# Chapter 2. Metric Spaces

## Inner Product Spaces

**2.1 Definition:** Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $U$ be a vector space over $\mathbb{F}$. An **inner product** on $U$ (over $\mathbb{F}$) is a function $\langle\ ,\ \rangle : U\times U\to\mathbb{F}$ (meaning that if $u,v\in U$ then $\langle u,v\rangle\in\mathbb{F}$) such that for all $u,v,w\in U$ and all $t\in\mathbb{F}$ we have

(1) (Sesquilinearity) $\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle\ ,\ \langle tu,v\rangle=t\,\langle u,v\rangle$,
$\langle u,v+w\rangle=\langle u,v\rangle+\langle u,w\rangle\ ,\ \langle u,tv\rangle=\overline{t}\,\langle u,v\rangle$,

(2) (Conjugate Symmetry) $\langle u,v\rangle=\overline{\langle v,u\rangle}$, and

(3) (Positive Definiteness) $\langle u,u\rangle\in\mathbb{R}$ with $\langle u,u\rangle\geq 0$ and $\langle u,u\rangle=0\iff u=0$.

For $u,v\in U$, $\langle u,v\rangle$ is called the **inner product** of $u$ with $v$. We say that $u$ and $v$ are **orthogonal** when $\langle u,v\rangle=0$. An **inner product space** (over $\mathbb{F}$) is a vector space over $\mathbb{F}$ equipped with an inner product. Given two inner product spaces $U$ and $V$ over $\mathbb{F}$, a linear map $L:U\to V$ is called a **homomorphism** of inner product spaces (or we say that $L$ **preserves inner product**) when $\big\langle L(x),L(y)\big\rangle=\langle x,y\rangle$ for all $x,y\in U$. A bijective homomorphism is called an **isomorphism**.

**2.2 Example:** The **standard inner product** on $\mathbb{F}^n$ is given by

$$\langle u,v\rangle=v^*u=\sum_{k=1}^n u_k\overline{v_k}.$$

**2.3 Example:** We write $\mathbb{F}^\omega$ to denote the **space of sequences** in $\mathbb{F}$, and we write $\mathbb{F}^\infty$ to denote the **space of eventually zero sequences** in $\mathbb{F}$, that is

$$\mathbb{F}^\omega=\big\{u=(u_1,u_2,u_3,\cdots)\,\big|\text{ each }u_k\in\mathbb{F}\big\}$$
$$\mathbb{F}^\infty=\big\{u\in\mathbb{F}^\omega\,|\,\exists n\in\mathbb{Z}^+\ \forall k\geq n\ \ u_k=0\big\}.$$

Recall that $\mathbb{F}^\infty$ is a countable-dimensional vector space with standard basis $\{e_1,e_2,e_3,\cdots\}$ where $e_1=(1,0,0,\cdots)$, $e_2=(0,1,0,\cdots)$ and so on. Note that $\{e_1,e_2,e_3,\cdots\}$ spans $\mathbb{F}^\infty$ (and *not* all of $\mathbb{F}^\omega$) because linear combinations are given by *finite* sums (not by infinite series). The **standard inner product** on $\mathbb{F}^\infty$ is given by

$$\langle u,v\rangle=v^*u=\sum_{k=1}^\infty u_k\overline{v_k}\,.$$

Note that the sum here does make sense because only finitely many of the terms are nonzero (but we cannot use the same formula to give an inner product on $\mathbb{F}^\omega$).

**2.4 Example:** For $a,b\in\mathbb{R}$ with $a\leq b$, we write

$$\mathcal{B}[a,b]=\mathcal{B}\big([a,b],\mathbb{F}\big)=\big\{f:[a,b]\to\mathbb{F}\,\big|\,f\text{ is bounded}\big\}$$
$$\mathcal{C}[a,b]=\mathcal{C}\big([a,b],\mathbb{F}\big)=\big\{f:[a,b]\to\mathbb{F}\big|f\text{ is continuous}\big\}.$$

The **standard inner product** on $\mathcal{C}[a,b]$ is given by

$$\langle f,g\rangle=\int_a^b f\,\overline{g}=\int_a^b f(x)\,\overline{g(x)}\,dx.$$

Note that this is positive definite because $\langle f,f\rangle=\int_a^b\big|f(x)\big|^2dx\geq 0$ and if $\int_a^b\big|f(x)\big|^2dx=0$ then we must have $f(x)=0$ for all $x\in[a,b]$, using the fact that if $g$ is non-negative and continuous on $[a,b]$ with $\int_a^b g(x)\,dx=0$, then we must have $g(x)=0$ for all $x\in[a,b]$.

**2.5 Theorem:** *Let $U$ be an inner product space and let $u, v \in U$. If $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in U$, or if $\langle x, u\rangle = \langle x, v\rangle$ for all $x \in U$, then $u = v$.*

Proof: Suppose that $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in U$. Then $\langle u - v, x\rangle = \langle u, x\rangle - \langle v, x\rangle = 0$ for all $x \in U$. In particular, taking $x = u - v$ we have $\langle u - v, u - v\rangle = 0$ so that $u = v$, by positive definiteness. Similarly, if $\langle x, u\rangle = \langle x, v\rangle$ for all $x \in U$ then $u = v$.

**2.6 Definition:** Let $U$ be an inner product space. For $u \in U$, we define the **norm** (or **length**) of $u$ to be

$$\|u\| = \sqrt{\langle u, u\rangle}.$$

**2.7 Theorem:** *(Basic Properties of Inner Product and Norm) Let $U$ be an inner product space. For $u, v \in U$ and $t \in \mathbb{R}$ we have*

*(1) (Scaling) $\|tu\| = |t|\,\|u\|$,*
*(2) (Positive Definiteness) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$,*
*(3) $\|u \pm v\|^2 = \|u\|^2 \pm 2\,\mathrm{Re}\,\langle u, v\rangle + \|v\|^2$,*
*(4) (Pythagoras' Theorem) If $\mathbb{F} = \mathbb{R}$ then $\langle u, v\rangle = 0 \iff \|u + v\|^2 = \|u\|^2 + \|v\|^2$,*
*(5) (Parallelogram Law) $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$,*
*(6) (Polarization Identity) If $\mathbb{F} = \mathbb{R}$ then $\langle u, v\rangle = \frac{1}{4}\big(\|u + v\|^2 - \|u - v\|^2\big)$ and if $\mathbb{F} = \mathbb{C}$ then $\langle u, v\rangle = \frac{1}{4}\big(\|u + v\|^2 + i\|u + iv\|^2 - \|u - v\|^2 - i\|u - iv\|^2\big)$,*
*(7) (The Cauchy-Schwarz Inequality) $\big|\langle u, v\rangle\big| \leq \|u\|\,\|v\|$ with $|\langle u, v\rangle| = \|u\|\,\|v\|$ if and only if $\{u, v\}$ is linearly dependent, and*
*(8) (The Triangle Inequality) $\big|\|u\| - \|v\|\big| \leq \|u + v\| \leq \|u\| + \|v\|$.*

Proof: You will have already seen a proof in a linear algebra course, but let us remind you of some of the proofs. The first 6 parts are all easy to prove. To prove Part 7, suppose first that $\{u, v\}$ is linearly dependent. Then one of $u$ and $v$ is a multiple of the other, say $v = tu$ with $t \in \mathbb{F}$. Then we have $|\langle u, v\rangle| = |\langle u, tu\rangle| = \big|\,\overline{t}\,\langle u, u\rangle\big| = |t|\,\|u\|^2 = \|u\|\,\|tu\| = \|u\|\,\|v\|$. Next suppose that $\{u, v\}$ is linearly independent. Then $1 \cdot v + t \cdot u \neq 0$ for all $t \in \mathbb{F}$, so in particular $v - \frac{\langle v,u\rangle}{\|u\|^2}\,u \neq 0$. Thus we have

$$\begin{aligned} 0 < \Big\|v - \tfrac{\langle v,u\rangle}{\|u\|^2}\,u\Big\|^2 &= \|v\|^2 - 2\,\mathrm{Re}\,\big\langle v, \tfrac{\langle v,u\rangle}{\|u\|^2}\,u\big\rangle + \Big\|\tfrac{\langle v,u\rangle}{\|u\|^2}\,u\Big\|^2 \\ &= \|v\|^2 - 2\,\mathrm{Re}\,\tfrac{\overline{\langle v,u\rangle}\langle v,u\rangle}{\|u\|^2} + \tfrac{|\langle v,u\rangle|^2}{\|u\|^2} = \|v\|^2 - \tfrac{|\langle v,u\rangle|^2}{\|u\|^2} \end{aligned}$$

so that $\frac{|\langle u,v\rangle|^2}{|u|^2} = \frac{|\langle v,u\rangle|^2}{\|u\|^2} < \|v\|^2$, and hence $|\langle u, v\rangle| \leq \|u\|\,\|v\|$. This proves Part 7.

