# Chapter 5. Connectedness and Compactness

## Connectedness

**5.1 Definition:** Let $X$ be a metric space and let $A \subseteq X$. For sets $U, V \subseteq X$, we say that $U$ and $V$ **separate** $A$ (in $X$) when

$$U \cap A \neq \emptyset\,,\ V \cap A \neq \emptyset\,,\ U \cap V = \emptyset \text{ and } A \subseteq U \cup V.$$

We say that $A$ is **connected** (in $X$) when there do not exist open sets $U$ and $V$ in $X$ which separate $A$. We say that $A$ is **disconnected** (in $X$) when it is not connected, that is when there do exist open sets $U$ and $V$ in $X$ which separate $A$.

**5.2 Example:** Show that the connected sets in $\mathbb{R}$ are the intervals.

Solution: Recall (or verify) that the intervals in $\mathbb{R}$ are the sets with the intermediate value property: for all $a, b, \in A$ and all $x \in \mathbb{R}$, if $a < x < b$ then $x \in A$. Let $A \subseteq \mathbb{R}$. Suppose that $A$ is not an interval. Then $A$ does not have the intermediate value property so we can choose $a, b \in A$ and $u \in \mathbb{R}$ with $a < u < b$ and $u \notin A$. Then $U = (-\infty, u)$ and $V = (u, \infty)$ separate $A$ and so $A$ is disconnected.

Suppose, conversely, $A$ is disconnected. Choose open sets $U$ and $V$ which separate $A$. Choose $a \in U$ and $b \in V$. Note that $a \neq b$ since $U \cap V = \emptyset$. Suppose that $a < b$ (the case that $b < a$ is similar). Let $u = \sup\big(U \cap [a, b]\big)$. Note that $u \neq a$ since we can choose $\delta > 0$ such that $[a, a+\delta) \subseteq U \cap [a, b]$ and then we have $u = \sup\big(U \cap [a, b]\big) \geq a + \delta$. Note that $u \neq b$ since we can choose $\delta > 0$ such that $(b-\delta, b] \subseteq V \cap [a, b]$ and then we have $u = \sup\big(U \cap [a, b]\big) \leq b - \delta$ since $U \cap V = \emptyset$. Thus we have $a < u < b$. Note that $u \notin U$ since if we had $u \in U$ we could choose $\delta > 0$ such that $(u-\delta, u+\delta) \subseteq U \cap [a, b]$ which contradicts the fact that $u = \sup\big(U \cap [a, b]\big)$. Note that $u \notin V$ since if we had $u \in V$ then we could choose $\delta > 0$ such that $(u-\delta, u+\delta) \subseteq V \cap [a, b]$ which contradicts the fact that $u = \sup\big(U \cap [a, b]\big)$ because $U \cap V = \emptyset$. Since $u \notin U$ and $u \notin V$ and $A \subseteq U \cap V$ we have $u \notin A$, so $A$ does not have the intermediate value property, and so $A$ is not an interval.

**5.3 Example:** Show that the non-empty connected sets in $\mathbb{Q}$ are the one-point sets.

Solution: Every one-point set (in any metric space) is clearly connected. Suppose that $A \subseteq \mathbb{Q}$ contains at least two points, say $a, b \in A$ with $a < b$. We choose an irrational number $r \in (a, b)$, and then the open sets $U = \big\{x \in \mathbb{Q} \big| x < r\big\}$ and $\big\{x \in \mathbb{Q} \big|\ x > r\big\}$ separate $A$ in $\mathbb{Q}$.

**5.4 Theorem:** *Let $X$ and $Y$ be metric spaces, let $f : X \to Y$, and let $A \subseteq X$. If $f$ is continuous and $A$ is connected in $X$ then $f(A)$ is connected in $Y$.*

Proof: Suppose that $f$ is continuous and $f(A)$ is disconnected. Choose open sets $U$ and $V$ in $Y$ which separate $f(A)$ in $Y$, that is $U \cap f(A) \neq \emptyset$, $V \cap f(A) \neq \emptyset$, $U \cap V = \emptyset$ and $f(A) \subseteq U \cup V$. Since $f$ is continuous, the sets $f^{-1}(U)$ and $f^{-1}(V)$ are open in $X$. Since $U \cap f(A) \neq \emptyset$ and $V \cap f(A) \neq \emptyset$, we have $f^{-1}(U) \cap A \neq \emptyset$ and $f^{-1}(V) \cap A \neq \emptyset$. Since $U \cap V = \emptyset$, we have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Since $f(A) \subseteq U \cup V$, we have $A \subseteq f^{-1}(U) \cup f^{-1}(V)$. Thus the open sets $f^{-1}(U)$ and $f^{-1}(V)$ separate $A$ in $X$, so $A$ is disconnected in $X$.

**5.5 Theorem:** *Let $X$ be a metric space and let $A \subseteq P \subseteq X$. Then $A$ is connected in $P$ if and only if $A$ is connected in $X$.*

Proof: Suppose that $A$ is not connected in $X$. Choose open sets $U$ and $V$ in $X$ which separate $A$ in $X$, that is $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$. Let $E = U \cap P$ and $F = V \cap P$. Note that $E$ and $F$ are open in $P$ and $E$ and $F$ separate $A$ in $P$.

