}{\ell}$ small enough such that the rectangle $R = [a-\delta, a+\delta] \times [b-k\delta, b+k\delta]$ is contained in $B\big((a, b), r\big)$. Verify as an exercise (Using the Mean Value Theorem) that if $f(x)$ is any solution to the given differential equation with $f(a) = b$ then the graph of f must be contained in the rectangle R. Let

$$X = \big\{f \in \mathcal{C}[a-\delta, a+\delta] \,\big|\, \text{Graph}(f) \subseteq R\big\}.$$

Verify that X is a closed subspace of the metric space $\mathcal{C}[a-\delta, a+\delta]$ (using the supremum metric) and so X is complete. Define $G : X \to \mathcal{C}[a-\delta, a+\delta]$ by

$$G(f)(x) = b + \int_a^x F\big(t, f(t)\big)\, dt.$$

Note that $G(X) \subseteq X$ because for all $f \in X$ and $x \in [a-\delta, a+\delta]$ we have

$$\big|G(f)(x) - b\big| = \left|\int_a^x F(t, f(t))dt\right| \leq \left|\int_a^x k\, dt\right| = k|x - a| \leq k\delta.$$

Note that G is a contraction map on X, with contraction constant $c = \ell\delta < 1$ because, for all $f, g \in X$ and all $x \in [a-\delta, a+\delta]$, we have

$$\begin{aligned}\big|G(f)(x) - G(g)(x)\big| &= \left|\int_a^x \big(F(t, f(t)) - F(t, g(t))\big)dt\right| \leq \left|\int_a^x \big|F(t, f(t)) - F(t, g(t))\big|dt\right| \\ &\leq \left|\int_a^x \ell\big|f(t) - g(t)\big|\, dt\right| \leq \left|\int_a^x \ell\, \|f - g\|_\infty\, dt\right| \\ &= \ell|x - a|\, \|f - g\|_\infty \leq \ell\delta\, ||f - g||_\infty.\end{aligned}$$

By the Banach Fixed-Point Theorem, the map G has a unique fixed point $f \in X$, and this function $f \in X$ is the unique solution to the above integral equation, which is equivalent to the given differential equation.

6.9 Definition: Let X be a set and let $S \subseteq \mathcal{F}(X) = \mathcal{F}(X, \mathbb{R})$. We say that S is **pointwise bounded** when for every $x \in X$ there exists $m = m(x) > 0$ such that $|f(x)| \leq m$ for every function $f \in S$. We say that S is **uniformly bounded** when there exists $m > 0$ such that $|f(x)| \leq m$ for every $x \in X$ and every $f \in S$.

Let X be a metric space and let $S \subseteq \mathcal{C}(X) = \mathcal{C}(X, \mathbb{R})$. We say that S is **equicontinuous** when for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $f \in S$ and for all $x, y \in X$, if $d(x, y) < \delta$ then $d\big(f(x), f(y)\big) < \epsilon$

6.10 Note: When X is a compact metric space, by the Extreme Value Theorem, every continuous function $f : X \to \mathbb{R}$ is also bounded, so we have $\mathcal{C}(X) = \mathcal{C}_b(X)$, which is a complete metric space using the supremum norm. Unless otherwise stated, when we refer to the metric space $\mathcal{C}(X)$ it is understood that we are using the supremum metric.

6.11 Note: When X is a compact metric space and $S \subseteq \mathcal{C}(X)$, note that S is uniformly bounded if and only if S is bounded as a subspace of the metric space $\mathcal{C}(X)$.

6.12 Theorem: *Let X be a compact metric space and let (f_n) be a sequence in $\mathcal{C}(X)$. If the sequence (f_n) converges in the metric space $\mathcal{C}(X)$ (equivalently, if the sequence (f_n) converges uniformly on X) then the set $\{f_n\}$ is equicontinuous.*

Proof: Suppose (f_n) converges in $\mathcal{C}(X)$. Let $\epsilon > 0$. Since (f_n) converges in $\mathcal{C}(X)$ we can choose $\ell \in \mathbb{Z}^+$ such that for all $n, m \geq \ell$ we have $\|f_n - f_m\|_\infty < \frac{\epsilon}{3}$. Since X is compact, each of the functions f_n is uniformly continuous on X. Choose $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $\big|f_n(x) - f_n(y)\big| < \epsilon$ for each $n < \ell$ and we have $\big|f_\ell(x) - f_\ell(y)\big| < \frac{\epsilon}{3}$. Then for all $n \geq \ell$ and all $x, y \in X$ with $d(x, y) < \delta$ we have

$$\big|f_n(x) - f_n(y)\big| \leq \big|f_n(x) - f_\ell(x)\big| + \big|f_\ell(x) - f_\ell(y)\big| + \big|f_\ell(y) - f_n(y)\big| < \epsilon.$$

6.13 Corollary: *Let X be a compact metric space. Then every compact set $S \subseteq \mathcal{C}(X)$ is equicontinuous.*

Proof: Let $S \subseteq \mathcal{C}(X)$. Suppose that S is not equicontinuous. Choose $\epsilon > 0$ such that for all $\delta > 0$ there exists $f \in S$ and there exist $x, y \in X$ with $d(x, y) < \delta$ such that $\big|f(x) - f(y)\big| \geq \epsilon$. For each $n \in \mathbb{Z}^+$, choose $f_n \in S$ such that there exist $x, y \in X$ with $d(x, y) < \frac{1}{2^n}$ such that $\big|f_n(x) - f_n(y)\big| \geq \epsilon$. Then no subsequence of (f_n) can possibly converge in S (using the supremum metric) and so S cannot be compact.