Using Parts 3 and 7, and the inequality $|\mathrm{Re}\,(z)\| \leq \|z\|$ for $z \in \mathbb{C}$ (which follows from Pythagoras' Theorem in $\mathbb{R}^2$), we have

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + 2\,\mathrm{Re}\,\langle u, v\rangle + \|v\|^2 \leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 = \big(\|u\| + \|v\|\big)^2. \end{aligned}$$

Taking the square root on both sides gives $\|u + v\| \leq \|u\| + \|v\|$. Finally note that $\|u\| = \|(u+v) - v\| \leq \|u+v\| + \|-v\| = \|u+v\| + \|v\|$ so that we have $\|u\| - \|v\| \leq \|u+v\|$, and similarly $\|v\| - \|u\| \leq \|u + v\|$, hence $\big|\|u\| - \|v\|\big| \leq \|u + v\|$. This proves Part 8.

## Normed Linear Spaces

**2.8 Definition:** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U$ be a vector space over $\mathbb{F}$. A **norm** on $U$ is a function $\| \ \| : U \to \mathbb{R}$ (meaning that if $u \in U$ then $\|u\| \in \mathbb{R}$) such that for all $u, v \in U$ and all $t \in \mathbb{R}$ we have

(1) (Scaling) $\|tu\| = |t| \, \|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$, and
(3) (Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$.

For $u \in U$, the real number $\|u\|$ is called the **norm** (or **length**) of $u$, and we say that $u$ is a **unit vector** when $\|u\| = 1$. A **normed linear space** (over $\mathbb{R}$) is a vector space equipped with a norm. Given two normed linear spaces $U$ and $V$ over $\mathbb{R}$, a linear map $L : U \to V$ is called a **homomorphism** of normed linear spaces (or we say that $L$ **preserves norm**) when $\big\|L(x)\big\| = \|x\|$ for all $x \in U$. A bijective homomorphism is called an **isomorphism**.

**2.9 Example:** The standard inner product on $\mathbb{F}^n$ induces the **standard norm** on $\mathbb{F}^n$, which is also called the **2-norm** on $\mathbb{F}^n$, given by

$$\|u\|_2 = \|u\| = \sqrt{\langle u, u\rangle} = \Big( \sum_{k=1}^{n} |u_k|^2 \Big)^{1/2}.$$

We also define the **1-norm** and the **supremum norm** (also called the **infinity norm**) on $\mathbb{F}^n$ by

$$\|u\|_1 = \sum_{k=1}^{n} |u_k| \, ,$$
$$\|u\|_\infty = \max\big\{ |u_1|, |u_2|, \cdots, |u_n| \big\}.$$

**2.10 Example:** The standard inner product on $\mathbb{F}^\infty$ induces the **standard norm**, also called the **2-norm**, on $\mathbb{F}^\infty$ given by

$$\|u\|_2 = \|u\| = \sqrt{\langle u, u\rangle} = \Big( \sum_{k=1}^{\infty} |u_k|^2 \Big)^{1/2}.$$

We also define the **1-norm** and the **supremum norm** (also called the **infinity norm**) on $\mathbb{F}^\infty$ by

$$\|u\|_1 = \sum_{k=1}^{\infty} |u_k| \, ,$$
$$\|u\|_\infty = \sup\big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\} = \max\big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\}.$$

**2.11 Definition:** For $u \in \mathbb{F}^w$, we define the **1-norm** of $u$, the **2-norm** of $u$, and the **supremum norm** (or **infinity norm**) of $u$ to be the extended real numbers

$$\|u\|_1 = \sum_{k=1}^{\infty} |u_k| \ , \ \|u\|_2 = \Big( \sum_{k=1}^{\infty} |u_k|^2 \Big)^{1/2} \ \text{ and } \ \|u\|_\infty = \sup\big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\}$$

Note that these can be infinite (so they are not actually norms according to Definition 2.8), with $\|u\|_\infty = \infty$ in the case that $\big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\}$ is not bounded above (by a real number). Define

$$\ell_1 = \ell_1(\mathbb{F}) = \big\{ u \in \mathbb{F}^\omega \,\big|\, \|u\|_1 < \infty \big\},$$
$$\ell_2 = \ell_2(\mathbb{F}) = \big\{ u \in \mathbb{F}^\omega \,\big|\, \|u\|_2 < \infty \big\},$$
$$\ell_\infty = \ell_\infty(\mathbb{F}) = \big\{ u \in \mathbb{F}^\omega \,\big|\, \|u\|_\infty < \infty \big\}.$$

For $p = 1, 2, \infty$, we shall show (in Theorem 2.14 below) that the $p$-norm is a (well-defined, finite-valued) norm on $\ell_p$.

**2.12 Example:** For the sequence $(u_k)_{k\geq 1}$ in $\mathbb{R}$ given by $u_k = \frac{1}{2^k}$, we have

$$\|u\|_1 = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \ , \ \|u\|_2 = \Big(\sum_{k=1}^{\infty} \frac{1}{4^k}\Big)^{1/2} = \frac{1}{\sqrt{3}} \ , \text{ and } \ \|u\|_\infty = |u_1| = \tfrac{1}{2}.$$

For the sequence $(v_k)_{k\geq 1}$ given by $v_k = \frac{1}{k}$, we have

$$\|v\|_1 = \sum_{k=1}^{\infty} \frac{1}{k} = \infty \ , \ \|v\|_2 = \Big(\sum_{k=1}^{\infty} \frac{1}{k^2}\Big)^{1/2} < \infty \ , \text{ and } \ \|v\|_\infty = |v_1| = 1$$

$\big($in fact $\|v\|_2 = \frac{\pi}{\sqrt{6}}\big)$. For the sequence $(w_k)_{k\geq 1}$ given by $w_k = \frac{1}{\sqrt{k}}$ we have

$$\|w\|_1 = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty \ , \ \|w\|_2 = \Big(\sum_{k=1}^{\infty} \frac{1}{k}\Big)^{1/2} = \infty \ , \text{ and } \ \|w\|_\infty = |w_1| = 1.$$

**2.13 Theorem:** *We have* $\mathbb{F}^\infty \subseteq \ell_1 \subseteq \ell_2 \subseteq \ell_\infty \subseteq \mathbb{F}^\omega$.

Proof: If $u \in \mathbb{F}^\infty$ then $\|u\|_1 = \sum_{k=1}^{\infty} |u_k| < \infty$ (because only finitely many of the terms are nonzero) and so $u \in \ell_1$. Thus we have $\mathbb{F}^\infty \subseteq \ell_1$.

Suppose that $u \in \ell_1$. Since $\|u\|_1 = \sum |u_k| < \infty$, we know that $|u_k| \to 0$ (by the Divergence Test from calculus) so we can choose $m \in \mathbb{Z}^+$ such that when $k \geq m$ we have $|a_k| \leq 1$. Then for $k \geq m$ we have $|a_k|^2 \leq |a_k|$. Since $\sum |a_k|$ converges and $|a_k|^2 \leq |a_k|$ for $k \geq m$, it follows that $\sum |a_k|^2$ converges by the Comparison Test (from calculus). Thus $\|u\|_2 = \Big(\sum_{k=1}^{\infty} |a_k|^2\Big)^{1/2} < \infty$ and so $u \in \ell_2$. Thus we have $\ell_1 \subseteq \ell_2$.

