Chapter 3. Limits and Continuity

Limits of Sequences

3.1 Definition: Let $r \in \mathbb{Z}$ and let $(x_n)_{n\geq r}$ be a sequence in a metric space X. We say that the sequence $(x_n)_{n\geq r}$ is **bounded** when the set $\{x_n\}_{n\geq r}$ is bounded, that is when there exists $a \in X$ and $R > 0$ such that $x_n \in B(a, R)$ for all indices $n \geq r$.

For $a \in X$, we say that the sequence $(x_n)_{n\geq r}$ **converges** to a (or that the **limit** of x_n is equal to a) and we write $\lim\limits_{n\to\infty} x_n = a$ (or we write $x_n \to a$) when

$$\forall \epsilon > 0\ \exists m \in \mathbb{Z}_{\geq r}\ \forall n \in \mathbb{Z}_{\geq r} \big(n \geq m \Longrightarrow d(x_n, a) < \epsilon\big).$$

We say that the sequence $(x_n)_{n\geq r}$ **converges** (in X) when it converges to some point $a \in X$, and otherwise we say that $(x_n)_{n\geq r}$ **diverges** (in X).

3.2 Theorem: *(Basic Properties of Limits of Sequences) Let $(x_n)_{n\geq r}$ be a sequence in a metric space X, and let $a \in X$.*

(1) If $(x_n)_{n\geq r}$ converges then its limit is unique.
(2) If $s \geq r$ and $y_n = x_n$ for all $n \geq s$, then $(x_n)_{n\geq r}$ converges if and only if $(y_n)_{n\geq s}$ converges and, in this case, $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$.
(3) If $(x_{n_k})_{k\geq s}$ is a subsequence of $(x_n)_{n\geq r}$, and $\lim\limits_{n\to\infty} x_n = a$, then $\lim\limits_{k\to\infty} x_{n_k} = a$.
(4) If $(x_n)_{n\geq r}$ converges then it is bounded.
(5) We have $\lim\limits_{n\to\infty} x_n = a$ in X if and only if $\lim\limits_{n\to\infty} d(x_n, a) = 0$ in $\mathbb{R}$.

Proof: We prove Parts 1, 4 and 5 and leave the proofs of the other parts as an exercise. To prove Part 1, suppose that $x_n \to a$ in X and $x_n \to b$ in X. We need to show that $a = b$. Suppose, for a contradiction, that $a \neq b$, and note that $d(a, b) > 0$. Since $x_n \to a$ and $x_n \to b$, we can choose $m \in \mathbb{Z}_{\geq r}$ such that when $n \geq m$ we have $d(x_n, a) < \frac{d(a,b)}{2}$, and $d(x_n, b) < \frac{d(a,b)}{2}$. Then we have $d(a, b) \leq d(a, x_m) + d(x_m, b) < \frac{d(a,b)}{2} + \frac{d(a,b)}{2} = d(a, b)$, giving the desired contradiction.

To prove Part 4, suppose that $(x_n)_{n\geq r}$ converges, say $x_n \to a$ in X. Choose $m \in \mathbb{Z}_{\geq r}$ such that when $n \geq m$ we have $d(x_n, a) < 1$. Then for all $n \in \mathbb{Z}_{\geq r}$ we have $d(x_n, a) \leq R$ where $R = \max\big\{d(x_r, a), d(x_{r+1}, a), \cdots, d(x_{m-1}, a), 1\big\}$ so that $x_n \in B(a, R+1)$.

To prove Part 5, note that since $d(x_n, a) \geq 0$ we have $d(x_n, a) = \big|d(x_n, a) - 0\big|$ and so

$$\begin{aligned}
\lim_{n\to\infty} x_n = a \text{ in } X &\iff \forall \epsilon > 0\ \exists m \in \mathbb{Z}_{\geq r}\ \forall n \in \mathbb{Z}_{\geq r}\ d(x_n, a) < \epsilon \\
&\iff \forall \epsilon > 0\ \exists m \in \mathbb{Z}_{\geq r}\ \forall n \in \mathbb{Z}_{\geq r}\ \big|d(x_n, a) - 0\big| < \epsilon \\
&\iff \lim_{n\to\infty} d(x_n, a) = 0 \text{ in } \mathbb{R}.
\end{aligned}$$

3.3 Note: Because of Part 2 of the above theorem, the initial index r of a sequence $(x_n)_{n\geq r}$ does not effect whether or not the sequence converges and it does not effect the limit. For this reason, we often omit the initial index r from our notation and write (x_n) for the sequence $(x_n)_{n\geq r}$. Also, we often choose a specific value of r, usually $r = 1$, in the statements or the proofs of various theorems with the understanding that any other initial value would work just as well.

3.4 Theorem: *(Components of Sequences in $\mathbb{F}^m$) Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{F}^m$, say $x_n=\big(x_{n,1},x_{n,2},\cdots,x_{n,m}\big)\in\mathbb{F}^m$, and let $a=(a_1,a_2,\cdots,a_m)\in\mathbb{F}^m$. Then using any of the metrics d_1, d_2 or d_∞ in $\mathbb{F}^m$, we have $\lim\limits_{n\to\infty}x_n=a$ in $\mathbb{F}^m$ if and only if $\lim\limits_{n\to\infty}x_{n,k}=a_k$ in $\mathbb{F}$ for all indices k with $1\leq k\leq m$.*

Proof: Let $p=1$, 2 or ∞. Suppose $\lim\limits_{n\to\infty}x_n=a$ in $(\mathbb{F}^m,d_p)$. Let $1\leq k\leq m$ and let $\epsilon>0$. Choose $\ell\in\mathbb{Z}^+$ such that when $n\geq\ell$ we have $d_p(x_n,a)<\epsilon$, that is $\|x_n-a\|_p<\epsilon$. Then when $n\geq\ell$ we have

$$|x_{n,k}-a_k|=\big(|x_{n,k}-a_k|^p\big)^{1/p}\leq\Big(\sum_{j=1}^m|x_{n,j}-a_j|^p\Big)^{1/p}=\|x_n-a\|_p<\epsilon\,,$$

and so $x_{n,k}\to a_k$ in $\mathbb{F}$, as required.