6.14 Theorem: *Let X be a compact metric space and let (f_n) be a sequence in $\mathcal{C}(X)$. If the set $\{f_n\}$ is pointwise bounded and equicontinuous then the set $\{f_n\}$ is uniformly bounded and the sequence (f_n) has a convergent subsequence in $\mathcal{C}(X)$.*

Proof: Suppose that the set $\{f_n\}$ is pointwise bounded and equicontinuous. We claim that the set $\{f_n\}$ is uniformly bounded. Since $\{f_n\}$ is equicontinuous, we can choose $\delta > 0$ such that for all $n \in \mathbb{Z}^+$ and for all $x, y \in X$ with $d(x, y) < \delta$ we have $\big|f_n(x) - f_n(y)\big| < 1$. Since X is compact, we can choose $a_1, a_2, \cdots, a_\ell \in X$ such that $X = B(a_1, \delta) \cup \cdots \cup B(a_\ell, \delta)$. Since $\{f_n\}$ is pointwise bounded, we can choose $m > 0$ such that for each index k with $1 \leq k \leq \ell$, and for all $n \in \mathbb{Z}^+$, we have $\big|f_n(a_k)\big| \leq m$. Let $n \in \mathbb{Z}^+$ and $x \in X$. Choose an index k with $1 \leq k \leq \ell$ such that $x \in B(a_k, \delta)$. Since $d(x, a_k) < \delta$ we have $\big|f_n(x) - f_n(a_k)\big| < 1$ and so $\big|f_n(x)\big| \leq \big|f_n(x) - f_n(a_k)\big| + \big|f_n(a_k)\big| < 1 + m$. Since $n \in \mathbb{Z}^+$ and $x \in X$ were arbitrary, the set $\{f_n\}$ is uniformly bounded, as claimed.

It remains to show that the sequence (f_n) has a convergent subsequence in $\mathcal{C}(X)$. Since X is compact, and hence separable, we can choose a countable dense subset $A \subseteq X$, say $A = \{a_1, a_2, a_3, \cdots\}$. We claim that the sequence $(f_n)_{n\geq 1}$ has a subsequence $(f_{n_k})_{k\geq 1}$ which converges pointwise on A. Since the real-valued sequence $\big(f_n(a_1)\big)_{n\geq 1}$ is bounded, we can choose a subsequence, which we shall write as $\big(f_{1,k}\big)_{k\geq 1} = \big(f_{1,1}, f_{1,2}, f_{1,3}, \cdots\big)$, of the sequence of functions $(f_n)_{n\geq 1}$ such that the real-valued sequence $\big(f_{1,k}(a_1)\big)_{k\geq 1}$ converges. Since the real-valued sequence $\big(f_{1,k}(a_2)\big)_{k\geq 1}$ is bounded, we can choose a subsequence $\big(f_{2,k}\big)$ of the sequence of functions $\big(f_{1,k}\big)$ such that the real-valued sequence $\big(f_{2,k}(a_2)\big)$ converges. Note that since $\big(f_{2,k}(a_1)\big)$ is a subsequence of the convergent sequence $\big(f_{1,k}(a_1)\big)$, it also converges. By recursively repeating this procedure, we construct sequences $(f_{n,k})_{k\geq 1}$ for each $n \geq 1$, such that $\big(f_{n+1,k}\big)_{k\geq 1}$ is a subsequence of $\big(f_{n,k}\big)_{k\geq 1}$ and the real-valued sequences $\big(f_{n,k}(a_j)\big)_{k\geq 1}$ converge for all j with $1 \leq j \leq n$. Let $\big(f_{n_k}\big)_{k\geq 1}$ denote the sequence $\big(f_{1,1}, f_{2,2,}, f_{3,3,}, \cdots\big)$, note that this is a subsequence of the original sequence (f_n), and the real-valued sequences $\big(f_{n_k}(a_j)\big)_{k\geq 1}$ converge for all indices $j \in \mathbb{Z}^+$, so the subsequence (f_{n_k}) converges pointwise on A, as required.

Finally, we claim that the above subsequence (f_{n_k}) converges in $\mathcal{C}(X)$. Let $\epsilon > 0$. Since the set $\{f_n\}$ is equicontinuous we can choose $\delta > 0$ such that for all $n \in \mathbb{Z}^+$ and all $x, y \in X$ with $d(x,y) < \delta$ we have $\big|f_n(x) - f_n(y)\big| < \frac{\epsilon}{3}$. Since A is dense in X, the set $\mathcal{U} = \big\{B(a_n,\delta)\big|n \in]Z^+\big\}$ is an open cover of X. Since X is compact, we can choose a finite subcover of $\mathcal{U}$, so we can choose $a_1, a_2, \cdots, a_p \in X$ such that $X = B(a_1,\delta) \cup \cdots \cup B(a_p,\delta)$. Since the sequences $\big(f_{n_k}(a_j)\big)_{k\geq 1}$ all converge, we can choose $m \in \mathbb{Z}^+$ such that for all $j \in \mathbb{Z}^+$ with $1 \leq j \leq p$ and all $k, \ell \in \mathbb{Z}^+$ with $k, \ell \geq m$ we have $\big|f_{n_k}(a_j) - f_{n_\ell}(a_j)\big| < \frac{\epsilon}{3}$. Let $x \in X$ and let $k, \ell \in \mathbb{Z}^+$ with $k, \ell \geq m$. Choose an index j with $1 \leq j \leq p$ such that $x \in B(a_j, \delta)$. Then we have

$$\big|f_{n_k}(x) - f_{n_\ell}(x)\big| \leq \big|f_{n_k}(x) - f_{n_k}(a_j)\big| + \big|f_{n_k}(a_j) - f_{n_\ell}(a_j)\big| + \big|f_{n_\ell}(a_j) - f_{n_\ell}(x)\big| < \epsilon.$$