Suppose $u \in \ell_2$. Since ${\|u\|_2}^2 = \sum_{k=1}^{\infty} |a_k|^2 < \infty$ we have $|a_k|^2 \to 0$ (by the Divergence Test) hence also $|a_k| \to 0$. Choose $m \in \mathbb{Z}^+$ such that when $k \geq m$ we have $|a_k| \leq 1$. Then the set $\big\{|a_k| \,\big|\, k \in \mathbb{Z}^+\big\}$ is bounded above by $M = \max\big\{|a_1|, |a_2|, \cdots, |a_{m-1}|, 1\big\}$, and so we have $\|u\|_\infty \leq M$, and hence $u \in \ell_\infty$. Thus we have $\ell_2 \subseteq \ell_\infty$.

Finally note that $\ell_\infty \subseteq \mathbb{F}^\omega$, by definition.

**2.14 Theorem:**

*(1) The space $\ell_2$ is an inner product space with inner product defined by*

$$\langle u, v\rangle = \sum_{k=1}^{\infty} u_k \overline{v_k}.$$

*(2) For $p = 1, 2, \infty$, the space $\ell_p$ is a normed linear space with norm given by $\|u\|_p$.*

Proof: To prove Part 1, we must verify that if $u, v \in \ell_2$ then the sum $\sum_{k=1}^{\infty} u_k \overline{v_k}$ converges so that the inner product is well-defined. Let $u, v \in \ell_2$. We claim that $\sum u_k \overline{v_k}$ converges absolutely, that is $\sum |u_k v_k|$ converges. For $n \in \mathbb{Z}^+$, let $x = (|u_1|, |u_2|, \cdots, |u_n|) \in \mathbb{R}^n$ and $y = (|v_1|, |v_2|, \cdots, |v_n|) \in \mathbb{R}^n$, and note that $\|x\|_2 = \Big(\sum_{k=1}^{n} |u_k|^2\Big)^{1/2} \leq \Big(\sum_{k=1}^{\infty} |u_k|^2\Big)^{1/2} = \|u\|_2$ and similarly $\|y\|_2 \leq \|v\|_2$. By applying the Cauchy-Schwarz Inequality in $\mathbb{R}^n$ we have $\sum_{k=1}^{n} |u_k v_k| = \big|\langle x, y\rangle\big| \leq \|x\|_2 \, \|y\|_2 \leq \|u\|_2 \|v\|_2$. By the Monotone Convergence Theorem, since $\sum_{k=1}^{n} |u_k v_k| \leq \|u\|_2 \|v\|_2$ for every $n \in \mathbb{Z}^+$, it follows that $\sum |u_k v_k|$ converges with $\sum_{k=1}^{\infty} |u_k v_k| \leq \|u\|_2 \|v\|_2$. Thus $\sum u_k \overline{v_k}$ converges absolutely, as claimed.

All students will have seen that absolute convergence implies convergence for sequences in $\mathbb{R}$, that is if for a sequence $(x_k)$ in $\mathbb{R}$, if $\sum |x_k|$ converges then so does $\sum x_k$. Let us show that the same is true for a sequence $(z_k)$ in $\mathbb{C}$. Suppose that $\sum |z_k|$ converges, where $z_k = x_k + iy_k$ with $x_k, y_k \in \mathbb{R}$. Since $|x_k| \leq |z_k|$ and $|y_k| \leq |z_k|$ for all $k$, it follows that $\sum |x_k|$ and $\sum |y_k|$ both converge (by the Comparison Test) and hence $\sum x_k$ and $\sum y_k$ both converge (since absolute convergence implies convergence for sequences in $\mathbb{R}$). Since $\sum x_k$ and $\sum y_k$ converge, it follows that $\sum z_k$ converges in $\mathbb{C}$ $\big($indeed if $u_n \to u$ in $\mathbb{R}$ and $v_n \to v$ in $\mathbb{R}$, then $u_n + iv_n \to u + iv$ in $\mathbb{C}$ because $\big|(u_n + iv_n) - (u + iv)\big| \leq |u_n - u| + |v_n - v|\big)$.

Thus, whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, since $\sum u_k \overline{v_k}$ converges absolutely, it follows that $\sum u_k \overline{v_k}$ converges so that $\langle u, v\rangle = \sum_{k=1}^{\infty} u_k \overline{v_k} \in \mathbb{F}$ (so the inner product is well-defined).

We leave it as an exercise to verify that the 3 properties which define an inner product (in Definition 2.1) are all satisfied.

Because $\langle u, v\rangle = \sum_{k=1}^{\infty} u_k \overline{v_k}$ gives a (well-defined, finite-valued) inner product on $\ell_2$, it follows (from Theorem 2.7) that this inner product induces a (well-defined, finite-valued) norm given by $\|u\| = \sqrt{\langle u, u\rangle} = \Big(\sum_{k=1}^{\infty} |u_k|^2\Big)^{1/2}$. This is the formula we used to define the 2-norm, so the 2-norm is a norm on $\ell_2$. To complete the proof of Part 2 of the theorem, it remains to show that $\|u\|_1$ and $\|u\|_\infty$ are norms on $\ell_1$ and $\ell_\infty$. We leave this as an exercise $\big($but we remark that that unlike the situation for the inner product $\langle u, v\rangle$, we do not need to verify that $\|u\|_1$ and $\|u\|_\infty$ are finite-valued because this is immediate from the definition of $\ell_1$ and $\ell_\infty\big)$.

**2.15 Example:** For $a, b \in \mathbb{R}$ with $a \leq b$, recall that

$$\mathcal{B}[a,b] = \mathcal{B}\big([a,b],\mathbb{F}\big) = \big\{f : [a,b] \to \mathbb{F} \,\big|\, f \text{ is bounded}\big\},$$
$$\mathcal{C}[a,b] = \mathcal{C}\big([a,b],\mathbb{F}\big) = \big\{f : [a,b] \to \mathbb{F} \big| f \text{ is continuous}\big\}.$$

For $f \in \mathcal{C}[a,b]$, we define the **1-norm** and the **2-norm** of $f$ to be

$$\|f\|_1 = \int_a^b |f|\,,$$
$$\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}.$$

and for $f \in \mathcal{B}[a,b]$, we define the **supremum norm** (also called the **infinity norm**) of $f$ to be

$$\|f\|_\infty = \sup\Big\{\big|f(x)\big| \,\Big|\, a \leq x \leq b\Big\}.$$

We leave it as an exercise to show that these are indeed norms (in particular, show that the 1-norm is positive-definite). The 2-norm on $\mathcal{C}[a,b]$ is induced by the inner product on $\mathcal{C}[a,b]$ given by

$$\langle f, g\rangle = \int_a^b f\,\overline{g} = \int_a^b f(x)\,\overline{g(x)}\,dx\,.$$

## Metric Spaces

**2.16 Definition:** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U$ be a normed linear space. For $u, v \in U$, we define the **distance** between $u$ and $v$ to be

$$d(u,v) = \|v - u\| \in \mathbb{R}.$$

**2.17 Theorem:** *Let $U$ be as normed linear space. For all $u, v, w \in U$,*
*(1) (Symmetry) $d(u,v) = d(v,u)$,*
*(2) (Positive Definiteness) $d(u,v) \geq 0$ with $d(u,v) = 0 \iff u = v$, and*
*(3) (Triangle Inequality) $d(u,w) \leq d(u,v) + d(v,w)$.*

Proof: The proof is left as an easy exercise.