Suppose, conversely, that for all k with $1\leq k\leq m$ we have $\lim\limits_{n\to\infty}x_{n,k}=a_k$ in $\mathbb{F}$. Let $\epsilon>0$. Choose $\ell\in\mathbb{Z}^+$ such that for all $n\geq\ell$ we have $|x_{n,k}-a_k|<\frac{\epsilon}{m}$ for $1\leq k\leq m$. Then, letting e_k denote the k^{th} standard basis vector in $\mathbb{F}^m$, when $n\geq\ell$ we have

$$\begin{aligned}\|x_n-a\|_p&=\Big\|\sum_{k=1}^m(x_{n,k}-a_k)\,e_k\Big\|_p\leq\sum_{k=1}^m\big\|(x_{n,k}-a_k)e_k\big\|_p\\&=\sum_{k=1}^m\big|x_{n,k}-a_k\big|\,\|e_k\|_p=\sum_{k=1}^m\big|x_{n,k}-a_k\big|<\sum_{k=1}^m\tfrac{\epsilon}{m}=\epsilon\end{aligned}$$

so that $x_n\to a$ in $\mathbb{F}^m$, as required.

3.5 Note: When $(x_n)_{n\geq1}$ is a sequence in $\mathbb{F}^\infty$, ℓ_1, ℓ_2 or ℓ_∞, each term x_n is itself a sequence (so that (x_n) is a sequence of sequences) and we can write $x_n=(x_{n,k})_{k\geq1}$. We have sequences $x_1=(x_{1,1},x_{1,2},\cdots)$, $x_2=(x_{2,1},x_{2,2},x_{2,3},\cdots)$, and $x_3=(x_{3,1},x_{3,2},\cdots)$ and so on. This is not the same thing as a subsequence of a sequence (x_n) in $\mathbb{F}$, which is a single sequence $(x_{n_k})_{k\geq1}=(x_{n_1},x_{n_2},x_{n_3},\cdots)$.

3.6 Theorem: $\big($*Components of Sequences in* $\ell_p\big)$. *Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ and let $p=1$, 2 or ∞. Let $(x_n)_{n\geq1}$ be a sequence in ℓ_p, say $x_n=(x_{n,k})_{k\geq1}\in\ell_p$, and let $a=(a_k)_{k\geq1}\in\ell_p$. If $\lim\limits_{n\to\infty}x_n=a$ in (ℓ_p,d_p) then $\lim\limits_{n\to\infty}x_{n,k}=a_k$ in $\mathbb{F}$ for all $k\in\mathbb{Z}^+$.*

Proof: The proof is the same as the first half of the proof of Theorem 3.4. Suppose that $\lim\limits_{n\to\infty}x_n=a$ in (ℓ_p,d_p). Let $k\in\mathbb{Z}^+$ and let $\epsilon>0$. Choose $m\in\mathbb{Z}^+$ such that when $n\geq m$ we have $\|x_n-a\|_p<\epsilon$. Then when $n\geq m$ we have

$$|x_{n,k}-a_k|=\big(|x_{n,k}-a_k|^p\big)^{1/p}\leq\Big(\sum_{j=1}^\infty|x_{n,j}-a_j|^p\Big)^{1/p}=\|x_n-a\|_p<\epsilon$$

and so $x_{n,k}\to a_k$ in $\mathbb{F}$, as required.

3.7 Note: Unlike the case in $\mathbb{F}^m$, in the infinite-dimensional spaces ℓ_p, when $x_{n,k}\to a_k$ in $\mathbb{F}$ for all indices k, it does not necessarily follow that $x_n\to a$ in (ℓ_p,d_p). For example, you can verify, as an exercise, that when $x_n=e_n$ (the n^{th} standard basis vector in $\mathbb{F}^\infty$), we have $\lim\limits_{n\to\infty}x_{n,k}=0$ in $\mathbb{F}$ for all $k\in\mathbb{Z}^+$, but $\lim\limits_{n\to\infty}x_n\neq0$ in (ℓ_p,d_p) for $p=1,2,3$.

3.8 Exercise: For each $n\in\mathbb{Z}^+$, let $x_n\in\mathbb{R}^\infty$ be the sequence given by $x_n=\frac{1}{n}\sum\limits_{k=1}^n e_k$, that is by $x_n=(x_{n,k})_{k\geq1}=\big(\frac1n,\frac1n,\cdots,\frac1n,0,0,0,\cdots\big)$ with n non-zero terms. Show that (x_n) converges in $(\mathbb{R}^\infty,d_2)$ but diverges in $(\mathbb{R}^\infty,d_1)$.