6.15 Theorem: *(The Arzela-Ascoli Theorem) Let X be a compact metric space and let $S \subseteq \mathcal{C}(X)$, using the supremum metric.*

(1) S is compact if and only if S is closed, pointwise bounded, and equicontinuous.
(2) If S is pointwise bounded and equicontinuous, then $\overline{S}$ is compact.

Proof: To prove Part 1, suppose that S is compact. Then we know that S is closed and bounded and we know (from Corollary 6.13) that S is equicontinuous. Since S is bounded, using the supremum metric, it follows that S is uniformly bounded, hence also pointwise bounded.

Suppose, conversely, that S is closed, pointwise bounded, and equicontinuous. Let (f_n) be a sequence in S. Since S is pointwise bounded and equicontinuous, the subset $\{f_n\}$ is also pointwise bounded and equicontinuous. By the above theorem, the sequence (f_n) has a convergent subsequence (f_{n_k}) in $\mathcal{C}(X)$. Since S is closed, the limit of this subsequence lies in S. This proves that every sequence in S has a subsequence which converges in S, and so S is compact.

This completes the proof of Part 1, and we leave the proof of Part 2 as an exercise.

6.16 Theorem: *(Peano) Let $U \subseteq \mathbb{R}^2$ be open, let $(a,b) \in U$, and let $F : U \to \mathbb{R}$ be continuous. Then there exists $d > 0$ such that the differential equation $\frac{dy}{dx} = F(x,y)$ has a solution $y = f(x)$ with $f(a) = b$ which is defined for all $x \in [a-d, a+d]$.*

Proof: Choose a closed rectangle Q with $(a,b) \in Q \subseteq U$. Since Q is compact, $\left|F(x,y)\right|$ attains its maximum value on Q, let $M = \max\left\{|F(x,y)| \,\middle|\, (x,y) \in Q\right\}$. Choose $d > 0$ so that $R = [a-d, a+d] \times [b-Md, b+Md] \subseteq Q$.

Fix $n \in \mathbf{Z}^+$. Since R is compact so that F is uniformly continuous on R, we can choose $\delta > 0$ so that for all $(x_1,y_1), (x_2,y_2) \in R$,

$$\left|(x_1,y_1) - (x_2,y_2)\right| < \delta \Longrightarrow \left|F(x_1,y_1) - F(x_2,y_2)\right| \leq \tfrac{1}{n}.$$

Choose $\ell \in \mathbb{Z}^+$ so $\frac{d}{\ell} < \frac{\delta}{M+1}$ and let $c_k = a + \frac{kd}{\ell}$ for $0 \leq k < \ell$ so $a = c_0 < c_1 < \cdots < c_\ell = a+d$ with $c_{k+1} - c_k = \frac{d}{\ell} < \frac{\delta}{M+1}$ for all $0 \leq k < \ell$. Let $f = f_n : [a, a+d] \to \mathbb{R}$ be the continuous, piecewise linear function with $f(a) = b$ such that $f'(x) = F\big(c_k, f(c_k)\big)$ for all $x \in (c_k, c_{k+1})$ $\big($the function $f = f_n$ is constructed recursively by beginning with $f(a) = b$ and then, having defined $f(x)$ for all $x \in [a, c_k]$, define $f(x) = f(c_k) + F\big(c_k, f(c_k)\big)(x - c_k)$ for all $x \in [c_k, c_{k+1}]\,\big)$.

Claim 1: we claim that for all $x_1, x_2 \in [a, a+d]$ we have

$$\left|f(x_1) - f(x_2)\right| \leq M|x_1 - x_2|.$$

Let $x_1, x_2 \in [a, a+d]$ with $x_1 \leq x_2$. For $0 \leq k < p$, let $m_k = F(c_k, f(c_k))$ and note that $|m_k| = \left|F(c_k, f(c_k))\right| \leq M$ for all k. When $x_1, x_2 \in [c_k, c_{k+1}]$ with $x_1 \leq x_2$, we have $f(x_2) = f(x_1) + m_k(x_2 - c_k)$ so that $\left|f(x_2) - f(x_1)\right| = \left|m_k(x_2 - c_k)\right| \leq M(x_2 - x_1)$, and when $x_1 \in [c_j, c_{j+1}]$ and $x_2 \in [c_k, c_{k+1}]$ with $j < k$ we have

$$f(x_2) = f(x_1) + m_j(c_{j+1} - x_1) + m_{j+1}(c_{j+2} - c_{j+1}) + \cdots + m_{k-1}(c_k - c_{k-1}) + m_k(x_2 - c_k)$$

so $\left|f(x_2) - f(x_1)\right| \leq M(c_{j+1} - x_1) + M(c_{j+2} - c_{j+1}) + \cdots + M(c_k - c_{k-1}) + M(x_2 - c_k)$, that is $\left|f(x_2) - f(x_1)\right| \leq M(x_2 - x_1)$, as required.