**2.18 Definition:** Let $X$ be a set. A **metric** on $X$ is a map $d : X \times X \to \mathbb{R}$ such that for all $a, b, c \in X$ we have
(1) (Symmetry) $d(a,b) = d(b,a)$,
(2) (Positive Definiteness) $d(a,b) \geq 0$ with $d(a,b) = 0 \iff a = b$, and
(3) (Triangle Inequality) $d(a,c) \leq d(a,b) + d(b,c)$.

For $a, b \in X$, $d(a,b)$ is called the **distance** between $a$ and $b$. A **metric space** is a set $X$ which is equipped with a metric $d$, and we sometimes denote the metric space by $X$ and sometimes by the pair $(X, d)$. Given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, a map $f : X \to Y$ is called a **homomorphism** of metric spaces (or we say that $f$ is **distance preserving**) when $d_Y\big(f(a), f(b)\big) = d_X(a,b)$ for all $a, b \in X$. A bijective homomorphism is called an **isomorphism** or an **isometry**.

**2.19 Note:** Every inner product space is also a normed linear space, using the induced norm given by $\|u\| = \sqrt{\langle u, u\rangle}$. Every normed linear space is also a metric space, using the induced metric given by $d(u,v) = \|v - u\|$. If $U$ is an inner product space then every subspace of $U$ is also an inner product space using (the restriction of) the same inner product used in $U$. If $U$ is a normed linear space then every subspace of $U$ is also a normed linear space using the same norm. If $X$ is a metric space then so is every subset of $X$ using the same metric.

**2.20 Example:** In $\mathbb{F}^n$ (or in any subset $X \subseteq \mathbb{F}^n$), the **standard metric** (also called the **2-metric**) is given by

$$d(a,b) = d_2(a,b) = \|a - b\|_2 = \Big(\sum_{k=1}^{n} |a_k - b_k|^2\Big)^{1/2}.$$

We also have the **1-metric** and the **supremum metric** (or the **infinity metric**) given by

$$d_1(a,b) = \|a - b\|_1 = \sum_{k=1}^{\infty} |a_k - b_k| \quad \text{and}$$
$$d_\infty(a,b) = \|a - b\|_\infty = \max\big\{|a_k - b_k| \,\big|\, 1 \leq k \leq n\big\}.$$

**2.21 Exercise:** In $\mathbb{R}^3$, let $u = (1,2,5)$ and $v = (3,5,-1)$. Find $d_1(u,v)$, $d_2(u,v)$ and $d_\infty(u,v)$.

**2.22 Example:** In $\ell_1$ $\big($or in any subset $X \subseteq \ell_1\big)$, we have the **1-metric** given by

$$d_1(a,b) = \|a-b\|_1 = \sum_{k=1}^{\infty} |a_k - b_k| \,.$$

In $\ell_2$ $\big($or in any nonempty subset $X \subseteq \ell_2\big)$ we have the **2-metric** given by

$$d_2(a,b) = \|a-b\|_2 = \Big( \sum_{k=1}^{\infty} |a_k - b_k|^2 \Big)^{1/2}.$$

In $\ell_\infty$ $\big($or in any nonempty subset $X \subseteq \ell_\infty\big)$ we have the **supremum metric** (or the **infinity metric**) given by

$$d_\infty(a,b) = \|a-b\|_\infty = \sup\big\{ |a_k - b_k| \,\big|\, k \in \mathbb{Z}^+ \big\}.$$

Since $\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_2 \subseteq \ell_\infty$, we could (if we wanted) use any of the metrics $d_p$ in the space $\mathbb{R}^\infty$ (just as we can use any of the metrics $d_p$ in $\mathbb{R}^n$). We could also use any of the metrics $d_p$ in the space $\ell_1$, and we could use either of the metrics $d_2$ or $d_\infty$ in the space $\ell_2$.

**2.23 Exercise:** Let $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ be the sequences in $\ell_1$ given by $u_k = \frac{1}{2^k}$ and $v_k = \frac{1}{3^k}$. Find $d_1(u,v)$, $d_2(u,v)$ and $d_\infty(u,v)$.

**2.24 Example:** Let $a,b \in \mathbb{R}$ with $a \leq b$. In $\mathcal{C}[a,b]$ $\big($or in any subset $X \subseteq \mathcal{C}[a,b]\big)$, we have the **1-metric** and the **2-metric**, given by

$$d_1(f,g) = \|f-g\|_1 = \int_a^b |f-g| = \int_a^b \big|f(x)-g(x)\big|\,dx\,,$$
$$d_2(f,g) = \|f-g\|_2 = \left( \int_a^b |f-g|^2 \right)^{1/2} = \left( \int_a^b \big|f(x)-g(x)\big|^2 dx \right)^{1/2},$$

and in $\mathcal{B}[a,b]$ $\big($or in any subset $X \subseteq \mathcal{B}[a,b]\big)$ we have the supremum metric (also called the **infinity metric**) given by

$$d_\infty(f,g) = \|f-g\|_\infty = \sup\Big\{ \big|f(x)-g(x)\big| \,\Big|\, a \leq x \leq b \Big\}.$$

**2.25 Exercise:** Define $f,g : [0,1] \to \mathbb{R}$ be $f(x) = x$ and $g(x) = x^2$. Find $d_1(f,g)$, $d_2(f,g)$ and $d_\infty(f,g)$.

**2.26 Example:** For any nonempty set $X \neq \emptyset$, the **discrete metric** on $X$ is given by $d(x,y) = 1$ for all $x,y \in X$ with $x \neq y$ and $d(x,x) = 0$ for all $x \in X$.

**2.27 Remark:** There are, in fact, a ridiculously vast number of metrics that one could define on $\mathbb{R}$. For example, if we let $f : \mathbb{R} \to \mathbb{R}$ be any bijective map then we can define a metric on $\mathbb{R}$ by $d(x,y) = |f(x) - f(y)|$. But in this course, we shall usually concern ourselves with the metrics described in the above examples.

## Open and Closed Sets in Metric Spaces

**2.28 Definition:** Let $X$ be a metric space. For $a \in X$ and $0 < r \in \mathbb{R}$, the **open ball**, the **closed ball**, and the (open) **punctured ball** in $X$ centred at $a$ of radius $r$ are defined to be the sets

$$B(a,r) = B_X(a,r) = \{x \in X \mid d(x,a) < r\},$$
$$\overline{B}(a,r) = \overline{B}_X(a,r) = \{x \in X \mid d(x,a) \leq r\},$$
$$B^*(a,r) = B^*_X(a,r) = \{x \in X \mid 0 < d(x,a) < r\}.$$

When the metric on $X$ is denoted by $d_p$ with $1 \leq p \leq \infty$, we often write $B(a,r)$, $\overline{B}(a,r)$ and $B^*(a,r)$ as $B_p(a,r)$, $\overline{B}_p(a,r)$ and $B^*_p(a,r)$. For $A \subseteq X$, we say that $A$ is **bounded** when $A \subseteq B(a,r)$ for some $a \in X$ and some $0 < r \in \mathbb{R}$.

**2.29 Exercise:** Draw a picture of the open balls $B_1(0,1)$, $B_2(0,1)$ and $B_\infty(0,1)$ in $\mathbb{R}^2$ (using the metrics $d_1$, $d_2$ and $d_\infty$).

**2.30 Definition:** Let $X$ be a metric space. For $A \subseteq X$, we say that $A$ is **open** (in $X$) when for every $a \in A$ there exists $r > 0$ such that $B(a,r) \subseteq A$, and we say that $A$ is **closed** (in $X$) when its complement $A^c = X \setminus A$ is open in $X$.

**2.31 Example:** Let $X$ be a metric space and let $a \in X$. Show that $\{a\}$ is closed in $X$.

Solution: To show that $\{a\}$ is closed, we shall show that $\{a\}^c = X \setminus \{a\}$ is open. Let $b \in X \setminus \{a\}$. Let $r = d(a,b)$ and note that since $b \neq a$ we have $r > 0$. Let $x \in B(b,r)$. Then $d(x,b) < r = d(a,b)$. Since $d(x,b) \neq d(a,b)$ we have $x \neq a$ so that $x \in X \setminus \{a\}$. Thus $B(b,r) \subseteq X \setminus \{a\}$. This proves that $X \setminus \{a\}$ is open, and so $\{a\}$ is closed.