3.9 Definition: Let A be a set, let X be a metric space, and let $f_n, g : A \to X$. We say that the sequence (f_n) **converges pointwise** to g on A, and we write $f_n \to g$ pointwise on A, when

$$\forall x\in A\ \forall \epsilon>0\ \exists m\in\mathbb{Z}^+\ \forall n\in\mathbb{Z}^+\big(n\geq m \Longrightarrow d(f_n(x),g(x))<\epsilon\big).$$

We say that the sequence (f_n) **converges uniformly** to g on A, and we write $f_n \to g$ uniformly on A, when

$$\forall \epsilon>0\ \exists m\in\mathbb{Z}^+\ \forall n\in\mathbb{Z}^+\ \forall x\in A\ \big(n\geq m \Longrightarrow d(f_n(x),g(x))<\epsilon\big).$$

3.10 Remark: In the definition of the limit of a sequence in a metric space X (Definition 3.1), we can replace the strict inequality $d(x_n,a)<\epsilon$ by the inequality $d(x_n,a)\leq\epsilon$ without changing the meaning. In other words, for a sequence $(x_n)_{n\geq p}$ in X and an element $a\in X$ we have

$$\lim_{n\to\infty} x_n = a \text{ in } X \iff \forall\epsilon>0\ \exists m\in\mathbb{Z}_{\geq p}\forall n\in\mathbb{Z}_{\geq p}\big(n\geq m \Longrightarrow d(x_n,a)\leq\epsilon\big).$$

The same holds for various other definitions, such as the defiinition of uniform convergence.

3.11 Remark: Note that for $h\in\mathcal{B}[a,b]$ and $r>0$, we have

$$\|h\|_\infty \leq r \iff \sup\big\{|h(x)|\,\big|\,a\leq x\leq b\big\}\leq r \iff |h(x)|\leq r \text{ for all } x\in[a,b].$$

We also remark that we would not have equivalence if we replaced $\leq r$ by $<r$, as we only have a one way implication: if $|h(x)|<r$ for all $x\in[a,b]$ then $\sup\big\{|h(x)|\,\big|\,a\leq x\leq b\big\}\leq r$.

3.12 Theorem: *(Limits in $\mathcal{B}[a,b]$ and Uniform Convergence) Let $(f_n)_{n\geq1}$ be a sequence in $\mathcal{B}[a,b]$, and let $g\in\mathcal{B}[a,b]$. Then $f_n\to g$ in $\big(\mathcal{B}[a,b],d_\infty\big)$ if and only if $f_n\to g$ uniformly on $[a,b]$.*

Proof: This follows immediately from the definition of uniform convergence and from the two preceding remarks. Indeed we have

$$\begin{aligned} f_n\to g \text{ in } \mathcal{B}[a,b] &\iff \forall\epsilon>0\ \exists m\in\mathbb{Z}^+\ \forall n\in\mathbb{Z}^+\big(n\geq m\Longrightarrow \|f_n-g\|_\infty\leq\epsilon\big)\\ &\iff \forall\epsilon>0\ \exists m\in\mathbb{Z}^+\ \forall n\in\mathbb{Z}^+\big(n\geq m\Longrightarrow \big|f_n(x)-g(x)\big|\leq\epsilon \text{ for all } x\in[a,b]\big)\\ &\iff f_n\to g \text{ uniformly on } [a,b]. \end{aligned}$$

3.13 Remark: For a metric space X whose elements are functions, such as $\mathcal{B}[a,b]$ or $\mathcal{C}[a,b]$, a sequence in X is a sequence of functions, so we can consider several different notions of convergence for sequences of functions, including pointwise convergence, uniform convergence, and convergence in the metric space. The above theorem shows that convergence in the metric space $\mathcal{B}[a,b]$ $\big($hence also in $\mathcal{C}[a,b]\big)$ using the supremum metric d_∞, is the same thing as uniform convergence. One might ask whether convergence in $\mathcal{C}[a,b]$ using the metrics d_1 or d_2 implies, or is implied by, pointwise convergence. The answer is negative, as the following exercises illustrate.

3.14 Exercise: Define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=1-nx$ for $0\leq x\leq\frac1n$ and $f_n(x)=0$ for $\frac1n\leq x\leq1$. Show that $f_n\to0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2, but $f_n\not\to0$ pointwise on $[0,1]$.

3.15 Exercise: Define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=n^2x-n^3x^2$ for $0\leq x\leq\frac1n$ and $f_n(x)=0$ for $\frac1n\leq x\leq1$. Show that $f_n\to0$ pointwise on $[0,1]$ but $f_n\not\to0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2.

3.16 Exercise: Define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=\sqrt{n}\,x^n$. Show that $(f_n)_{n\geq1}$ converges in $\big(\mathcal{C}[0,1],d_1\big)$ but diverges in $\big(\mathcal{C}[0,1],d_2\big)$.

Limits and Closed Sets

3.17 Theorem: *(The Sequential Characterization of Limit Points and Closed Sets) Let X be a metric space, let $a \in X$, and let $A \subseteq X$.*

(1) $a \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$ in X.

(2) $a \in \overline{A}$ if and only if there exists a sequence (x_n) in A with $\lim\limits_{n\to\infty} x_n = a$ in X.

(3) A is closed in X if and only if for every sequence (x_n) in A which converges in X, we have $\lim\limits_{n\to\infty} x_n \in A$.

Proof: We prove Parts 1 and 3 and leave the proof of Part 2 as an exercise. Suppose that $a \in A'$ (which means that for every $r > 0$ we have $B^*(a,r) \cap A \neq \emptyset$). For each $n \in \mathbb{Z}^+$, choose $x_n \in B^*\big(a, \frac{1}{n}\big) \cap A$, that is choose $x_n \in A \setminus \{a\}$ with $d(x_n, a) < \frac{1}{n}$. Then $(x_n)_{n\geq 1}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$.

Suppose, conversely, that $(x_n)_{n\geq 1}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$. Let $r > 0$. Choose $m \in \mathbb{Z}^+$ such that $d(x_n, a) < r$ for all $n \geq m$. Since $x_m \in A \setminus \{a\}$ with $d(x_m, a) < r$, we have $x_m \in B^*(a,r) \cap A$ and so $B^*(a,r) \cap A \neq \emptyset$. This proves Part 1.