Claim 2: we claim that when $x \in [c_k, c_{k+1}]$ we have

$$\int_{c_k}^{x} \left|m_k - F\big(t, f(t)\big)\right| dt \leq \tfrac{d}{n}.$$

Let $x \in [c_k, c_{k+1}]$ and let $t \in [c_k, x]$. Then $|c_k - t| \leq c_{k+1} - c_k = \frac{d}{\ell} < \frac{\delta}{M+1}$. By Claim 1, we have $\left|f(c_k) - f(t)\right| < \frac{M\delta}{M+1}$, so

$$\left|(c_k, f(c_k)) - (t, f(t))\right| \leq |c_k - t| + \left|f(c_k) - f(t)\right| < \tfrac{\delta}{M+1} + \tfrac{M\delta}{M+1} = \delta.$$

By the choice of δ, we have $\left|m_k - F\big(t, f(t)\big)\right| = \left|F\big(c_k, f(c_k)\big) - F\big(t, f(t)\big)\right| \leq \frac{1}{n}$. This holds for all $t \in [c_k, x]$, so

$$\int_{c_k}^{x} \left|m_k - F\big(t, f(t)\big)\right| dt \leq \int_{c_k}^{x} \tfrac{1}{n}\, dt = \tfrac{1}{n}\,(x - c_k) \leq \tfrac{d}{n}$$

as claimed.

Claim 3. we claim that for all $x \in [a, a+d]$ we have

$$\left|\big(f(x) - f(a)\big) - \int_a^x F\big(t, f(t)\big)\, dt\right| \leq \tfrac{d}{n}.$$

Let $x \in [a, a+d]$. Choose an index k so that $x \in [c_k, c_{k+1}]$. Then

$$f(x) - f(a) = \sum_{j=0}^{k-1} m_j(c_{j+1} - c_j) + m_k(x - c_k) = \sum_{j=0}^{k-1} \int_{c_j}^{c_{j+1}} m_j\, dt + \int_{c_k}^{x} m_k\, dt$$

and so,
$$\begin{aligned}
&\left|\big(f(x)-f(a)\big) - \int_a^x F\big(t,f(t)\big)\,dt\right| \\
&\quad = \left|\sum_{j=0}^{k-1}\int_{c_j}^{c_{j+1}}\Big(m_j - F\big(t,f(t)\big)\Big)\,dt + \int_{c_k}^{x}\Big(m_k - F\big(t,f(t)\big)\Big)\,dt\right| \\
&\quad \le \sum_{j=0}^{k-1}\int_{c_j}^{c_{j+1}}\Big|m_j - F\big(t,f(t)\big)\Big|\,dt + \int_{c_k}^{x}\Big|m_k - F\big(F\big(t,f(t)\big)\Big|\,dt \\
&\quad \le \sum_{j=0}^{k-1}\tfrac{1}{n}\,(c_{j+1}-c_j) + \tfrac{1}{n}\,(x-c_k) = \tfrac{1}{n}\,(x-c_0) = \tfrac{1}{n}\,(x-a) \le \tfrac{d}{n}\,,
\end{aligned}$$
(where we used Claim 2 on the final line above), as claimed.

We repeat the above construction for every $n \in \mathbb{Z}^+$ to obtain a sequence of functions $(f_n)_{n\ge1}$. Each function f_n satisfies Claims 1, 2 and 3. Let $S = \big\{f_n \,\big|\, n\in\mathbb{Z}^+\big\} \subseteq \mathcal{C}[a,a{+}d]$. Note that by Claim 1, the set S is equicontinuous $\big($indeed given $\epsilon>0$, if $|x_1-x_2| < \frac{\epsilon}{M}$ then $\big|f_n(x_1)-f_n(x_2)\big| \le M|x_1-x_2| < \epsilon\big)$ and the set S uniformly bounded $\big($indeed since $f_n(a)=b$ and $\big|f_n(x)-f_n(a)\big| \le M|x-a| \le Md$ we have $b-Md \le f_n(x) \le b+Md$ for all $x\big)$. By the Arzela-Ascoli Theorem, the closure $\overline{S}$ of S in $\big(\mathcal{C}[a,a{+}d], d_\infty\big)$ is compact. Thus we can choose a subsequence $(f_{n_k})_{k\ge1}$ of $(f_n)_{n\ge1}$ which converges in $\overline{S} \subseteq \mathcal{C}[a,a{+}d]$ using the metric d_∞, that is $(f_{n_k})_{k\ge1}$ converges uniformly on $[a,a{+}d]$ to some continuous function $g:[a,a{+}d]\to\mathbb{R}$.