**2.32 Example:** Let $X$ be a metric space. Show that for $a \in X$ and $0 < r \in \mathbb{R}$, the set $B(a,r)$ is open and the set $\overline{B}(a,r)$ is closed.

Solution: Let $a \in X$ and let $r > 0$. We claim that $B(a,r)$ is open. We need to show that for all $b \in B(a,r)$ there exists $s > 0$ such that $B(b,s) \subseteq B(a,r)$. Let $b \in B(a,r)$ and note that $d(a,b) < r$. Let $s = r - d(a,b)$ and note that $s > 0$. Let $x \in B(b,s)$, so we have $d(x,b) < s$. Then, by the Triangle Inequality, we have

$$d(x,a) \leq d(x,b) + d(b,a) < s + d(a,b) = r$$

and so $x \in B(a,r)$. This shows that $B(b,s) \subseteq B(a,r)$ and hence $B(a,r)$ is open.

Next we claim that $\overline{B}(a,r)$ is closed, that is $\overline{B}(a,r)^c$ is open. Let $b \in \overline{B}(a,r)^c$, that is let $b \in X$ with $b \notin \overline{B}(a,r)$. Since $b \notin \overline{B}(a,r)$ we have $d(a,b) > r$. Let $s = d(a,b) - r > 0$. Let $x \in B(b,s)$ and note that $d(x,b) < s$. Then, by the Triangle Inequality, we have

$$d(a,b) \leq d(a,x) + d(x,b) < d(x,a) + s$$

and so $d(x,a) > d(a,b) - s = r$. Since $d(x,a) > r$ we have $x \notin \overline{B}(a,r)$ and so $x \in \overline{B}(a,r)^c$. This shows that $B(b,s) \subseteq \overline{B}(a,r)^c$ and it follows that $\overline{B}(a,r)^c$ is open and hence that $\overline{B}(a,r)$ is closed.

**2.33 Example:** In $\mathbb{R}$ (using its standard metric), an open ball is the same thing as a bounded non-degenerate open interval, and a closed ball is the same thing as a bounded non-degenerate closed interval. The unbounded open intervals $(a,\infty)$, $(-\infty,b)$ are open, and the unbounded closed intervals $[a,\infty)$ and $(-\infty,b]$ are closed. The degenerate closed intervals $[a,a] = \{a\}$ are closed. The degenerate interval $(a,a) = \emptyset$ and the interval $(-\infty,\infty) = \mathbb{R}$ are both open and closed (see Theorem 2.35 below). The bounded non-degenerate half-open intervals $[a,b)$ and $(a,b]$ are neither open nor closed.

**2.34 Remark:** It is often fairly difficult to determine whether a given set is open or closed (or neither or both) directly from the definition of open and closed sets. We will be able to do this more easily after we have discussed limits of sequences and continuous functions in the next chapter.

**2.35 Theorem:** *(Basic Properties of Open Sets) Let $X$ be a metric space.*

*(1) The sets $\emptyset$ and $X$ are open in $X$.*
*(2) If $S$ is a set of open sets in $X$ then the union $\bigcup S = \bigcup\limits_{U\in S} U$ is open in $X$.*
*(3) If $S$ is a finite set of open sets in $X$ then the intersection $\bigcap S = \bigcap\limits_{U\in S} U$ is open in $X$.*

Proof: The empty set is open because any statement of the form "for all $x\in\emptyset$ $F$" (where $F$ is any statement) is considered to be true (by convention). The set $X$ is open because given $a \in X$ we can choose any value of $r > 0$ and then we have $B(a,r) \subseteq X$ by the definition of $B(a,r)$. This proves Part 1.

To prove Part 2, let $S$ be any set of open sets in $X$. Let $a \in \bigcup S = \bigcup_{U\in S} U$. Choose an open set $U \in S$ such that $a \in U$. Since $U$ is open we can choose $r > 0$ such that $B(a,r) \subseteq U$. Since $U \in S$ we have $U \subseteq \bigcup S$. Since $B(a,r) \subseteq U$ and $U \subseteq \bigcup S$ we have $B(a,r) \subseteq \bigcup S$. Thus $\bigcup S$ is open, as required.

To prove Part 3, let $S$ be a finite set of open sets in $X$. If $S = \emptyset$ then we use the convention that $\bigcap S = X$, which is open. Suppose that $S \neq \emptyset$, say $S = \{U_1, U_2, \cdots, U_m\}$ where each $U_k$ is an open set. Let $a \in \bigcap S = \bigcap_{k=1}^m U_k$. For each index $k$, since $a \in U_k$ we can choose $r_k > 0$ so that $B(a,r_k) \subseteq U_k$. Let $r = \min\{r_1, r_2, \cdots, r_m\}$. Then for each index $k$ we have $B(a,r) \subseteq B(a,r_k) \subseteq U_k$. Since $B(a,r) \subseteq U_k$ for every index $k$, it follows that $B(a,r) \subseteq \bigcap_{k=1}^m U_k = \bigcap S$. Thus $\bigcap S$ is open, as required.

**2.36 Theorem:** *(Basic Properties of Closed Sets) Let $X$ be a metric space.*

*(1) The sets $\emptyset$ and $X$ are closed in $X$.*
*(2) If $S$ is a set of closed sets in $X$ then the intersection $\bigcap S = \bigcap\limits_{K\in S} K$ is closed in $X$.*
*(3) If $S$ is a finite set of closed sets in $X$ then the union $\bigcup S = \bigcup\limits_{K\in S} K$ is closed in $X$.*

Proof: This follows from Theorem 2.35, by taking complements using the fact that for a set $S$ of subsets of $X$ we have $\big(\bigcup\limits_{A\in S} A\big)^c = \bigcap\limits_{A\in S} A^c$ and $\big(\bigcap\limits_{A\in S} A\big)^c = \bigcup\limits_{A\in S} A^c$ (these rules are called DeMorgan's Laws, and you should convince yourself that they are true if you have not seen them).

**2.37 Example:** When $X$ is a metric space, $a \in X$ and $r > 0$, the punctured ball $B^*(a,r)$ is open (by Part 3 of Theorem 2.35) because $B^*(a,r) = B(a,r) \cap \{a\}^c$, and the sets $B(a,r)$ and $\{a\}^c$ are both open.

**2.38 Example:** In $\mathbb{R}$, note that $\bigcap\limits_{n=1}^{\infty}\big(-\frac{1}{n}, 1+\frac{1}{n}\big) = [0,1]$, which is closed and not open, so the intersection of an infinite set of open sets is not always open. Similarly, note that $\bigcup\limits_{n=1}^{\infty}\big[\frac{1}{n}, 1-\frac{1}{n}\big] = (0,1)$, which is open and not closed, so the union of an infinite set of closed sets is not always closed.,

## Topological Spaces

**2.39 Definition:** A **topology** on a set $X$ is a set $T$ of subsets of $X$ such that
(1) $\emptyset \in T$ and $X \in T$,
(2) for every set $S \subseteq T$ we have $\bigcup S \in T$, and
(3) for every finite subset $S \subseteq T$ we have $\bigcap S \in T$.

A **topological space** is a set $X$ with a topology $T$. When $X$ is a metric space, the set of all open sets in $X$ is a topology on $X$, which we call the **metric topology** (or the topology **induced** by the metric). When $X$ is any topological space, the sets in the topology $T$ are called the **open sets** in $X$ and their complements are called the **closed sets** in $X$. When $S$ and $T$ are both topologies on a set $X$ with $S \subseteq T$, we say that the topology $T$ is **finer** than the topology $S$, and that the topology $S$ is **coarser** than the topology $T$. When $S \subsetneqq T$ we say that $T$ is **strictly finer** than $S$ and that $S$ is **strictly coarser** than $T$.