Suppose, conversely, that there exist sets $E, F \subseteq P$ which are open in $P$ and which separate $A$ in $P$, that is $A \cap E \neq \emptyset$, $A \cap F \neq \emptyset$, $E \cap F = \emptyset$ and $A \subseteq E \cup F$. Choose open sets $U, V \subseteq X$ such that $E = U \cap P$ and $F = V \cap P$. Note that it is possible that $U \cap V \neq \emptyset$ and so $U$ and $V$ might not separate $A$ in $X$. For this reason, we shall construct open subsets $U_0 \subseteq U$ and $V_0 \subseteq V$ which do separate $A$ in $X$. For each $a \in E$ choose $r_a > 0$ such that $B(a, 2r_a) \subseteq U$ and then let $U_0 = \bigcup_{a \in E} B(a, r_a)$. Note that $U_0$ is open in $X$ (since it is a union of open sets in $X$) and that we have $E \subseteq U_0 \subseteq U$. Similarly, for each $b \in F$ choose $s_b > 0$ so that $B(b, 2s_b) \subseteq V$, and then let $V_0 = \bigcup_{b \in F} B(b, s_b)$. Note that $V_0$ is open in $X$ and $F \subseteq V_0 \subseteq V$. We claim that the open sets $U_0$ and $V_0$ separate $A$ in $X$. Since $E \subseteq U_0$ and $F \subseteq V_0$ we have $\emptyset \neq A \cap E \subseteq A \cap U_0$, $\emptyset \neq A \cap F \subseteq A \cap V_0$ and $A \subseteq E \cup F \subseteq U_0 \cup V_0$. It remains to show that $U_0 \cap V_0 = \emptyset$. Suppose, for a contradiction, that $U_0 \cap V_0 \neq \emptyset$. Choose $x \in U_0 \cap V_0$. Since $x \in U_0 = \bigcup_{a \in E} B(a, r_a)$ we can choose $a \in E$ such that $x \in B(a, r_a)$. Similarly, we can choose $b \in F$ so that $x \in B(b, s_b)$. Suppose that $r_a \geq s_b$ (the case that $s_b \geq r_a$ is similar). By the Triangle Inequality, it follows that $|b - a| \leq |b - x| + |x - a| < s_b + r_a \leq 2r_a$ and so we have $b \in B(a, 2r_a) \subseteq U$. Since $b \in F \subseteq P$ and $b \in U$ we have $b \in U \cap P = E$. Thus we have $b \in E \cap F$ which contradicts the fact that $E \cap F = \emptyset$, and so $U_0 \cap V_0 = \emptyset$, as required.

**5.6 Corollary:** *Let $X$ be a metric space and let $A \subseteq X$. Then $A$ is connected in $X$ if and only if $A$ is connected in itself if and only if the only subsets of $A$ which are both open and closed in $A$ are the sets $\emptyset$ and $A$.*

Proof: By the above theorem, with $P = A$, we see that $A$ is connected in $X$ if and only if $A$ is connected in itself. If there is a set $U$ in $A$, with $\emptyset \subsetneqq U \subsetneqq A$, which is both open and closed in $A$, then its complement $V = U^c = A \setminus U$ is also both open and closed in $A$, and then $U$ and $V$ separate $A$ so that $A$ is disconnected (in itself). Conversely, if $A$ is disconnected (in itself) then we can choose open sets $U$ and $V$ in $A$ which separate $A$ (that is $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = A$) and then each of the sets $U$ and $V$ is both open and closed, and neither is empty, and neither is equal to all of $A$.

**5.7 Remark:** Because of the above theorem and corollary, when $A$ is a connected subset of a metric space $X$, we do not normally say that $A$ is connected in $X$; we simply say that $A$ is connected. Also, again because of the above corollary, we can extend our definition of connectedness so that it applies to topological spaces:

**5.8 Definition:** For a topological space $X$, we say that $X$ is **disconnected** when it is the union of two disjoint nonempty open sets, otherwise, we say that $X$ is **connected**.

**5.9 Theorem:** *Let $X$ be a metric space. The union of any set of connected sets in $X$, which share a common point, is connected.*

Proof: Let $a \in X$ and for each $k \in K$ (where $K$ is any set), let $A_k \subseteq X$ be connected in $X$ with $a \in A_k$. Let $B = \bigcup_{k \in K} A_k$, and note that $a \in B$. Suppose, for a contradiction, that $B$ is not connected. Choose open sets $U, V \subseteq X$ which separate $B$ (so we have $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $U \cap V = \emptyset$ and $B \subseteq U \cup V$). Since $B \subseteq U \cup V$ and $a \in B$, we must have $a \in U$ or $a \in V$. Suppose that $a \in U$ (the case that $a \in V$ is similar). Let $k \in K$. Note that $A_k \cap U \neq \emptyset$ (since $a \in A_k \cap U$) and $U \cap V = \emptyset$ and $A_k \subseteq B \subseteq U \cup V$. Since $A_k$ is connected we must have $A_k \cap V = \emptyset$ (otherwise $U$ and $V$ would separate $A_k$). Since $A_k \subseteq U \cup V$ and $A_k \cap V = \emptyset$, we have $A \subseteq U$. Since $k \in K$ was arbitrary, we have $A_k \subseteq U$ for all $k \in K$, and hence $B = \bigcup_{k \in K} A_k \subseteq U$. Since $B \subseteq U$ and $U \cap V = \emptyset$, we have $B \cap V = \emptyset$, which gives the desired contradiction.