Claim 4: we claim that the above map $g:[a,a{+}d]\to\mathbb{R}$ is a solution to the given differential equation. First we note that when $\|f-g\|_\infty < \delta$, for all $t\in[a,a{+}d]$ we have $\big|(t,f(t))-(t,g(t))\big| = \big|f(t)-g(t)\big| \le \|f-g\|_\infty < \delta$ so that (by the choice of δ) we have $\big|F(t,f(t))-F(t,g(t))\big| \le \frac{1}{n}$ and hence
$$\int_a^x \Big|F\big(t,f(t)\big) - F\big(t,g(t)\big)\Big|\,dt \le \int_a^x \tfrac{1}{n}\,dt = \tfrac{1}{n}\,(x-a) \le \tfrac{d}{n}.$$
Given $\epsilon>0$ we can choose $k\in\mathbb{Z}^+$ such that $\|f_{n_k}-g\|_\infty < \delta$ and $\|f_{n_k}-g\|_\infty < \frac{\epsilon}{3}$ and $\frac{1}{n_k} < \frac{\epsilon}{3d}$. Write $n=n_k$ and $f=f_n=f_{n_k}$. Then for all $x\in[a,a{+}d]$ we have
$$\begin{aligned}
&\left|\big(g(x)-g(a)\big) - \int_a^x F\big(t,g(t)\big)\,dt\right| \le \Big|\big(g(x)-g(a)\big) - \big(f(x)-f(a)\big)\Big| \\
&\qquad + \left|\big(f(x)-f(a)\big) - \int_a^x F\big(t,f(t)\big)\,dt\right| + \left|\int_a^x F\big(t,f(t)\big)\,dt - \int_a^x F\big(t,g(t)\big)\,dt\right| \\
&\quad \le \big|g(x)-f(x)\big| + \left|\big(f(x)-f(a)\big) - \int_a^x F\big(t,f(t)\big)\,dt\right| + \int_a^x \Big|F\big(t,f(t)\big) - F\big(t,g(t)\big)\Big|\,dt \\
&\quad \le \|f-g\|_\infty + \tfrac{d}{n} + \tfrac{d}{n} < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.
\end{aligned}$$
Since $\epsilon>0$ was arbitrary, it follows that for all $x\in[a,a{+}d]$
$$g(x) = g(a) + \int_a^x F\big(t,g(t)\big)\,dt\,.$$
By the Fundamental Theorem of Calculus, g is differentiable wih $g'(x) = F\big(x,g(x)\big)$ for all $x\in[a,a{+}d]$, and so g is a solution of the given differential equation, as claimed.

Finally, we repeat the above procedure to obtain a solution $g:[a{-}\,d,a]\to\mathbb{R}$ then join the two solutions to obtain a solution $g:[a{-}d,a{+}d]\to\mathbb{R}$.

6.17 Definition: A (commutative) **algebra** over a field F is a vector space U with a binary multiplication operation such that for all $u, v, w \in U$ and all $t \in F$ we have $uv = vu$, $u(v+w) = uv + uw$, and $(tu)v = t(uv)$. A subspace $A \subseteq U$ is a **subalgebra** of U when it is an algebra using (the restriction of) the same operations used in U. Verify that a subset $A \subseteq U$ is a subalgebra of U when $0 \in A$ and for all $u, v \in A$ and all $t \in F$ we have $tu \in A$, $u + v \in A$ and $uv \in A$.

6.18 Example: When X is a metric space, the vector space $\mathcal{F}(X) = \mathcal{F}(X, \mathbb{R})$ of all functions $f: X \to \mathbb{R}$ is an algebra over $\mathbb{R}$, and $\mathcal{B}(X)$, $\mathcal{C}(X)$, and $\mathcal{C}_b(X)$ are all subalgebras.

6.19 Example: When $a \leq b$, the space $\mathcal{P}[a,b]$ of polynomial maps $f : [a,b] \to \mathbb{R}$ and the space $\mathcal{C}^1[a,b]$ of continuously differentiable maps are subalgebras of the algebra $\mathcal{C}[a,b]$ of continuous maps $f : [a,b] \to \mathbb{R}$, and the space $\mathcal{R}[a,b]$ of Riemann integrable functions is a subalgebra of the algebra $\mathcal{B}[a,b]$ of bounded functions $f : [a,b] \to \mathbb{R}$.

6.20 Example: Show that $f(x) = |x|$ lies in the closure of $\mathcal{P}[-1,1]$ in $\big(\mathcal{C}[-1,1], d_\infty\big)$.

Solution: Let $\epsilon > 0$ and let $a = \frac{\epsilon}{2}$. Let $g(x) = \sqrt{x + a^2}$ and let $p_n(x)$ be the n^{th} Taylor polynomial for $g(x)$ centred at 1: to be explicit, for $\left|\frac{x-1}{1+a^2}\right| < 1$ we have

$$g(x) = \big((x-1) + (1+a^2)\big)^{1/2} = \sqrt{1+a^2}\,\left(1 + \tfrac{x-1}{1+a^2}\right)^{1/2} = \sqrt{1+a^2}\,\sum_{k=1}^{\infty} \binom{1/2}{k}\left(\tfrac{x-1}{1+a^2}\right)^k,$$

and we have

$$p_n(x) = \sqrt{1+a^2}\,\sum_{k=0}^{n} \binom{1/2}{k}\left(\tfrac{x-1}{1+a^2}\right)^k.$$

Note that $p_n \to g$ pointwise for $\left|\frac{x-1}{1+a^2}\right| < 1$, that is for all $x \in (-a^2, 2+a^2)$, and $f_n \to g$ uniformly on $[0,2]$ (hence also on $[0,1]$). Choose $n \in \mathbb{Z}^+$ such that $\big|p_n(x) - g(x)\big| < a = \frac{\epsilon}{2}$ for all $x \in [0,1]$. Also note that for all $x \in \mathbb{R}$ we have

$$\Big||x| - g(x^2)\Big| = \sqrt{x^2+a^2} - \sqrt{x^2} = \frac{a^2}{\sqrt{x^2+a^2} + \sqrt{x^2}} \leq a = \tfrac{\epsilon}{2},$$

so for all $x \in [-1,1]$, we have $x^2 \in [0,1]$, hence

$$\Big||x| - p_n(x^2)\Big| \leq \Big||x| - g(x^2)\Big| + \Big|g(x^2) - p_n(x^2)\Big| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

6.21 Definition: Let $A \subseteq \mathcal{C}(X)$. We say that A **separates points** when for all $x, y \in X$ with $x \neq y$ there exist $f \in A$ with $f(x) \neq f(y)$. We say that A **vanishes nowhere** when for all $x \in X$ there exists $f \in A$ such that $f(x) \neq 0$. Note that if $1 \in A$ (where 1 denotes the constant function) the A vanishes nowhere.