To prove Part 3, suppose that A is closed in X. Let $(x_n)_{n\geq 1}$ be a sequence in A which converges in X, and let $a = \lim\limits_{n\to\infty} x_n \in X$. Suppose, for a contradiction, that $a \notin A$. Since $a \notin A$ we have $A = A \setminus \{a\}$ so in fact (x_n) is a sequence in $A \setminus \{a\}$. Since (x_n) is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$, it follows from Part 1 that $a \in A'$. Since A is closed we have $A' \subseteq A$ and so $a \in A$ giving the desired contradiction.

Suppose, conversely, that for every sequence in A which converges in X, the limit of the sequence lies in A. Let $a \in A'$. By Part 1, we can choose a sequence (x_n) in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$. Then (x_n) is a sequence in A which converges in X, so its limit lies in A, that is $a \in A$. Since $a \in A'$ was arbitrary, this shows that $A' \subseteq A$, and so A is closed. This proves Part 3.

3.18 Example: Let U be a normed linear space, let $a \in U$ and let $r > 0$. Show that $\overline{B(a,r)} = \overline{B}(a,r)$ (so the closed ball is equal to the closure of the open ball).

Solution: We saw, in Example 2.32, that $\overline{B}(a,r)$ is closed. Since $\overline{B}(a,r)$ is closed and $B(a,r) \subseteq \overline{B}(a,r)$, it follows that $\overline{B(a,r)} \subseteq \overline{B}(a,r)$. Let $b \in \overline{B}(a,r)$, that is let $b \in U$ with $\|b - a\| \leq r$. If $\|b - a\| < r$ then we have $b \in B(a,r) \subseteq \overline{B(a,r)}$. Suppose that $\|b - a\| = r$. For $n \in \mathbb{Z}^+$, let $x_n = a + \big(1 - \frac{1}{n}\big)(b - a) \in U$. Note that

$$\|x_n - a\| = \big\|\big(1 - \tfrac{1}{n}\big)(b - a)\big\| = \big(1 - \tfrac{1}{n}\big)\|b - a\| = \big(1 - \tfrac{1}{n}\big)r < r$$

so that $x_n \in B(a,r)$. Note that

$$\|x_n - b\| = \big\|\tfrac{1}{n}(a - b)\big\| = \tfrac{1}{n}\|a - b\| = \tfrac{r}{n} \to 0 \text{ in } \mathbb{R}$$

so that we have $x_n \to b$ in U (by Part 6 of Theorem 3.2). Since (x_n) is a sequence in $B(a,r)$ with $x_n \to b$ in U, it follows that $b \in \overline{B(a,r)}$ by Part 2 of the above theorem.

3.19 Example: In the previous example, it might have seemed intuitively obvious that $\overline{B(a,r)} = \overline{B}(a,r)$, but in fact this is not true in all metric spaces. For example in $\mathbb{Z}$ (using the same standard metric used in $\mathbb{R}$) we have $B(0,1) = \{0\}$ and $\overline{B(0,1)} = \{0\}$, but $\overline{B}(0,1) = \{-1, 0, 1\}$.

3.20 Exercise: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Recall that $\mathbb{F}^\infty \subseteq \ell_1 \subseteq \ell_2 \subseteq \ell_\infty$. Determine whether $\mathbb{F}^\infty$ is closed in (ℓ_1, d_1). Determine which of the spaces $\mathbb{F}^\infty$ and ℓ_1 is closed in (ℓ_2, d_2). Determine which of the spaces $\mathbb{F}^\infty$, ℓ_1 and ℓ_2 is closed in (ℓ_∞, d_∞)

3.21 Exercise: Let $\mathbb{F} = \mathbb{R}$ and let

$$\begin{aligned} \mathcal{R}[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is Riemann integrable} \big\}, \\ \mathcal{P}[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is a polynomial} \big\}, \\ \mathcal{C}^1[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is continuously differentiable} \big\}. \end{aligned}$$

Note that

$$\mathcal{P}[a,b] \subseteq \mathcal{C}^1[a,b] \subseteq \mathcal{C}[a,b] \subseteq \mathcal{R}[a,b] \subseteq \mathcal{B}[a,b].$$

Determine which of the above spaces are closed in the metric space $\mathcal{B}[a,b]$, using the supremum metric d_∞ (we deal with the space $\mathcal{C}[a,b]$ in the following example).

3.22 Example: Let $\mathbb{F} = \mathbb{R}$. Show that $\mathcal{C}[a,b]$ is closed in the metric space $\big(\mathcal{B}[a,b], d_\infty\big)$.

Solution: Let (f_n) be a sequence in $\mathcal{C}[a,b]$ which converges in the metric space $\big(\mathcal{B}[a,b], d_\infty\big)$. Let $g = \lim\limits_{n\to\infty} f_n$ in $\big(\mathcal{B}[a,b], d_\infty\big)$. By Theorem 3.12, we know that $f_n \to g$ uniformly on $[a,b]$. Since each function f_n is continuous on $[a,b]$, and $f_n \to g$ uniformly on $[a,b]$, it follows that g is continuous on $[a,b]$, that is $g \in \mathcal{C}[a,b]$. By the Sequential Characterization of Closed Sets (Part 3 of Theorem 3.17), it follows that $\mathcal{C}[a,b]$ is closed in $\mathcal{B}[a,b]$.