**5.10 Definition:** Let $X$ be a metric space. Define a relation $\cong$ on $X$ by stipulating that for $a, b \in X$ we have $a \cong b$ if and only if there exists a connected set $A \subseteq X$ such that $a, b \in A$. Note that $\cong$ is an equivalence relation $\big($which means that for all $a, b, c \in X$ we have $a \cong a$, and $a \cong b \implies b \cong a$, and $(a \cong b$ and $b \cong c) \implies a \cong c\big)$. Recall the the **equivalence class** of $a \in X$ is the set

$$[a] = \big\{x \in X \,\big|\, x \cong a\big\}.$$

Recall (or verify) that the equivalence classes are disjoint, with $[a] = [b] \iff a \cong b$, and that $X$ is equal to the disjoint union of the equivalence classes. The equivalence classes of $X$, under this equivalence relation $\cong$, are called the (connected) **components** of $X$.

**5.11 Theorem:** *Let $X$ be a metric space. The connected components of $X$ are connected, and every connected subset of $X$ is contained in one of the connected components of $X$.*

Proof: First let us show that every connected subset of $X$ is contained in one of the components. Let $P \subseteq X$ be connected. If $P$ is empty then of course it is contained in one of the components of $X$. Suppose $P \neq \emptyset$ and let $p \in P$. Since the components cover $X$, we can choose $a \in X$ such that $p \in [a]$. We claim that $P \subseteq [a]$. Let $x \in P$. Since $p \in P$ and $x \in P$ and $P$ is connected, we have $x \cong p$ (by the definition of the relation $\cong$). Since $p \in [a]$ we have $p \cong a$ hence $[p] = [a]$. Since $x \cong p$ we have $x \in [p] = [a]$. Since $x \in P$ was arbitrary, $P \subseteq [a]$, as claimed.

Now let us show that the components of $X$ are connected. Let $a \in X$. We claim that $[a]$ is connected. For each $x \in [a]$, we have $x \cong a$ and so (by the definition of $\cong$) we can choose a connected set $A_x \subseteq X$ with $x, a \in A_x$. As shown above, $A_x$ is contained in one of the components of $X$, and since $a \in A_x \cap [a]$, that component must be $[a]$, so we have $A_x \subseteq [a]$. Since $A_x \subseteq [a]$ for every $x \in [a]$, we see that $[a] = \bigcup_{x \in [a]} A_x$. By the above lemma (since the sets $A_x$ are connected with $a \in A_x$ for every $x \in [a]$) the set $\bigcup_{x \in X} A_x$ is connected.

## Path Connectedness

**5.12 Definition:** Let $X$ be a metric space (or a topological space) and let $A \subseteq X$. For $a, b \in A$, a (continuous) **path** from $a$ to $b$ in $A$ is a continuous map $\alpha : [0,1] \to A$ with $\alpha(0) = a$ and $\alpha(1) = b$. When there exists a path from $a$ to $b$ in $A$, we write $a \sim b$ in $A$. We say that $A$ is **path connected** (in $X$) when it has the property that for all $a, b \in A$, we have $a \sim b$ in $A$.

**5.13 Remark:** It is clear from the definition that $A$ is path connected in $X$ if and only if $A$ is path connected in itself (because the continuity of a map $\alpha : [0,1] \to A$ is unchanged if we regard $\alpha$ as a map $\alpha : [0,1] \to X$). Because of this, we do not normally say that $A$ is path connected in $X$; we simply say that $A$ is path connected.

**5.14 Example:** When $X$ is a normed linear space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and $A \subseteq X$, note that if $A$ is convex then $A$ is path connected (because when $A$ is convex, the map $\alpha : [0,1] \to X$ given by $\alpha(t) = a + t(b-a)$ is continuous and takes values in the set $A$). In particular, note that the open and closed balls $B(a,r)$ and $\overline{B}(a,r)$ are path-connected.

**5.15 Theorem:** *Let $X$ and $Y$ be metric spaces (or topological spaces), let $A \subseteq X$, and let $f : A \subseteq X \to Y$. If $f$ is continuous on $A$, and $A$ is path connected, then $f(A)$ is path connected.*

Proof: Suppose that $f$ is continuous on $A$ and $A$ is path connected. Let $c, d \in f(A)$. Choose $a, b \in A$ with $f(a) = c$ and $f(b) = d$. Since $A$ is path connected, we can choose a continuous map $\alpha : [0,1] \to A$ with $\alpha(0) = a$ and $\alpha(1) = b$. Then the map $\beta : [0,1] \to f(A)$ given by $\beta(t) = f\big(\alpha(t)\big)$ is continuous with $\beta(0) = c$ and $\beta(1) = d$, and so $f(A)$ is path-connected.

**5.16 Theorem:** *Let $X$ be a metric space (or a topological space). The relation $\sim$ on $X$ (given by stipulating that $a \sim b$ when there exists a path from $a$ to $b$ in $X$) is an equivalence relation on $X$.*

Proof: Let $a, b, c \in X$. We have $a \sim a$ because we can define $\alpha : [0,1] \to X$ by $\alpha(t) = a$ for all $t$, and then $\alpha$ is continuous with $\alpha(0) = a$ and $\alpha(1) = a$.

Suppose that $a \sim b$. Let $\alpha$ be a path from $a$ to $b$, so $\alpha : [0,1] \to X$ is continuous with $\alpha(0) = a$ and $\alpha(1) = b$. Define $\beta : [0,1] \to X$ by $\beta(t) = \alpha(1-t)$. Note that $\beta$ is continuous since it is the composite of the continuous map $\alpha$ with the continuous map $s : [0,1] \to [0,1]$ given by $s(t) = 1 - t$, and note that we have $\beta(0) = \alpha(1) = b$ and $\beta(1) = \alpha(0) = a$. Thus $\beta$ is a path in $X$ from $b$ to $a$ and so $b \sim a$.