6.22 Theorem: *(The Stone-Weierstrass Theorem for Real Valued Functions) Let X be a compact metric space and let $A \subseteq \mathcal{C}(X) = \mathcal{C}(X, \mathbb{R})$ be a subalgebra. If A separates points and vanishes nowhere then $\overline{A} = \mathcal{C}(X)$ (using the supremum metric d_∞).*

Proof: Note first that $\overline{A}$ is also a subalgebra of $\mathcal{C}(X)$. Indeed given $f, g \in \overline{A}$ and $c \in \mathbb{R}$, we can choose sequences (f_n) and (g_n) in A such that $f_n \to f$ and $g_n \to g$ in $\mathcal{C}(X)$ (that is $f_n \to f$ and $g_n \to g$ uniformly on X), and then we have $cf_n \to cf$, $f_n + g_n \to f + g$ and $f_n g_n \to fg$ uniformly on X, and hence $cf \in \overline{A}$, $f + g \in \overline{A}$ and $fg \in \overline{A}$. Also note that $\overline{A}$ separates points and vanishes nowhere, and so we may assume, without loss of generality, that A is closed.

Next we claim that if $f \in A$ then we also have $|f| \in A$. Let $f \in A \subseteq \mathcal{C}(X)$. Choose $m > 0$ with $m \geq \|f\|_\infty$. Let $g = \frac{1}{m} f$ and note that $g \in A$ with $\|g\|_\infty \leq 1$, that is $g(x) \in [-1, 1]$ for all $x \in X$. Let $\epsilon > 0$. By Example 6.20, we can choose a polynomial $p_0(x) = a_0 + a_1 x + \cdots + a_n x^n$ such that $\big|p_0(u) - |u|\big| \leq \frac{\epsilon}{2}$ for all $u \in [-1, 1]$. Let $p(x) = p_0(x) - a_0$ and note that $\big|p(u) - |u|\big| \leq \epsilon$ for all $u \in [-1, 1]$. For all $x \in X$, we have $g(x) \in [-1, 1]$ and so $\big|p\big(g(x)\big) - |g(x)|\big| < \epsilon$. Note that the function $h(x) = p\big(g(x)\big) = a_1 g(x) + a_2 g(x)^2 + \cdots a_n g(x)^n$ lies in A (because $g \in A$ and A is an algebra). This shows that for every $\epsilon > 0$ we can find $h \in A$ with $\big|h - |g|\big| < \epsilon$, and (since A is closed) it follows that $|g| \in A$ and hence $|f| = m|g| \in A$.

Next we note that if $f, g \in A$ then we also have $\max\{f, g\} \in A$ and $\min\{f, g\} \in A$ because

$$\max\{f, g\} = \frac{f + g}{2} + \frac{|f - g|}{2} \quad \text{and} \quad \min\{f, g\} = \frac{f + g}{2} - \frac{|f - g|}{2}$$

and it follows, inductively, that if $f_1, f_2, \cdots, f_n \in A$ then we have $\max\{f_1, \cdots, f_n\} \in A$ and $\min\{f_1, \cdots, f_n\} \in A$.

We claim that for all $r, s \in \mathbb{R}$ and for all $a, b \in X$ with $a \neq b$, there is a function $g \in A$ with $g(a) = r$ and $g(b) = s$. Let $r, s \in \mathbb{R}$ and let $a, b \in X$ with $a \neq b$. Since A separates points, we can choose $h \in A$ with $h(a) \neq h(b)$. Since A vanishes nowhere, we can choose $k, \ell \in A$ with $k(a) \neq 0$ and $\ell(b) \neq 0$. Define $u, v \in A$ by

$$u(x) = \big(h(x) - h(b)\big)k(x) \quad \text{and} \quad v(x) = \big(h(a) - h(x)\big)\ell(x)$$

and note that $u(a) \neq 0$ and $u(b) = 0$ while $v(a) = 0$ and $v(b) \neq 0$. Then define $g \in A$ by

$$g(x) = r\, \frac{u(x)}{u(a)} + s\, \frac{v(x)}{v(b)}$$

to obtain $g(a) = r$ and $g(b) = s$, as required.