**2.40 Example:** Show that on $\mathbb{F}^n$, the metrics $d_1$, $d_2$ and $d_\infty$ all induce the same topology.

Solution: For $a, x \in \mathbb{F}^n$ we have

$$\max_{1\leq k\leq n} |x_k - a_k| \leq \Big(\sum_{k=1}^n |x_k - a_k|^2\Big)^{1/2} \leq \sum_{k=1}^n |x_k - a_k| \leq n \max_{1\leq k\leq n} |x_k - a_k|$$

and so

$$d_\infty(a,x) \leq d_2(a,x) \leq d_1(a,x) \leq n\, d_\infty(a,x).$$

It follows that for all $a \in \mathbb{F}^n$ and $r > 0$ we have

$$B_\infty(a,r) \supseteq B_2(a,r) \supseteq B_1(a,r) \supseteq B_\infty\big(a, \tfrac{r}{n}\big).$$

Thus for $U \subseteq \mathbb{F}^n$, if $U$ is open in $\mathbb{F}^n$ using $d_\infty$ then it is open using $d_2$, and if $U$ is open using $d_2$ then it is open using $d_1$, and if $U$ is open using $d_1$ then it is open using $d_\infty$.

**2.41 Example:** Show that on the space $\mathcal{C}[a,b]$, the topology induced by the metric $d_\infty$ is strictly finer than the topology induced by the metric $d_1$.

Solution: For $f, g \in \mathcal{C}[a,b]$ we have

$$d_1(f,g) = \int_a^b |f-g| \leq \int_a^b \max_{a\leq x\leq b} \big|f(x) - g(x)\big| = (b-a)\, d_\infty(f,g).$$

It follows that for $f \in \mathcal{C}[a,b]$ and $r > 0$ we have

$$B_\infty(f,r) \subseteq B_1\big(f, (b-a)r\big).$$

Thus for $U \subseteq \mathcal{C}[a,b]$, if $U$ is open using $d_1$ then $U$ is also open using $d_\infty$, and so the topology induced by the metric $d_\infty$ is finer (or equal to) the topology induced by $d_1$.

On the other hand, we claim that for $f \in \mathcal{C}[a,b]$ and $r > 0$, the set $B_\infty(f,r)$ is not open in the topology induced by $d_1$. Fix $g \in B_\infty(f,r)$ and let $s > 0$. Choose a bump function $h \in \mathcal{C}\big([a,b],\mathbb{R}\big)$ with $h \geq 0$, $\int_a^b h < s$ and $\max_{a\leq x\leq b} h(x) > 2r$ $\big($for example, choose $c \in (a,b)$ with $c - a < \frac{s}{2r}$ and then define $h$ by $h(x) = 3r\big(1 - \frac{x-a}{c-a}\big)$ for $a \leq x \leq c$ and $h(x) = 0$ for $c \leq x \leq b\big)$. Then we have $g + h \in B_1(g,s)$ but $g + h \notin B_\infty(f,r)$. It follows that $B_\infty(f,r)$ is not open in the topology induced by $d_1$, as claimed.

**2.42 Example:** For any set $X$, the **trivial topology** on $X$ is the the topology in which the only open sets in $X$ are the sets $\emptyset$ and $X$, and the **discrete topology** on $X$ is the topology in which every subset of $X$ is open. Note that the discrete metric on a nonempty set $X$ induces the discrete topology on $X$.

## Interior and Closure

**2.43 Definition:** Let $X$ be a metric space (or a topological space) and let $A \subseteq X$. The **interior** and the **closure** of $A$ (in $X$) are the sets

$$A^o = \bigcup \left\{ U \subseteq X \,\middle|\, U \text{ is open, and } U \subseteq A \right\},$$
$$\overline{A} = \bigcap \left\{ K \subseteq X \,\middle|\, K \text{ is closed and } A \subseteq K \right\}.$$

**2.44 Example:** Show that $\overline{\mathbb{Q}} = \mathbb{R}$ where $\overline{\mathbb{Q}}$ is the closure of $\mathbb{Q}$ in $\mathbb{R}$.

Solution: Let $S = \left\{ K \subseteq \mathbb{R} \,\middle|\, K \text{ is closed in } \mathbb{R} \text{ and } \mathbb{Q} \subseteq K \right\}$ so that $\overline{\mathbb{Q}} = \bigcap S = \bigcap\limits_{K \in S} K$. It is immediate that $\overline{\mathbb{Q}} \subseteq \mathbb{R}$ (since every $K \in S$ is a subset of $\mathbb{R}$), so we need to show that $\mathbb{R} \subseteq \overline{\mathbb{Q}}$. Let $a \in \mathbb{R}$. To show that $a \in \overline{\mathbb{Q}}$ we need to show that $a \in K$ for every $K \in S$. Let $K \in S$, that is let $K$ be a closed set in $\mathbb{R}$ with $\mathbb{Q} \subseteq K$. Suppose, for a contradiction, that $a \notin K$. Then $a \in K^c = \mathbb{R} \setminus K$, which is open. Choose $r > 0$ so that $B(a,r) \subseteq K^c$, that is $B(a,r) \cap K = \emptyset$, that is $(a-r, a+r) \cap K = \emptyset$. Since $\mathbb{Q} \subseteq K$, we also have $(a-r, a+r) \cap \mathbb{Q} = \emptyset$. This contradicts the fact that for all $u, v \in \mathbb{R}$ with $u < v$, there exists $x \in \mathbb{Q}$ with $u < x < v$.

**2.45 Definition:** Let $X$ be a metric space (or a topological space) and let $A \subseteq X$. We say that $A$ is **dense** in $X$ when $\overline{A} = X$.

**2.46 Theorem:** *Let $X$ be a metric space (or a topological space) and let $A \subseteq X$.*

*(1) The interior of $A$ is the largest open set which is contained in $A$. In other words, $A^o \subseteq A$ and $A^o$ is open, and for every open set $U$ with $U \subseteq A$ we have $U \subseteq A^o$.*

*(2) The closure of $A$ is the smallest closed set which contains $A$. In other words, $A \subseteq \overline{A}$ and $\overline{A}$ is closed, and for every closed set $K$ with $A \subseteq K$ we have $\overline{A} \subseteq K$.*

Proof: Let $S = \left\{ U \subseteq X \,\middle|\, U \text{ is open, and } U \subseteq A \right\}$. Note that $A^o$ is open (by Part 2 of Theorem 2.35 or by Part 2 of Definition 2.39) because $A^o$ is equal to the union of $S$, which is a set of open sets. Also note that $A^o \subseteq A$ because $A^o$ is equal to the union of $S$, which is a set of subsets of $A$. Finally note that for any open set $U$ with $U \subseteq A$ we have $U \in S$ so that $U \subseteq \bigcup S = A^o$. This completes the proof of Part 1, and the proof of Part 2 is similar.

**2.47 Corollary:** *Let $X$ be a metric space (or a topological space) and let $A \subseteq X$.*
*(1) $(A^o)^o = A^o$ and $\overline{\overline{A}} = \overline{A}$.*
*(2) $A$ is open if and only if $A = A^o$*
*(3) $A$ is closed if and only if $A = \overline{A}$.*

Proof: The proof is left as an exercise.

## Interior Points, Limit Points and Boundary Points

**2.48 Definition:** Let $X$ be a metric space and let $A \subseteq X$. An **interior point** of $A$ is a point $a \in A$ such that for some $r > 0$ we have $B(a,r) \subseteq A$. A **limit point** of $A$ is a point $a \in X$ such that for every $r > 0$ we have $B^*(a,r) \cap A \neq \emptyset$. An **isolated point** of $A$ is a point $a \in A$ which is not a limit point of $A$. A **boundary point** of $A$ is a point $a \in X$ such that for every $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$. The set of all limit points of $A$ is denoted by $A'$. The **boundary** of $A$, denoted by $\partial A$, is the set of all boundary points of $A$.