Finally, suppose that $a \sim b$ and $b \sim c$. Let $\alpha$ be a path from $a$ to $b$ in $X$ and let $\beta$ be a path from $b$ to $c$ in $X$. Define $\gamma : [0,1] \to X$ by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{, for } 0 \le t \le \frac{1}{2}, \\ \beta(2t-1) & \text{, for } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that $\gamma(0) = \alpha(0) = a$, $\gamma\left(\frac{1}{2}\right) = \alpha(1) = \beta(0) = b$, and $\gamma(1) = \beta(1) = c$. We claim that $\gamma$ is continuous. Note that the sets $A = \left[0, \frac{1}{2}\right]$ and $B = \left[\frac{1}{2}, 1\right]$ are closed in $[0,1]$ with $A \cup B = [0,1]$, and the restriction of $\gamma$ to $A$ is given by $\alpha(2t)$, which is continuous (being the composite of two continuous functions), and the restriction of $\gamma$ to $B$ is given by $\beta(2t-1)$, which is also continuous. Let $C \subseteq X$. Since $\alpha$ is continuous, the set $\alpha^{-1}(C)$ is closed in $\left[0, \frac{1}{2}\right]$, hence also in $[0,1]$, and since $\beta$ is continuous, the set $\beta^{-1}(C)$ is closed in $\left[\frac{1}{2}, 1\right]$, hence also in $[0,1]$, and so the set $\gamma^{-1}(C) = \alpha^{-1}(C) \cup \beta^{-1}(C)$ is closed in $[0,1]$ (since it is the union of two closed sets). Thus $\gamma$ is continuous by Theorem 3.29 (the Topological Characterization of Continuity).

**5.17 Definition:** Let $X$ be a metric space (or a topological space). The equivalence classes $[a] = \{x \in X | x \sim a\}$ are called the **path components** of $X$. Recall (or verify) that since $\sim$ is an equivalence relation on $X$, the path components of $X$ are disjoint with $[a] = [b] \iff a \sim b$, and $X$ is equal to the disjoint union of its path components.

**5.18 Theorem:** *Let $X$ be a metric space (or a topological space). The path components of $X$ are path connected, and every path connected subset of $X$ is contained in one of the path components of $X$.*

Proof: The proof is left as an exercise.

**5.19 Theorem:** *(Path Connectedness Implies Connectedness) Let $X$ be a metric space (or a topological space). If $X$ is path connected then $X$ is connected.*

Proof: Suppose that $X$ is path connected. Suppose, for a contradiction, that $X$ is not connected. Choose nonempty disjoint open sets $U$ and $V$ in $X$ such that $X = U \cup V$. Choose $a \in U$ and $b \in V$. Choose a path $\alpha : [0,1] \to X$ from $a$ to $b$ in $X$. Since $\alpha$ is continuous, the sets $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are open in $[0,1]$. Since $\alpha(0) = a \in U$ and $\alpha(1) = b \in V$ we have $0 \in \alpha^{-1}(U)$ so that $\alpha^{-1}(U) \neq \emptyset$ and $1 \in \alpha^{-1}(V)$ so that $\alpha^{-1}(V) \neq \emptyset$. Since $X = U \cup V$ we have $[0\,1] = \alpha^{-1}(U) \cup \alpha^{-1}(V)$. Thus $[0,1]$ is the union of the disjoint nonempty subsets $f^{-1}(U)$ and $f^{-1}(V)$. This contradicts the fact that $[0,1]$ is connected.

**5.20 Corollary:** *In a metric space $X$, every path component is contained in one of the connected components, and every connected component is a disjoint union of the path components which it contains.*

**5.21 Note:** The converse of the above theorem does not always hold. For example, let $A = B \cup C$ with $B = \{(x,y) \in \mathbb{R}^2 \,|\, x > 0, y = \sin \frac{1}{x}\}$ and $C = \{(x,y) \in \mathbb{R}^2 \,|\, x = 0, -1 \leq y \leq 1\}$. As an exercise, verify that $A$ is connected and $B$ and $C$ are the path components of $A$.

**5.22 Theorem:** *Let $X$ be a normed linear space and let $A \subseteq X$. If $A$ is open in $X$ and $A$ is connected, then $A$ is path connected.*

Proof: Suppose that $A$ is open in $X$ and that $A$ is connected. Let $a \in A$. Let

$$U = \{b \in A | a \sim b\}.$$

We claim that $U$ is open in $A$. Let $b \in U$. Since $b \in A$ and $A$ is open in $X$, we can choose $r > 0$ so that $B(b,r) \subseteq A$. Let $c \in B(b,r)$. Since $b \in U$ we have $a \sim b$. Since $c \in B(b,r) \subseteq A$ we have $b \sim c$, indeed we can define $\alpha : [0,1] \to B(b,r) \subseteq A$ by $\alpha(t) = b + t(c-b)$ and then $\alpha$ is continuous with $\alpha(0) = b$ and $\alpha(1) = c$, and $\alpha(t) \in B(b,r)$ for all $t \in [0,1]$ because $\|\alpha(t) - b\| = \|t(c-b)\| = |t|\|c-b\| \leq \|c-b\| < r$. Since $a \sim b$ and $b \sim c$ we have $a \sim c$. Since $a \sim c$ we have $c \in U$, hence $B(b,r) \subseteq U$. This shows that $U$ is open.