We claim that for every $f \in \mathcal{C}(X)$, for every $a \in X$ and for every $\epsilon > 0$, there is a function $h \in A$ such that $h(a) = f(a)$ and $h(x) < f(x) + \epsilon$ for all $x \in X$. Let $f \in \mathcal{C}(X)$, let $a \in X$ and let $\epsilon > 0$. For each $b \in X$, by the previous claim we can choose $g_b \in A$ such that $g_b(a) = f(a)$ and $g_b(b) = f(b)$. For each $b \in X$, since f and g_b are continuous at b, we can choose $r_b > 0$ such that for all $x \in B(b, r_b)$ we have

$$\big|f(x) - f(b)\big| < \tfrac{\epsilon}{2} \quad \text{and} \quad \big|g_b(x) - g_b(b)\big| < \tfrac{\epsilon}{2}\ , \text{ hence } \big|g_b(x) - f(x)\big| < \epsilon.$$

Since X is compact and the set $\big\{B(b, r_b)\,\big|\,b \in X\big\}$ covers X, we can choose $b_1, b_2, \cdots, b_n \in X$ such that $X = \bigcup_{k=1}^{n} B(b_k, r_{b_k})$, and then we let

$$h = \min\big\{g_{b_1}, g_{b_2}, \cdots, g_{b_n}\big\} \in A.$$

For all $x \in X$ we can choose an index k such that $x \in B(b_k, r_{a_k})$ and then we have $h(x) \leq g_{b_k}(x) < f(x) + \epsilon$, as required.

Finally, we complete the proof by showing that for every $f \in \mathcal{C}[0,1]$ and every $\epsilon > 0$ there exists $g \in A$ such that $|g(x) - f(x)| < \epsilon$ for all $x \in X$. Let $f \in \mathcal{C}(X)$ and let $\epsilon > 0$. For each $a \in X$, by the previous claim we can choose $h_a \in A$ such that $h_a(a) = f(a)$ and $h_a(x) < f(x) + \epsilon$ for all $x \in X$. For each $a \in X$, since f and h_a are continuous at a, we can choose $s_a > 0$ such that for all $x \in B(a, s_a)$ we have

$$\big|f(x) - f(a)\big| < \tfrac{\epsilon}{2} \ \text{ and } \ \big|h_a(x) - h_a(a)\big| < \tfrac{\epsilon}{2} \ \text{ hence } \ \big|h_a(x) - f(x)\big| < \epsilon.$$

Since X is compact and $\big\{B(a_k, s_k)\,\big|\,a \in X\big\}$ covers X, we can choose $a_1, a_2, \cdots, a_m \in X$ such that $X = \bigcup_{k=1}^{m} B(a_k, s_{a_k})$, and then we choose

$$g = \max\big\{h_{a_1}, h_{a_2}, \cdots, h_{a_m}\big\} \in A.$$

For all $x \in X$ we can choose an index k such that $x \in B(a_k, s_{a_k})$ and we can choose an index ℓ such that $g(x) = h_{a_\ell}(x)$ and then we have

$$g(x) \geq h_{a_k}(x) > f(x) - \epsilon \ \text{ and } \ g(x) = h_{a_\ell}(x) < f(x) + \epsilon.$$

6.23 Corollary: *(The Weierstrass Approximation Theorem for Real Valued Functions) Let $X \subseteq \mathbb{R}^n$ be compact and let $f \in \mathcal{C}(X) = \mathcal{C}(X, \mathbb{R})$. Then for all $\epsilon > 0$ there exists a real polynomial p in n variables such that $\big|p(x) - f(x)\big| < \epsilon$ for all $x \in X$.*

Proof: Each polynomial p in n-variables determines a continuous function $p : X \to \mathbb{R}$. The set $\mathcal{P}(X)$ of such polynomial functions is a subalgebra of $\mathcal{C}(X)$ which separates points and vanishes nowhere, so $\mathcal{P}(X)$ is dense in $\mathcal{C}(X)$, using the metric d_∞. This means that given $f \in \mathcal{C}(X)$, for all $\epsilon > 0$ we can choose $p \in \mathcal{P}(X)$ such that $\|p - f\|_\infty < \epsilon$, and hence $\big|p(x) - f(x)\big| < \epsilon$ for all $x \in X$.

6.24 Corollary: *The space $\big(\mathcal{C}([a,b], \mathbb{R}), d_\infty\big)$ is separable, where $a, b \in \mathbb{R}$ with $a < b$.*

Proof: Let P be the set of polynomials with coefficients in $\mathbb{Q}$. Note that P is countable by Theorem 1.20 $\big($indeed, $\mathbb{Q}$ is countable by Part 4 of Theorem 1.20, hence $\mathbb{Q}^2, \mathbb{Q}^3, \cdots, \mathbb{Q}^n$ are all countable by Part 1 of Theorem 1.20 and by induction, hence the space P_n of polynomials over $\mathbb{Q}$ of degree at most n is countable since the map $F : \mathbb{Q}^{n+1} \to P_n$ given by $F(a_0, a_1, \cdots, a_{n+1}) = \sum_{k=0}^{n} a_k x^k$ is bijective, and hence $P = \bigcup_{n=0}^{\infty} P_n$ is countable by Part 3 of Theorem 1.20$\big)$. We claim that P is dense in $\mathcal{C}[a,b]$. Let $f \in \mathcal{C}[a,b]$ and let $\epsilon > 0$. By the Weierstrass Approximation Theorem we can choose a polynomial p with coefficients in $\mathbb{R}$ such that $\|p - f\|_\infty < \frac{\epsilon}{2}$, say $p(x) = \sum_{k=0}^{n} c_k x^k$ with each $c_k \in \mathbb{R}$. Let $m = \max\{|a|, |b|, 1\}$, for each index k, choose $a_k \in \mathbb{Q}$ with $|a_k - c_k| < \frac{\epsilon}{2(n+1)m^n}$ and let $g(x) = \sum_{k=0}^{n} a_k x^k$. Then for all $x \in [a,b]$ we have $|x| \leq m$ $\big($since $m \geq \max\{|a|, |b|\}\big)$ and hence for all $0 \leq k \leq n$ we have $|x|^k \leq m^k \leq m^n$ (since $m \geq 1$). Thus for all $x \in [a,b]$ we have

$$\big|g(x) - p(x)\big| = \Big|\sum_{k=0}^{n} (a_k - c_k) x^k\Big| \leq \sum_{k=0}^{n} |a_k - c_k|\,|x|^k \leq \sum_{k=0}^{n} \tfrac{\epsilon}{2(n+1)m^n}\, m^n = \tfrac{\epsilon}{2}.$$