Limits and Continuity of Functions

3.23 Definition: Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$, let $f : A \to Y$, let $a \in A'$, and let $b \in Y$. We say that the **limit** of $f(x)$ as x tends to a is equal to b, and we write $\lim_{x \to a} f(x) = b$, when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < d_X(x, a) < \delta \Longrightarrow d_Y\big(f(x), b\big) < \epsilon\right).$$

3.24 Theorem: *(The Sequential Characterization of Limits) Let X and Y be metric spaces, let $A \subseteq X$, let $f : A \to Y$, let $a \in A' \subseteq X$, and let $b \in Y$. Then $\lim_{x \to a} f(x) = b$ if and only if for every sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ we have $\lim_{n \to \infty} f(x_n) = b$.*

Proof: Suppose that $\lim_{x \to A} f(x) = b$. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \to a$. Let $\epsilon > 0$. Since $\lim_{x \to a} f(x) = b$ we can choose $\delta > 0$ such that $0 < d(x, a) < \delta \Longrightarrow d\big(f(x), b\big) < \epsilon$. Since $x_n \to a$ we can choose $m \in \mathbb{Z}^+$ such that $n \geq m \Longrightarrow d(x_n, a) < \delta$. For $n \geq m$ we have $d(x_n, a) < \delta$ and we have $x_n \neq a$ $\big($since (x_n) is a sequence in $A \setminus \{a\}\big)$, so that $0 < d(x_n, a) < \delta$, and hence $d\big(f(x_n), b\big) < \epsilon$. Thus $\lim_{n \to \infty} f(x_n) = b$, as required.

Suppose, conversely, that $\lim_{x \to a} f(x) \neq b$. Choose $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in A$ such that $0 < d(x, a) < \delta$ and $d\big(f(x), b\big) \geq \epsilon$. For each $n \in \mathbb{Z}^+$, choose $x_n \in A$ such that $0 < d(x_n, a) < \frac{1}{n}$ and $d\big(f(x_n), b\big) \geq \epsilon$. For each n, since $0 < d(x_n, a)$ we have $x_n \neq a$ so the sequence (x_n) lies in $A \setminus \{a\}$. Since $d(x_n, a) < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$, it follows that $x_n \to a$. Since $d\big(f(x_n), b\big) \geq \epsilon$ for all $n \in \mathbb{Z}^+$, it follows that $\lim_{n \to \infty} f(x) \neq b$. Thus we have found a sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ such that $\lim_{n \to \infty} f(x_n) \neq b$.

3.25 Definition: Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. For $a \in X$, we say that f is **continuous** at a when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$, if $d_X(x, a) < \delta$ then $d_Y\big(f(x), f(a)\big) < \epsilon$. We say that f is **continuous** (on X) when f is continuous at every point $a \in X$. We say that f is **uniformly continuous** (on X) when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, if $d_X(x, y) < \delta$ then $d_Y\big(f(x), f(y)\big) < \epsilon$. We say that f is **Lipschitz continuous** (on X) when there is a constant $\ell \geq 0$, called a **Lipschitz constant** for f, such that for all $x, y \in X$ we. have $d\big(f(x), f(y)\big) \leq \ell \cdot d(x, y)$. Note that if f is Lipschitz continuous then f is also uniformly continuous (indeed we can take $\delta = \frac{\epsilon}{\ell}$ in the definition of uniform continuity). A bijective map $f : X \to Y$ such that both f and f^{-1} are continuous is called a **homeomorphism**.

3.26 Note: Let X and Y be metric spaces and let $a \in X$. If a is a limit point of X then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$. If a is an isolated point of X then f is necessarily continuous at a, vacuously.

3.27 Theorem: *(The Sequential Characterization of Continuity) Let X and Y be metric spaces using metrics d_X and d_Y, let $f : X \to Y$, and let $a \in X$. Then f is continuous at a if and only if for every sequence (x_n) in X with $x_n \to a$ we have $\lim_{n \to \infty} f(x_n) = f(a)$.*

Proof: The proof is left as an exercise.

3.28 Exercise: Let X, Y and Z be metric spaces, let $f : X \to Y$, let $g : Y \to Z$. Show that if f is continuous at the point $a \in X$ and g is continuous at the point $f(a) \in Y$ then the composite function $g \circ f$ is continuous at a.

3.29 Theorem: *(The Topological Characterization of Continuity) Let X and Y be metric spaces and let $f : X \to Y$. Then*

(1) f is continuous (on X) if and only if $f^{-1}(V)$ is open in X for every open set V in Y,
(2) f is continuous (on X) if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof: To prove Part 1, suppose f is continuous in X. Let V be open in Y. Let $a \in f^{-1}(V)$ and let $f(a) \in V$. Since V is open, we can choose $\epsilon > 0$ such that $B\big(f(a), \epsilon\big) \subseteq V$. Since f is continuous at a we can choose $\delta > 0$ such that for all $x \in X$ with $d(x, a) < \delta$ we have $d\big(f(x), f(a)\big) < \epsilon$. Then we have $f\big(B(a, \delta)\big) \subseteq B\big(f(a), \epsilon\big) \subseteq V$ and so $B(a, \delta) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open in X, as required.

Suppose, conversely, that $f^{-1}(V)$ is open in X for every open set V in Y. Let $a \in X$ and let $\epsilon > 0$. Taking $V = B\big(f(a), \epsilon\big)$, which is open in Y, we see that $f^{-1}\big(B\big(f(a), \epsilon\big)\big)$ is open in X. Since $a \in f^{-1}\big(B(f(a), \epsilon\big)$ and $f^{-1}\big(B\big(f(a), \epsilon\big)\big)$ is open in X, we can choose $\delta > 0$ such that $B(a, \delta) \subseteq f^{-1}\big(B\big(f(a), \epsilon\big)\big)$. Then we have $f\big(B(a, \delta)\big) \subseteq B\big(f(a), \epsilon\big)$ or, in other words, for all $x \in X$, if $d(x, a) < \delta$ then $d\big(f(x), f(a)\big) < \epsilon$. Thus f is continuous at a hence, since a was arbitrary, f is continuous on X.