We claim that $U$ is also closed in $A$. Let $b \in A \setminus U$. Since $b \in A$ and $A$ is open in $X$, we can choose $r > 0$ so that $B(b,r) \subseteq A$. Let $c \in B(b,r)$. Since $b \notin U$ we have $a \not\sim b$. Since $c \in B(b,r) \subseteq A$ we have $b \sim c$, as above. It follows that $a \not\sim c$ since otherwise we would have $a \sim c$ and $c \sim b$ and hence $a \sim b$. Since $c \not\sim a$ we have $c \in A \setminus U$. Thus $B(b,r) \subseteq A \setminus U$. This shows that $A \setminus U$ is open so that $U$ is closed in $A$.

Since $A$ is connected, the only subsets of $A$ which are both open and closed are $\emptyset$ and $A$. Since $U$ is both open and closed we must have $U = \emptyset$ or $U = A$. Since $a \sim a$ we have $a \in U$ so $U \neq \emptyset$ and so $U = A$. Since $A = U = \{b \in A | a \sim b\}$ we have $a \sim b$ for every $b \in A$. Thus $A$ is path connected.

## Compactness

**5.23 Definition:** Let $X$ be a metric space (or a topological space) and let $A \subseteq X$. An **open cover** for $A$ (in X) is a set $S$ of open sets in $X$ such that $A \subseteq \bigcup S = \bigcup_{U \in S} U$. When $S$ is an open cover for $A$ in $X$, a **subcover** of $S$ for $A$ is a subset $T \subseteq S$ such that $A \subseteq \bigcup T = \bigcup_{U \in T} U$. We say that $A$ is **compact** (in $X$) when every open cover for $A$ has a finite subcover.

**5.24 Example:** Recall that for $A \subseteq \mathbb{R}^n$, the Heine-Borel Theorem states that $A$ is compact if and only if $A$ is closed and bounded. Note that this also holds for $A \subseteq \mathbb{C}^n$ because $(\mathbb{C}^n, d_2) = (\mathbb{R}^{2n}, d_2)$.

**5.25 Example:** When $X$ is a metric space and $A \subseteq X$ is closed and bounded, it is *not* always the case that $A$ is compact. For example, if $X$ is any infinite set and $d$ is the discrete metric on $X$, then every infinite subset $A \subseteq X$ is closed and bounded but not compact. In particular, closed unit balls are not compact, indeed for all $a \in X$ we have $\overline{B}(a, 1) = X$.

**5.26 Theorem:** *Let $A \subseteq X \subseteq Y$ where $Y$ is a metric space (or a topological space). Then $A$ is compact in $X$ if and only if $A$ is compact in $Y$.*

Proof: Suppose that $A$ is compact in $X$. Let $T$ be an open cover for $A$ in $Y$. For each $V \in T$, let $U_V = V \cap X$. Note that each set $U_V$ is open in $X$ by Theorem 2.51 (or by Definition 2.52). Since $A \subseteq X$ and $A \subseteq \bigcup_{V \in T} V$, we also have $A \subseteq \bigcup_{V \in T}(V \cap X) = \bigcup_{V \in T} U_V$. Thus the set $S = \big\{U_V \big| V \in T\big\}$ is an open cover for $A$ in $X$. Since $A$ is compact in $X$ we can choose a finite subcover, say $\big\{U_{V_1}, \cdots U_{V_n}\big\}$ of $S$, where each $V_i \in T$. Since $A \subseteq \bigcup_{i=1}^n U_{V_i} = \bigcup_{i=1}^n (V_i \cap X)$, we also have $A \subseteq \bigcup_{i=1}^n V_i$ and so $\{V_1, \cdots, V_n\}$ is a finite subcover of $T$.

Suppose, conversely, that $A$ is compact in $Y$. Let $S$ be an open cover for $A$ in $X$. For each $U \in S$, by Theorem 2.51 (or by Definition 2.52) we can choose an open set $V_U$ in $Y$ such that $U = V_U \cap X$. Then $T = \big\{V_U \big| U \in S\big\}$ is an open cover of $A$ in $Y$. Since $A$ is compact in $Y$ we con choose a finite subcover, say $\big\{V_{U_1}, \cdots, V_{U_n}\big\}$ of $T$, where each $U_i \in S$. Then we have $A \subseteq \bigcup_{i=1}^n (V_{U_i} \cap X) = \bigcup_{i=1}^n U_i$ and so $\{U_1, \cdots, U_n\}$ is a finite subcover of $S$.

**5.27 Remark:** Let $A \subseteq X$ where $X$ is a metric space (or a topological space). By the above theorem, note that $A$ is compact in $X$ if and only if $A$ is compact in itself. For this reason, we do not usually say that $A$ is compact in $X$, we simply say that $A$ is compact.

**5.28 Theorem:** *Let $X$ be a metric space and let $A \subseteq X$. If $A$ is compact then $A$ is closed and bounded.*

Proof: Suppose that $A$ is compact. We claim that $A$ is closed. Let $a \in A^c$. For each $x \in A$, let $r_x = d(a, x) > 0$, let $U_x = B\big(a, \frac{r_x}{2}\big)$, and let $V_x = B\big(x, \frac{r_x}{2}\big)$ so that $U_x$ and $V_x$ are disjoint. Note that the set $S = \big\{V_x \big| x \in A\big\}$ is an open cover for $A$. Since $A$ is compact we can choose a finite subcover, say $\{V_{x_1}, \cdots, V_{x_n}\}$ where each $x_i \in A$. Let $r = \min\{r_{x_1}, \cdots, r_{x_n}\}$ so that $B\big(a, \frac{r}{2}\big) \subseteq U_{x_i}$ for all $i$, and hence $B\big(a, \frac{r}{2}\big)$ is disjoint from each set $V_{x_i}$. Since $B\big(a, \frac{r}{2}\big)$ is disjoint from each set $V_{x_i}$ and the sets $V_{x_i}$ cover $A$, it follows that $B\big(a, \frac{r}{2}\big)$ is disjoint from $A$, hence $B\big(a, \frac{r}{2}\big) \subseteq A^c$. Thus $A^c$ is open, hence $A$ is closed.