**2.49 Theorem:** *(Properties of Interior, Limit and Boundary Points) Let $X$ be a metric space and let $A \subseteq X$.*
*(1) $A^o$ is equal to the set of all interior points of $A$.*
*(2) $A$ is closed if and only if $A' \subseteq A$.*
*(3) $\overline{A} = A \cup A'$.*
*(4) $\partial A = \overline{A} \setminus A^o$.*

Proof: We leave the proofs of Parts 1 and 4 as exercises. To prove Part 2, note that when $a \notin A$ we have $B(a,r) \cap A = B^*(a,r) \cap A$ and so

$$\begin{aligned}
A \text{ is closed} &\iff A^c \text{ is open} \\
&\iff \forall a \in A^c \; \exists r > 0 \; B(a,r) \subseteq A^c \\
&\iff \forall a \in X \; \big(a \notin A \Longrightarrow \exists r > 0 \; B(a,r) \subseteq A^c \\
&\iff \forall a \in X \; \big(a \notin A \Longrightarrow \exists r > 0 \; B(a,r) \cap A = \emptyset\big) \\
&\iff \forall a \in X \; \big(a \notin A \Longrightarrow \exists r > 0 \; B^*(a,r) \cap A = \emptyset\big) \\
&\iff \forall a \in X \; \big(\forall r > 0 \; B^*(a,r) \cap A \neq \emptyset \Longrightarrow a \in A\big) \\
&\iff \forall a \in X \; \big(a \in A' \Longrightarrow a \in A\big) \\
&\iff A' \subseteq A.
\end{aligned}$$

To prove Part 3 we shall prove that $A \cup A'$ is the smallest closed set which contains $A$. It is clear that $A \cup A'$ contains $A$. We claim that $A \cup A'$ is closed, that is $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$, that is let $a \in X$ with $a \notin A$ and $a \notin A'$. Since $a \notin A$, we note that $B^*(a,r) \cap A = B(a,r) \cap A$. Since $a \notin A$ and $a \notin A'$ we can choose $r > 0$ so that $B(a,r) \cap A = \emptyset$. We claim that because $B(a,r) \cap A = \emptyset$ it follows that $B(a,r) \cap A' = \emptyset$. Suppose, for a contradiction, that $B(a,r) \cap A' \neq \emptyset$. Choose $b \in B(a,r) \cap A'$. Since $b \in B(a,r)$ and $B(a,r)$ is open, we can choose $s > 0$ so that $B(b,s) \subseteq B(a,r)$. Since $b \in A'$ it follows that $B(b,s) \cap A \neq \emptyset$. Choose $x \in B(b,s) \cap A$. Then we have $x \in B(b,s) \subseteq B(a,r)$ and $x \in A$ and so $x \in B(a,r) \cap A$, which contradicts the fact that $B(a,r) \cap A = \emptyset$. Thus $B(a,r) \cap A' = \emptyset$, as claimed. Since $B(a,r) \cap A = \emptyset$ and $B(a,r) \cap A' = \emptyset$ it follows that $B(a,r) \cap (A \cup A') = \emptyset$ hence $B(a,r) \subseteq (A \cup A')^c$. Thus proves that $(A \cup A')^c$ is open, and hence $A \cup A'$ is closed.

It remains to show that for every closed set $K$ in $X$ with $A \subseteq K$ we have $A \cup A' \subseteq K$. Let $K$ be a closed set in $X$ with $A \subseteq K$. Note that since $A \subseteq K$ it follows that $A' \subseteq K'$ because if $a \in A'$ then for all $r > 0$ we have $B^*(a,r) \cap A \neq \emptyset$ hence $B^*(a,r) \cap K \neq \emptyset$ and so $a \in K'$. Since $K$ is closed we have $K' \subseteq K$ by Part 2. Since $A' \subseteq K'$ and $K' \subseteq K$ we have $A' \subseteq K$. Since $A \subseteq K$ and $A' \subseteq K$ we have $A \cup A' \subseteq K$, as required. This completes the proof of Part 3.

## Open and Closed Sets in Subspaces

**2.50 Note:** Let $X$ be a metric space and let $P \subseteq X$. Note that $P$ is also a metric space using (the restriction of) the metric used in $X$. For $a \in P$ and $0 < r \in \mathbb{R}$, note that the open and closed balls in $P$, centred at $a$ and of radius $r$, are related to the open and closed balls in $X$ by

$$B_P(a,r) = \big\{x \in P \,\big|\, d(x,a) < r\big\} = B_X(a,r) \cap P,$$
$$\overline{B}_P(a,r) = \big\{x \in P \,\big|\, d(x,a) \leq r\big\} = \overline{B}_X(a,r) \cap P.$$

**2.51 Theorem:** *Let $X$ be a metric space and let $A \subseteq P \subseteq X$.*
*(1) $A$ is open in $P$ if and only if there exists an open set $U$ in $X$ such that $A = U \cap P$.*
*(2) $A$ is closed in $P$ if and only if there exists a closed set $K$ in $X$ such that $A = K \cap P$.*

Proof: To prove Part 1, suppose first that $A$ is open in $P$. For each $a \in A$, choose $r_a > 0$ so that $B_P(a,r_a) \subseteq A$, that is $B_X(a,r_a) \cap P \subseteq A$, and let $U = \bigcup_{a\in A} B_X(a,r_a)$. Since $U$ is equal to the union of a set of open sets in $X$, it follows that $U$ is open in $X$. Note that $A \subseteq U \cap P$ and, since $B_X(a,r_a) \cap P \subseteq A$ for every $a \in A$, we also have $U \cap P = \Big(\bigcup_{a\in U} B_X(a,r_a)\Big) \cap P = \bigcup_{a\in A}\big(B_X(a,r_a)\cap P\big) \subseteq A$. Thus $A = U \cap P$, as required.

Suppose, conversely, that $A = U \cap P$ with $U$ open in $X$. Let $a \in A$. Since we have $a \in A = U \cap P$, we also have $a \in U$. Since $a \in U$ and $U$ is open in $X$ we can choose $r > 0$ so that $B_X(a,r) \subseteq U$. Since $B_X(a,r) \subseteq U$ and $U \cap P = A$ we have $B_P(a,r) = B_X(a,r) \cap P \subseteq U \cap P = A$. Thus $A$ is open, as required.

To prove Part 2, suppose first that $A$ is closed in $P$. Let $B$ be the complement of $A$ in $P$, that is $B = P \setminus A$. Then $B$ is open in $P$. Choose an open set $U$ in $X$ such that $B = U \cap P$. Let $K$ be the complement of $U$ in $X$, that is $K = X \setminus U$. Then $A = K \cap P$ since for $x \in X$ we have $x \in A \iff \big(x \in P \text{ and } x \notin B\big) \iff \big(x \in P \text{ and } x \notin U \cap P\big) \iff \big(x \in P \text{ and } x \notin U\big) \iff \big(x \in P \text{ and } x \in K\big) \iff x \in K \cap P$.

Suppose, conversely, that $K$ is a closed set in $X$ with $A = K \cap P$. Let $B$ be the complement of $A$ in $P$, that is $B = P \setminus A$, and let $U$ be the complement of $K$ in $X$, that is $U = X \setminus K$, and note that $U$ is open in $X$. Then we have $B = U \cap P$ since for $x \in P$ we have $x \in B \iff \big(x \in P \text{ and } x \notin A\big) \iff \big(x \in P \text{ and } x \notin K \cap P\big) \iff \big(x \in P \text{ and } x \notin K\big) \iff \big(x \in P \text{ and } x \in U\big) \iff x \in U \cap P$. Since $U$ is open in $X$ and $B = U \cap P$ we know that $B$ is open in $P$. Since $B$ is open in $P$, its complement $A = P \setminus B$ is closed in $P$.