This completes the proof of Part 1, and Part 2 follows by taking complements since for every set $B \subseteq Y$ we have $\big(f^{-1}(B)\big)^c = f^{-1}(B^c)$. Indeed for all $x \in A$ we have

$$x \in \big(f^{-1}(B)\big)^c \iff x \notin f^{-1}(B) \iff f(x) \notin B \iff f(x) \in B^c \iff x \in f^{-1}(B^c).$$

3.30 Definition: Let X and Y be topological spaces and let $f : X \to Y$. We say that f is **continuous** (on X) when $f^{-1}(V)$ is open in X for every open set V in Y. A bijective map $f : X \to Y$ such that both f and f^{-1} are continuous is called a **homeomorphism**.

3.31 Theorem: *(Composition of Continuous Functions) Let X, Y and Z be metric spaces (or topological spaces), let $f : X \to Y$, and let $g : Y \to Z$. If f and g are continuous then the composite function $g \circ f : X \to Z$ is continuous.*

Proof: Let $h = g \circ f : X \to Z$. If $W \subseteq Z$ is open in Z, then $g^{-1}(W)$ is open in Y (since g is continuous), hence $h^{-1}(W) = f^{-1}\big(g^{-1}(W)\big)$ is open in X (since f is continuous). Thus h is continuous, by Theorem 3.29 (or by Definition 3.30)

3.32 Example: Let $A = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, y < x^2\big\}$. Show that A is open in $\mathbb{R}^2$.

Solution: We remark that it is surprisingly difficult to show that A is open directly from the definition of an open set (as mentioned in Remark 2.34). But we can make use of the Topological Characterization of Continuity to give a quick proof. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = y - x^2$. Note that f is continuous (polynomial functions, and indeed all elementary functions, are continuous) and we have $A = \big\{(x, y) \,\big|\, f(x, y) < 0\big\} = f^{-1}(B)$ where B is the open interval $(-\infty, 0)$. Since B is open in $\mathbb{R}$ and f is continuous, it follows that $A = f^{-1}(B)$ is open in $\mathbb{R}^2$.

3.33 Example: Recall from Example 2.41 that every set $U \subseteq \mathcal{C}[a, b]$ which is open using the metric d_1 is also open using the metric d_∞, but not vice versa. It follows (from Theorem 3.29) that the identity map $I : \mathcal{C} \to \mathcal{C}[a, b]$ given by $I(f) = f$ is continuous as a map from the metric space $\big(\mathcal{C}[a, b], d_\infty\big)$ to the metric space $\big(\mathcal{C}[a, b], d_1\big)$, but not vice versa.

Continuity of Linear Maps

3.34 Note: If U and V are inner product spaces over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and $L:U\to V$ is an inner product space isomorphism, then L and its inverse preserve distance so they are both continuous (we can take $\delta=\epsilon$ in the definition of continuity), hence L is a homeomorphism.

If U and V are finite-dimensional inner product spaces with say $\dim U=n$ and $\dim V=m$, and if $\phi:\mathbb{F}^n\to U$ and $\psi:\mathbb{F}^m\to V$ are inner product space isomorphisms (obtained by choosing orthonormal bases for U and V) then a map $F:U\to V$ is continuous if and only if the composite map $\psi^{-1}F\phi:\mathbb{F}^n\to\mathbb{F}^m$ is continuous. In particular, if F is linear then F is continuous (since $\psi^{-1}F\phi:\mathbb{F}^n\to\mathbb{F}^m$ is linear, hence continuous).

We shall see below (in Corollary 3.39) that the same is true for finite dimensional normed linear spaces: every linear map between finite dimensional normed linear spaces is continuous. But this is not always true (see Example 3.33) for infinite dimensional spaces.

3.35 Theorem: *Let U and V be normed linear spaces over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ and let $F:U\to V$ be a linear map. Then the following are equivalent:*

(1) F is Lipschitz continuous on U,
(2) F is continuous at some point $a\in U$,
(3) F is continuous at 0, and
(4) $F\big(\overline{B}(0,1)\big)$ is bounded.

In this case, if $m\geq 0$ with $F\big(\overline{B}(0,1)\big)\subseteq B(0,m)$ then m is a Lipschitz constant for F.

Proof: It is clear that if F is Lipschitz continuous on U then F is continuous at some point $a\in U$ (indeed F is continuous at every point $a\in U$). Let us show that if F is continuous at some point $a\in U$ then F is continuous at 0. Suppose that F is continuous at $a\in U$. Let $\epsilon>0$. Since F is continuous at $a\in U$, we can choose $\delta_1>0$ such that for all $u\in U$ we have $\|u-a\|\leq\delta_1\implies\big\|F(u)-F(a)\big\|\leq 1$. Choose $\delta=\delta_1\epsilon$. Let $x\in U$ with $\|x-0\|<\delta$. If $x=0$ then $\big\|F(x)-F(0)\big\|=\|0\|=0$. Suppose that $x\neq 0$. Then for $u=a+\frac{\delta_1 x}{\|x\|}$ we have $\|u-a\|=\big\|\frac{\delta_1 x}{\|x\|}\big\|=\delta_1$ and so $\big\|F(u-a)\big\|=\big\|F(u)-F(a)\big\|\leq 1$, that is $\big\|F\big(\frac{\delta_1 x}{\|x\|}\big)\big\|\leq 1$ hence, by the linearity of F and the scaling property of the norm, we have

$$\big\|F(x)-F(0)\big\|=\big\|F(x)\big\|=\tfrac{\|x\|}{\delta_1}\big\|F\big(\tfrac{\delta_1 x}{\|x\|}\big)\big\|\leq\tfrac{\|x\|}{\delta_1}<\tfrac{\delta_1\epsilon}{\delta_1}=\epsilon.$$