We claim that $A$ is bounded. Let $a \in A$. For each $n \in \mathbb{Z}^+$, let $U_n = B(a, n)$. Then the set $S = \{U_1, U_2, U_3, \cdots\}$ is an open cover for $A$. Since $A$ is compact, we can choose a finite subcover, say $\{U_{n_1}, U_{n_2}, \cdots, U_{n,\ell}\} \subseteq S$, with each $n_i \in \mathbb{Z}^+$. Let $m = \max\{n_1, n_2, \cdots, n_\ell\}$ so that $U_{n_i} \subseteq U_m$ for all indices $i$. Then we have $A \subseteq \bigcup_{i=1}^{\ell} U_{n_i} = U_m = B(a, m)$ and so $A$ is bounded.

**5.29 Theorem:** *Let $X$ be a metric space (or a topological space) and let $A \subseteq X$. If $X$ is compact and $A$ is closed in $X$, then $A$ is compact.*

Proof: Suppose that $X$ is compact and $A$ is closed in $X$. Let $S$ be an open cover for $A$. Then $S \cup \{A^c\}$ is an open cover for $X$. Since $X$ is compact, we can choose a finite subcover $T$ of $S \cup \{A^c\}$. Note that $T$ may or may not contain the set $A^c$ but, in either case, $T \setminus \{A^c\}$ is an open cover for $A$ with $T \setminus \{A^c\} \subseteq S$, so that $T \setminus \{A^c\}$ is a finite subcover of $S$.

**5.30 Corollary:** *Let $X$ be a metric space (or a topological space), let $A \subseteq X$ be closed, and let $K \subseteq X$ be compact. Then $A \cap K$ is compact.*

**5.31 Theorem:** *Let $X$ and $Y$ be metric spaces (or topological spaces) and let $f : X \to Y$. If $X$ is compact and $f$ is continuous then $f(X)$ is compact.*

Proof: Suppose that $X$ is compact and $f$ is continuous. Let $T$ be an open cover for $f(X)$ in $Y$. Since $f$ is continuous, so that $f^{-1}(V)$ is open in $X$ for each $V \in T$, the set $S = \left\{ f^{-1}(V) \middle| V \in T \right\}$ is an open cover for $X$. Since $X$ is compact, we can choose a finite subcover, say $\left\{ f^{-1}(V_1), f^{-1}(V_2), \cdots, f^{-1}(V_n) \right\}$ of $S$, with each $V_i \in T$. Then the set $\{V_1, V_2, \cdots, V_n\}$ is a finite subcover of $T$ for $f(X)$.

**5.32 Example:** Note that continuous maps do not necessarily send closed sets to closed sets. For example, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{2}{\pi} \tan^{-1}(x)$ sends the closed set $\mathbb{R}$ homeomorphically to the open interval $(-1, 1)$.

**5.33 Theorem:** *(The Extreme Value Theorem) Let $X$ be a compact metric space (or topological space) and let $f : X \to \mathbb{R}$ be continuous. Then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*

Proof: Since $X$ is compact and $f$ is continuous, it follows that $f(X)$ is compact in $\mathbb{R}$. Since $f(X)$ is compact, it is closed and bounded in $\mathbb{R}$. Since $f(X)$ is bounded in $\mathbb{R}$, it follows that $m = \inf f(X)$ and $M = \sup f(X)$ are both finite real numbers, and since $f(X)$ is closed in $\mathbb{R}$ it follows that $m \in f(X)$ and $M \in f(X)$ so that we can choose $a, b \in X$ such that $f(a) = m = \inf f(X)$ and $f(b) = M = \sup f(X)$.

**5.34 Theorem:** *Let $X$ and $Y$ be metric spaces with $X$ compact. Let $f : X \to Y$ be continuous and bijective. Then $f$ is a homeomorphism.*

Proof: Let $g = f^{-1} : Y \to X$. We need to prove that $g$ is continuous. Let $A \subseteq X$ be closed in $X$. Since $X$ is compact and $A \subseteq X$ is closed, it follows (from Theorem 5.29) that $A$ is compact. Since the map $f : A \to Y$ is continuous and $A$ is compact, it follows (from Theorem 5.31) that $f(A)$ is compact. Since $f(A)$ is compact it follows (from Theorem 5.28) that $f(A)$ is closed. Since $g = f^{-1}$ we have $g^{-1}(A) = f(A)$, which is closed. Since $g^{-1}(A)$ is closed in $Y$ for every closed set $A$ in $X$, it follows that $g$ is continuous, by the Topological Characterization of Continuity (Theorem 3.29).

**5.35 Example:** In the above theorem, the requirement that $X$ is compact is necessary. For example, if $X$ is the interval $X = [0, 2\pi)$ and $Y$ is the unit circle $Y = \left\{ z \in \mathbb{C} \middle| \|z\| = 1 \right\}$, then the map $f : X \to Y$ given by $f(t) = e^{i\,t}$ is continuous and bijective, but the inverse map is not continuous at 1.