Thus $\|g - p\|_\infty \leq \frac{\epsilon}{2}$ and hence $\|g - f\|_\infty \leq \|g - p\|_\infty + \|p - f\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

6.25 Exercise: Let $A = \big\{\sum_{k=1}^{n} f_k(x) g_k(y)\,\big|\,n \in \mathbb{Z}^+, f_k, g_k \in \mathcal{C}[0,1]\big\}$. Show that A is dense in $\mathcal{C}\big([0,1] \times [0,1]\big)$, using the metric d_∞.

6.26 Exercise: Let $A = \big\{b_0 + \sum_{k=1}^{n} (a_k \sin(kx) + b_k \cos(kx))\,\big|\,n \in \mathbb{Z}^+, a_k, b_k \in \mathbb{R}\big\}$. Show that for all $r \in [0, 2\pi]$, A is dense in $\mathcal{C}[0,r]$ but A is not dense in $\mathcal{C}[0,2\pi]$, using d_∞.

6.27 Theorem: *(The Stone-Weierstrass Theorem for Complex Valued Functions) Let X be a compact metric space and let $A \subseteq \mathcal{C}(X,\mathbb{C})$ be a complex subalgebra. If A separates points, vanishes nowhere, and is closed under conjugation (which means that if $p \in A$ then $\overline{p} \in A$), then $\overline{A} = \mathcal{C}(X,\mathbb{C})$, using the supremum metric d_∞.*

Proof: Let $A \subseteq \mathcal{C}(X,\mathbb{C})$ be a complex subalgebra. Suppose that A separates points, vanishes nowhere, and is closed under conjugation. Let $B = A \cap \mathcal{C}(X,\mathbb{R})$. Note that B is a real subalgebra of $\mathcal{C}(X,\mathbb{R})$. Note that given $p \in A$ with $p = u + iv$ where $u, v \in \mathcal{C}(X,\mathbb{R})$, since $u = \frac{1}{2}(p + \overline{p})$ and $v = \frac{1}{2i}(p - \overline{p})$, it follows that $u, v \in B$ because A is an algebra and A is closed under conjugation. We claim that B separates points and vanishes nowhere. To show that B separates points, let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since A separates points, we can choose $p \in A$ such that $p(x_1) \neq p(x_2)$. Write $p = u + iv$ with $u, v \in \mathcal{C}(X,\mathbb{R})$. As shown above, we have $u, v \in B$. Since $u(x_1) + iv(x_1) = p(x_1) \neq p(x_2) = u(x_2) + iv(x_2)$, it follows that either $u(x_1) \neq u(x_2)$ or $v(x_1) \neq v(x_2)$, and so B separates points, as claimed. To show that B vanishes nowhere, let $x \in X$. Since A vanishes nowhere we can choose $p \in A$ such that $p(x) \neq 0$. Write $p = u + iv$ with $u, v \in \mathcal{C}(X,\mathbb{R})$, and note that $u, v \in B$. Since $0 \neq p(x) = u(x) + iv(x)$, either we have $u(x) \neq 0$ or we have $v(x) \neq 0$, and so B vanishes nowhere, as claimed. Since B is a real subalgebra of $\mathcal{C}(X,\mathbb{R})$ which separates points and vanishes nowhere, the Stone-Weierstrass Theorem for Real Functions implies that B is dense in $\big(\mathcal{C}(X,\mathbb{R}), d_\infty\big)$. It follows easily that A is dense in $\big(\mathcal{C}(X,\mathbb{C}), d_\infty\big)$: indeed given $h \in \mathcal{C}(X,\mathbb{C})$, say $h = f + ig$ with $f, g \in \mathcal{C}(X,\mathbb{R})$, and given $\epsilon > 0$, we can choose $u, v \in B$ such that $\|u - f\|_\infty < \frac{\epsilon}{2}$ and $\|v - g\|_\infty < \frac{\epsilon}{2}$, and then $p = u + iv \in A$ with $\|p - h\|_\infty = \big\|(u - f) + i(v - g)\big\|_\infty \leq \|u - f\|_\infty + \|i(v - g)\|_\infty = \|u - f\|_\infty + \|v - g\|_\infty < \epsilon$.

6.28 Corollary: *(Weierstrass Approximation Theorem for Complex Valued Functions) Let $X \subseteq \mathbb{C}^n$ be compact and let $f \in \mathcal{C}(X,\mathbb{C})$. Then for all $\epsilon > 0$ there exists a complex polynomial p in the $2n$ variables $z_1, \overline{z}_1, \cdots, z_n, \overline{z}_n$ such that $\big|p(x) - f(x)\big| < \epsilon$ for all $x \in X$.*

Proof: The proof is left as an exercise.

6.29 Corollary: *The space $\big(\mathcal{C}([a,b],\mathbb{C}), d_\infty\big)$ is separable, where $a, b \in \mathbb{R}$ with $a < b$.*

Proof: The proof is left as an exercise.