Thus F is continuous at 0, as required

Next we show that if F is continuous at 0 then $F\big(\overline{B}(0,1)\big)$ is bounded. Suppose that F is continuous at 0. Choose $\delta>0$ so that for all $u\in U$ we have $\|u\|\leq\delta\implies\|F(u)\|\leq 1$. Let $m=\frac{1}{\delta}$. For $x\in U$, when $x=0$ we have $\|F(x)\|=0\leq m$ and when $0<\|x\|\leq 1$ we have

$$\|F(x)\|=\Big\|\tfrac{\|x\|}{\delta}\,F\big(\tfrac{\delta x}{\|x\|}\big)\Big\|=\tfrac{\|x\|}{\delta}\Big\|F\big(\tfrac{\delta x}{\|x\|}\big)\Big\|\leq\tfrac{\|x\|}{\delta}=m\|x\|\leq m.$$

Thus $F\big(\overline{B}(0,1)\big)$ is bounded, as required.

Finally we show that if $F\big(\overline{B}(0,1)\big)$ is bounded then F is Lipschitz continuous. Suppose that $F\big(\overline{B}(0,1)\big)$ is bounded. Choose $m>0$ so that $\|F(u)\|\leq m$ for all $u\in U$ with $\|u\|\leq 1$. Let $x,y\in U$. If $x=y$ then $\big\|F(x)-F(y)\big\|=0$. Suppose that $x\neq y$. Then we have $\big\|\frac{x-y}{\|x-y\|}\big\|=1$ so that $\big\|F\big(\frac{x-y}{\|x-y\|}\big)\big\|\leq m$ and so

$$\big\|F(x)-F(y)\big\|=\big\|F(x-y)\big\|=\|x-y\|\big\|F\big(\tfrac{x-y}{\|x-y\|}\big)\big\|\leq m\|x-y\|.$$

Thus F is Lipschitz continuous with Lipschitz constant m, as required.

3.36 Example: Let $\mathbb{F}=\mathbb{R}$ so $\mathcal{C}[a,b]=\big(\mathcal{C}[a,b],\mathbb{R}\big)$. Define $L:\big(\mathcal{C}[a,b],d_\infty\big)\to\big(\mathcal{C}[a,b],d_\infty\big)$ by $L(f)(x)=\displaystyle\int_a^x f(t)\,dt$. Show that L is Lipschitz continuous.

Solution: Let $f\in\mathcal{C}[a,b]$ with $\|f\|_\infty\leq 1$, that is with $\max\limits_{a\leq x\leq b}|f(x)|\leq 1$. Then

$$\big\|F(f)\big\|_\infty=\max_{a\leq x\leq b}\left|\int_a^x f(t)\,dt\right|\leq\max_{a\leq x\leq b}\int_a^x 1\,dt=\max_{a\leq x\leq b}|x-a|=|b-a|.$$

Thus $F\big(\overline{B}(0,1)\big)$ is bounded and so F is uniformly continuous.

3.37 Example: Let $\mathbb{F}=\mathbb{R}$. Let $\mathcal{C}^1[0,1]$ be the set of continuously differentiable maps $f:[0,1]\to\mathbb{R}$. Define $D:\big(\mathcal{C}^1[0,1],d_\infty\big)\to\big(\mathcal{C}[0,1],d_\infty\big)$ by $D(f)=f'$. Show that D is not continuous.

Solution: For $n\in\mathbb{Z}^+$, define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=x^n$. Then $f_n\in\mathcal{C}^1[a,b]$, and $\|f_n\|_\infty=\max\limits_{0\leq x\leq 1}|x^n|=1$ so that $f_n\in B(0,1)$, and $\big\|D(f_n)\big\|_\infty=\max\limits_{0\leq x\leq 1}\big|n\,x^{n-1}\big|=n$. Thus $D\big(\overline{B}(0,1)\big)$ is not bounded, so D is not continuous $\big($at any point $g\in\mathcal{C}[0,1]\big)$.

3.38 Theorem: *Let U be an n-dimensional normed linear space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $\{u_1,\cdots,u_n\}$ be any basis for U and let $\phi:\mathbb{F}^n\to U$ be the associated vector space isomorphism given by $\phi(t)=\sum\limits_{k=1}^n t_k u_k$. Then both ϕ and ϕ^{-1} are Lipschitz continuous.*

Proof: Let $M=\Big(\sum\limits_{k=1}^n\|u_k\|^2\Big)^{1/2}$. For $t\in\mathbb{F}^n$ we have

$$\begin{aligned}\big\|\phi(t)\big\|&=\Big\|\sum_{k=1}^n t_k u_k\Big\|\leq\sum_{k=1}^n|t_k|\,\|u_k\|\quad\text{, by the Triangle Inequality,}\\&\leq\Big(\sum_{k=1}^n|t_k|^2\Big)^{1/2}\Big(\sum_{k=1}^n\|u_k\|^2\Big)^{1/2}\quad\text{, by the Cauchy-Schwarz Inequality,}\\&=M\|t\|.\end{aligned}$$

For all $s,t\in\mathbb{F}^n$, $\big\|\phi(s)-\phi(t)\big\|=\big\|\phi(s-t)\big\|\leq M\,\|s-t\|$, so ϕ is Lipschitz continuous.