**5.36 Theorem:** *(The Lebesgue Number) Let $X$ be a compact metric space and let $S$ be an open cover for $X$. Then there exists a number $\lambda > 0$, which is called a* **Lebesgue number** *for the cover $S$, such that for all $a \in X$ there exists $U \in S$ such that $B(a, \lambda) \subseteq U$.*

Proof: For each $x \in X$, since $S$ is an open cover for $X$ we can choose $U_x \in S$ with $x \in U_x$ and then, since $U_x$ is open we can choose $r_x > 0$ so that $B(a, 2r_x) \subseteq U_x$. Note that the set $T = \{B(x, r_x) \big| x \in X\}$ is an open cover for $X$. Since $X$ is compact, we can choose a finite subcover, say $\{B(x_1, r_{x_1}), \cdots, B(x_n, r_{x_n})\}$ of $T$ for $X$, with each $x_i \in X$. Let $\lambda = \min\{r_{x_1}, \cdots, r_{x_n}\}$. We claim that $\lambda$ is a Lebesgue number for $S$. Let $a \in X$. Choose an index $i$ such that $a \in B(x_i, r_{x_i})$, and let $U = U_{x_i} \in S$. For all $y \in B(a, \lambda)$ we have $d(y, x_i) \leq d(y, a) + d(a, x_i) \leq \lambda + r_{x_i} \leq 2r_{x_i}$ and hence $y \in B(x_i, 2r_{x_i}) \subseteq U_{x_i} = U$. This shows that $B(a, \lambda) \subseteq U$, as required.

**5.37 Theorem:** *Let $X$ and $Y$ be metric spaces with $X$ compact and let $f : X \to Y$ be continuous. Then $f$ is uniformly continuous.*

Proof: We leave the proof as an exercise.

**5.38 Definition:** Let $X$ be a metric space. We say that $X$ is **totally bounded** when for every $\epsilon > 0$ there exists a finite subset $\{a_1, a_2, \cdots, a_n\} \subseteq X$ such that $X = \bigcup_{i=1}^{n} B(a_i, \epsilon)$. We say that $X$ has the **finite intersection property on closed sets** when for every set $T$ of closed sets in $X$, if every finite subset of $T$ has non-empty intersection, then $T$ has non-empty intersection.

**5.39 Theorem:** *Let $X$ be a metric space. Then the following are equivalent.*

*(1) $X$ is compact.*
*(2) $X$ has the finite intersection property on closed sets.*
*(3) Every sequence $(x_n)$ in $X$ has a convergent subsequence.*
*(4) Every infinite subset $A \subseteq X$ has a limit point.*
*(5) $X$ is complete and totally bounded.*

Proof: First we prove that (1) implies (2). Suppose that $X$ is compact. Let $T$ be a set of closed sets in $X$. Suppose that $T$ has empty intersection, that is suppose $\bigcap_{A \in T} A = \emptyset$. Then $\bigcup_{A \in T} A^c = X$ so the set $S = \{A^c \big| A \in T\}$ is an open cover for $X$. Since $X$ is compact, we can choose a finite subcover, say $\{{A_1}^c, \cdots, {A_n}^c\}$ of $S$ for $X$. Then we have $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$, showing that some finite subset of $T$ has empty intersection.

Next we prove that (2) implies (3). Suppose $X$ has the finite intersection property on closed sets. Let $(x_n)_{n \geq 1}$ be a sequence in $X$. For each $m \in \mathbb{Z}^+$, let $A_m = \overline{\{x_n | n > m\}}$ and note that each $A_m$ is closed with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Let $T = \{A_m | m \in \mathbb{Z}^+\}$. Note that every finite subset of $T$ has non-empty intersection because given $A_{m_1}, \cdots, A_{m_\ell} \in T$ we can let $m = \max\{m_1, \cdots, m_\ell\}$ and then we have $\bigcap_{i=1}^{\ell} A_{m_i} = A_m$ and we have $x_n \in A_m$. Since $X$ has the finite intersection property on closed sets, it follows that $T$ has non-empty intersection. Choose a point $a \in \bigcap_{m=1}^{\infty} A_m$. We construct a subsequence $(x_{n_k})_{k \geq 1}$ of $(x_n)_{n \geq 1}$ with $\lim_{k \to \infty} x_{n_k} = a$ as follows. Since $a \in A_1 = \overline{\{x_n | n > 1\}}$ we can choose $n_1 > 1$ such that $d(x_{n_1}, a) < 1$. Since $a \in A_{n_1} = \overline{\{x_n | n > n_1\}}$ we can choose $n_2 > n_1$ such that $d(x_{n_2}, a) < \frac{1}{2}$. Since $a \in A_{n_2} = \overline{\{x_n | n > n_2\}}$ we can choose $n_3 > n_2$ such that $d(x_{n_3}, a) < \frac{1}{3}$. Repeating this procedure, we can choose $1 < n_1 < n_2 < n_3 < \cdots$ such that $d(x_{n_k}, a) < \frac{1}{k}$ for all indices $k$, and then we have constructed a subsequence $(x_{n_k})$ such that $\lim_{k \to \infty} x_{n_k} = a$.

Next we prove that (3) implies (4). Suppose that every sequence $(x_n)$ in $X$ has a convergent subsequence. Let $A \subseteq X$ be an infinite subset. Choose a sequence $(x_n)$ in $A$ with the terms $x_n$ all distinct. Choose a convergent subsequence $(x_{n_k})$ of $(x_n)$ and let $a = \lim\limits_{k\to\infty} x_{n_k}$. Then $a$ is a limit point of the set $A$.