**2.52 Definition:** Let $X$ be a topological space and let $P \subseteq X$. Verify, as an exercise, that we can use the topology on $X$ to define a topology on $P$ as follows. Given a set $A \subseteq P$, we define $A$ to be **open** in $P$ when $A = U \cap P$ for some open set $U$ in $X$. The resulting topology on $P$ is called the **subspace topology**. The above theorem asserts that when $X$ is a metric space and $P \subseteq X$, the metric topology on $P$ (obtained by restricting the metric on $X$ to $P$) is the same as the subspace topology on $P$.

## Appendix: The $p$-Norms

**2.53 Definition:** Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. For $p\in(1,\infty)$ and for $x=(x_n)_{n\geq 1}\in\mathbb{F}^\omega$, define $\|x\|_p$ to be the extended real number

$$\|x\|_p=\Big(\sum_{n=1}^\infty |x_n|^p\Big)^{1/p}\in[0,\infty].$$

For $p\in(1,\infty)$, let

$$\ell_p=\ell_p(\mathbb{F})=\big\{x\in\mathbb{F}^\omega\,\big|\,\|x\|_p<\infty\big\}.$$

Also, for $x,y\in\mathbb{F}^\omega$ let $xy=(x_1y_1,x_2y_2,\cdots)$.

**2.54 Theorem:** *(The p-Norms) Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ and let $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$.*

*(1) For all $0\leq a,b\in\mathbb{R}$, we have $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$.*
*(2) (Hölder's Inequality) For all $x,y\in\mathbb{F}^\omega$ we have $\|xy\|_1\leq\|x\|_p\|y\|_q$.*
*(3) (Minkowski's Inequality) For all $x,y\in\mathbb{F}^\omega$ we have $\|x+y\|_p\leq\|x\|_p+\|y\|_p$.*

Proof: To prove Part 1, let $a,b\geq 0$. Note that for $p,q\in(1,\infty)$ we have

$$\tfrac{1}{p}+\tfrac{1}{q}=1\iff\tfrac{1}{q}=1-\tfrac{1}{p}=\tfrac{p-1}{p}\iff q(p-1)=p\iff p(q-1)=q.$$

For $x,y\geq 0$ we have

$$y=x^{p-1}\iff y^q=x^{q(p-1)}\iff y^q=x^p\iff y^{p(q-1)}=x^p\iff y^{q-1}=x$$

so the functions $f(x)=x^{p-1}$ and $g(y)=y^{q-1}$ are inverses of each other. By considering the area under $y=f(x)$ with $0\leq x\leq a$ and the area to the left of $y=f(x)$ with $0\leq y\leq b$ (the union of these two regions is the rectangle $0\leq x\leq a$, $0\leq y\leq b$) we see that

$$ab\leq\int_{x=0}^a x^{p-1}\,dx+\int_{y=0}^b y^{q-1}\,dy=\Big[\tfrac{1}{p}\,x^p\Big]_{x=0}^a+\Big[\tfrac{1}{q}\,y^q\Big]_{y=0}^b=\tfrac{a^p}{p}+\tfrac{b^q}{q}\,.$$

To prove Part 2, let $x,y\in\mathbb{F}^\omega$. If $x=0$ or $y=0$ then we have $\|xy\|_1=0=\|x\|_p\|y\|_q$, so suppose $x\neq 0$ and $y\neq 0$ (hence $\|x\|_p\neq 0$ and $\|y\|_q\neq 0$). If $\|x\|_p=\infty$ or $\|y\|_q=\infty$ then $\|xy\|_1\leq\infty=\|x\|_p\|y\|_q$, so suppose that $0\neq\|x\|_p<\infty$ and $0\neq\|y\|_q<\infty$. For each index $k$, apply Part 1 using $a=\frac{|x_k|}{\|x\|_p}$ and $b=\frac{|y_k|}{\|y\|_q}$ to get

$$\frac{|x_ky_k|}{\|x\|_p\|y\|_q}\leq\frac{|x_k|^p}{p\|x\|_p^p}+\frac{|y_k|^q}{q\|y\|_q^q}.$$

Sum over $k$ to get

$$\frac{\|xy\|_1}{\|x\|_p\|y\|_q}\leq\frac{\|x\|_p^p}{p\|x\|_p^p}+\frac{\|y\|_q^q}{q\|y\|_q^q}=\tfrac{1}{p}+\tfrac{1}{q}=1\,.$$

To prove Part 3, let $x,y\in\mathbb{F}^\omega$. If $\|x\|_p=\infty$ or $\|y\|_p=\infty$ then $\|x+y\|_p\leq\infty=\|x\|_p+\|y\|_q$. Suppose that $\|x\|_p<\infty$ and $\|y\|_p<\infty$. Since the function $f(x)=x^p$ is concave for $p>1$, we have $\big(\frac{a+b}{2}\big)^p\leq\frac{a^p+b^p}{2}$ for all $a,b\geq 0$, so for all $k\in\mathbb{Z}^+$ we have $\big|\frac{x_k+y_k}{2}\big|^p\leq\big(\frac{|x_k|+|y_k|}{2}\big)^p\leq\frac{|x_k|^p+|y_k|^p}{2}$, hence $|x_k+y_k|^p\leq 2^{p-1}\big(|x_k|^p+|y_k|^p\big)$. Sum over $k$ to get

$$\|x+y\|_p^p=\sum_{k=1}^\infty|x_k+y_k|^p\leq\sum_{k=1}^\infty 2^{p-1}\big(|x_k|^p+|y_k|^p\big)=2^{p-1}\big(\|x\|_p^p+\|y\|_p^p\big)<\infty.$$

Choose $q\in(1,\infty)$ so that $\frac{1}{p}+\frac{1}{q}=1$ hence $\frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}$. For each index $k$ we have

$$\begin{aligned}|x_k+y_k|^p&=|x_k+y_k|\,|x_k+y_k|^{p-1}\leq\big(|x_k|+|y_k|\big)|x_k+y_k|^{p-1}\\&=|x_k|\,|x_k+y_k|^{p-1}+|y_k|\,|x_k+y_k|^{p-1}.\end{aligned}$$

Sum over $k$ then apply Hölder's Inequality, writing $|x|$ for the sequence $|x| = \big(|x_1|, |x_2|, \cdots\big)$ and similarly $|y| = \big(|y_1|, |y_2|, \cdots\big)$ and $|x+y|^{p-1} = \big(|x_1+y_1|^{p-1}, |x_2+y_2|^{p-1}, \cdots\big)$, to get

$$\begin{aligned}\|x+y\|_p^p &\leq \Big\| |x|\,|x+y|^{p-1} \Big\|_1 + \Big\| |y|\,|x+y|^{p-1} \Big\|_1 \leq \|x\|_p \Big\| |x+y|^{p-1} \Big\|_q + \|y\|_p \Big\| |x+y|^{p-1} \Big\|_q \\ &= \Big(\|x\|_p + \|y\|_p\Big) \Big\| |x+y|^{p-1} \Big\|_q = \Big(\|x\|_p + \|y\|_p\Big) \Big( \sum_{k=1}^{\infty} |x+y|^{q(p-1)} \Big)^{1/q} \\ &= \Big(\|x\|_p + \|y\|_p\Big) \Big( \sum_{k=1}^{\infty} |x+y|^{p} \Big)^{(p-1)/p} = \Big(\|x\|_p + \|y\|_p\Big) \|x+y\|_p^{p-1}.\end{aligned}$$

If $\|x+y\|_p \neq 0$ then we can divide both sides by $\|x+y\|_p^{p-1}$ to get $\|x+y\|_p \leq \|x\|_p + \|y\|_p$, and if $\|x+y\|_p = 0$ then of course $\|x+y\|_p \leq \|x\|_p + \|y\|_q$.

**2.55 Definition:** Minkowski's Theorem shows that $\| \ \|_p$ satisfies the Triangle Inequality on $\ell_p$. It is easy to verify that it satisfies the other two properties which define a norm, and so $\| \ \|_p$ is a norm on $\ell_p$, which we call the ***p*-norm**.