Note that the map $N:U\to\mathbb{R}$ given by $N(x)=\|x\|$ is (uniformly) continuous, indeed we can take $\delta=\epsilon$ in the definition of continuity. Since ϕ and N are both continuous, so is the composite $G=N\circ\phi:\mathbb{F}^n\to\mathbb{R}$, which given by $G(t)=\big\|\phi(t)\big\|$. By the Extreme Value Theorem, the map G attains its minimum value on the unit sphere $\big\{t\in\mathbb{F}^n\big|\|t\|=1\big\}$, which is compact. Let $m=\min\limits_{\|t\|=1}G(t)=\min\limits_{\|t\|=1}\big\|\phi(t)\big\|$. Note that $m>0$ because when $t\neq 0$ we have $\phi(t)\neq 0$ (since ϕ is a bijective linear map) and hence $\|\phi(t)\|\neq 0$. For $t\in\mathbb{F}^n$, if $\|t\|>1$ then we have $\big\|\frac{t}{\|t\|}\big\|=1$ so, by the choice of m,

$$\big\|\phi(t)\big\|=\|t\|\,\Big\|\phi\big(\tfrac{t}{\|t\|}\big)\Big\|\geq\|t\|\cdot m>m.$$

It follows that for all $t\in\mathbb{F}^n$, if $\big\|\phi(t)\big\|\leq m$ then $\|t\|\leq 1$. Since ϕ is bijective, it follows that for $x\in U$, if $\|x\|\leq m$ then $\big\|\phi^{-1}(x)\big\|\leq 1$. Thus for all $x\in U$, if $x=0$ then $\|\phi^{-1}(x)\|=0=\frac{\|x\|}{m}$ and if $x\neq 0$ then since $\big\|\frac{mx}{\|x\|}\big\|=m$ we have

$$\big\|\phi^{-1}(x)\big\|=\tfrac{\|x\|}{m}\,\big\|\phi^{-1}\big(\tfrac{mx}{\|x\|}\big)\big\|\leq\tfrac{\|x\|}{m}.$$

For all $x,y\in U$, we have $\big\|\phi^{-1}(x)-\phi^{-1}(y)\big\|=\big\|\phi^{-1}(x-y)\big\|\leq\frac{1}{m}\,\|x-y\|$, so ϕ^{-1} is Lipschitz continuous.

3.39 Corollary: *When U and V are finite-dimensional normed linear spaces, every linear map $F : U \to V$ is Lipschitz continuous.*

Proof: Let U and V be finite-dimensional vector spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $F : U \to V$ be linear. Let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ be bases for U and V, and let $\phi : \mathbb{F}^n \to U$ and $\psi : \mathbb{F}^m \to V$ be the isomorphisms given by $\phi(t) = \sum_{k=1}^{n} t_k u_k$ and $\psi(s) = \sum_{k=1}^{m} s_k v_k$. Since ψ^{-1} and ϕ are both linear, the composite $G = \psi^{-1} F \phi : \mathbb{F}^n \to \mathbb{F}^m$ is linear, hence continuous (linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$, using the standard metric, are continuous). By the above theorem, we know that ψ and ϕ^{-1} are continuous, and so the composite map $F = \psi\, G \phi^{-1}$ is continuous, hence also Lipschitz continuous, by Theorem 3.35.

3.40 Corollary: *Any two norms on a finite-dimensional vector space U induce the same topology on U.*

Proof: Let U have two norms $\|\ \|_1$ and $\|\ \|_2$, inducing two metrics d_1 and d_2, determining two topologies on U. Let $I : (U, d_1) \to (U, d_2)$ be the identity map (given by $I(x) = x$), and let $J = I^{-1} : (U, d_2) \to (U, d_1)$ (so J is also the identity map). By the above corollary, I and J are continuous. Let $A \subseteq U$. If A is open in (U, d_1) then, since J is continuous, $J^{-1}(A)$ is open in (U, d_2), but $J^{-1}(A) = I(A) = A$ and so A is open in (U, d_2). Similarly, if A is open in (U, d_2) then $A = J(A) = I^{-1}(A)$ is open in (U, d_1).

3.41 Corollary: *Let U be a finite-dimensional vector space. Let $\|\ \|_1$ and $\|\ \|_2$ be two norms on U inducing the two metric d_1 and d_2 on U. Let $(x_n)_{n \geq 1}$ be a sequence in U, and let $a \in U$. Then $x_n \to a$ in (U, d_1) if and only if $x_n \to a$ in (U, d_2).*

Proof: Let $I : (U, d_1) \to (U, d_2)$ be the identity map (given by $I(x) = x$). By Corollary 3.38, I is Lipschitz continuous. Let $\ell \geq 0$ be a Lipschitz constant for I. Suppose that $x_n \to a$ in (U, d_1). Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that when $n \geq m$ we have $d_1(x_n, a) < \frac{\epsilon}{\ell+1}$. Then when $n \geq m$ we have $d_2(x_n, a) = d_2\big(I(x_n), I(a)\big) \leq \ell \cdot d_1(x_n, a) < \ell \cdot \frac{\epsilon}{\ell+1} < \epsilon$. Thus $x_n \to a$ in (U, d_2). Similarly, since the identity map $J : (U, d_2) \to (U, d_1)$ is Lipschitz continuous, it follows that if $x_n \to a$ in (U, d_2) then $x_n \to a$ in (U, d_1). We remark that I and J might have different Lipschitz constants (even though I and J are both the identity map from U to itself).