Now let us prove that (4) implies (5). Suppose that every infinite subset $A \subseteq X$ has a limit point. We claim that $X$ is complete. Let $(x_n)$ be a Cauchy sequence in $X$. We claim that $(x_n)$ has a convergent subsequence. If the set $\{x_n | n \in \mathbb{Z}^+\}$ is finite, then some term in the sequence occurs infinitely often, so we can choose indices $n_1 < n_2 < n_3 < \cdots$ such that $x_1 = x_2 = x_3 = \cdots$, and so in this case $(x_n)$ has a constant subsequence. Suppose the set $\{x_n | n \in \mathbb{Z}^+\}$ is infinite. Let $a$ be a limit point of the infinite set $A = \{x_n | n \in \mathbb{Z}^+\}$. Since $a$ is a limit point of the set $\{x_n\}$ we can choose indices $n_k$ with $n_1 < n_2 < n_3 < \cdots$ such that $0 < d(x_{n_k}, a) < \frac{1}{k}$ for each index $k$. Then $(x_{n_k})$ is a subsequence of $(x_n)$ with $\lim\limits_{k\to\infty} x_{n_k} = a$. Since the sequence $(x_n)$ is Cauchy and has a convergent subsequence, it follows, from Part 3 of Theorem 4.11, that the sequence $(x_n)$ converges. Thus $X$ is complete, as claimed.

Continuing our proof that (4) implies (5), suppose that $X$ is not totally bounded. Choose $\epsilon > 0$ such that there do not exist finitely many points $a_1, \cdots, a_n \in X$ for which $X = \bigcup_{i=1}^n B(a_i, \epsilon)$. Let $a_1 \in X$. Since $X \neq B(a_1, \epsilon)$ we can choose $a_2 \in X \setminus B(a_1, \epsilon)$. Since $X \neq B(a_1, \epsilon) \cup B(a_2, \epsilon)$ we can choose $a_3 \in X$ with $a_3 \notin B(a_1, \epsilon) \cup B(a_2, \epsilon)$. Repeat this procedure to choose points $a_1, a_2, a_3, \cdots$ with $a_{n+1} \notin \bigcup_{k=1}^n B(a_k, \epsilon)$. Then the set $A = \{a_n | n \in \mathbb{Z}^+\}$ is an infinite subset of $X$ which has no limit point.

Finally we prove that prove that (5) implies (1). Suppose that $X$ is complete and totally bounded. Suppose, for a contradiction, that $X$ is not compact, and choose an open cover $S$ for $X$ which has no finite subcover for $X$. Since $X$ is totally bounded, we can cover $X$ by finitely many balls of radius 1. Choose one of the balls, say $U_1 = B(a_1, 1)$ such that there is no finite subcover of $S$ for $U_1$ (if there was a finite subcover for each ball, then the union of all these subcovers would be a finite subcover for $X$). Since $X$ is totally bounded, we can cover $X$ (hence also $U_1$) by finitely many balls of radius $\frac{1}{2}$. Choose one of these balls, say $U_2 = B(a_2, \frac{1}{2})$ such that there is no finite subcover of $S$ for $U_1 \cap U_2$. Repeat the procedure to obtain balls $U_n = B\big(a_n, \frac{1}{n}\big)$ such that, for each $n$, there is no finite subcover of $S$ for $\bigcap_{k=1}^n U_k$. In particular, each intersection $\bigcap_{k=1}^n U_k$ is nonempty so we can choose an element $x_n \in \bigcap_{k=1}^n U_k$. Since for all $k, \ell \geq m$ we have $x_k, x_\ell \in U_m = B\big(a_m, \frac{1}{m}\big)$ it follows that $(x_n)$ is Cauchy. Since $X$ is complete, it follows that $(x_n)$ converges in $X$. Let $a = \lim\limits_{n\to\infty} x_n$. Since $S$ covers $X$ we can choose $U \in S$ with $a \in U$. Since $U$ is open we can choose $r > 0$ such that $B(a, r) \subseteq U$. Since $x_n \to a$ we can choose $m > \frac{3}{r}$ such that $d(x_m, a) < \frac{r}{3}$. Then for all $x \in U_m = B\big(a_m, \frac{1}{m}\big)$ we have $d(x, a) \leq d(x, a_m) + d(a_m, x_m) + d(x_m, a) < \frac{1}{m} + \frac{1}{m} + \frac{r}{3} < r$, and so $U_m \subseteq B(a, r) \subseteq U$. But then $S$ has a finite subcover for $U_m$, namely the singleton $\{U\}$, which contradicts the fact that $S$ has no finite subcover for $\bigcap_{k=1}^m U_k$.

**5.40 Example:** Let $\mathbb{F} = \mathbb{R}$. Show that in the metric space $\big(\mathcal{C}[0,1], d_\infty\big)$, the closed unit ball $\overline{B}(0,1)$ is not compact.

Solution: Let $f_n(x) = x^n$ for $n \in \mathbb{Z}^+$. Note that $\|f_n\|_\infty = 1$ so that each $f_n \in \overline{B}(0,1)$. Note that the pointwise limit of the sequence $(f_n)$ is the function $g : [0,1] \to \mathbb{R}$ given by $g(x) = 0$ when $x < 1$ and $g(1) = 1$, which is not continuous. If some subsequence $(f_{n_k})$ of $(f_n)$ were to converge in $\big(\mathcal{C}[0,1], d_\infty\big)$ then it would need to converge uniformly on $[0,1]$ to the function $g$. But this is not possible since the uniform limit of a sequence of continuous functions is always continuous. Thus $(f_n)$ has no convergent subsequence and so $\overline{B}(0,1)$ is not compact.