Chapter 2. Rings (Review)

Factorization in Commutative Rings

2.1 Definition: Let R be a ring. An ideal P in R is called **prime** when $P \neq R$ and for all ideals A and B in R, if $AB \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$. An ideal M in R is called **maximal** when $M \neq R$ and there is no ideal A in R with $M \subsetneqq A \subsetneqq R$.

2.2 Example: As an exercise, use the above definition to show that the maximal ideals in $\mathbb{Z}$ are the ideals of the form $\langle p \rangle$ with p prime, and the prime ideals in $\mathbb{Z}$ are the ideals of the form $\langle p \rangle$ with $p = 0$ or p prime.

2.3 Theorem: *Let R be a commutative ring with 1. Let P be an ideal in R with $P \neq R$. Then P is prime if and only if P has the property that for all $a, b \in R$, if $ab \in P$ then either $a \in P$ or $b \in P$.*

Proof: Since R is commutative with 1, we have $\langle a \rangle = \left\{ ar \middle| r \in R \right\}$ and $\langle b \rangle = \left\{ bs \middle| s \in R \right\}$ and so

$$\begin{aligned}\langle a \rangle \langle b \rangle &= \left\{ \sum_{i=1}^{n} a_i b_i \middle| a_i \in \langle a \rangle, b_i \in \langle b \rangle \right\} = \left\{ \sum_{i=1}^{n} (ar_i)(bs_i) \middle| r_i, s_i \in R \right\} \\ &= \left\{ \sum_{i=1}^{n} (ab)t_i \middle| t_i \in R \right\} = \langle ab \rangle.\end{aligned}$$

Suppose that P is prime. Let $a, b \in R$ with $ab \in P$. Then $\langle a \rangle \langle b \rangle = \langle ab \rangle \subseteq P$ and so, since P is prime, either $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$, and hence either $a \in P$ or $b \in P$.

Conversely, suppose that P has the property that for all $a, b \in R$, if $ab \in P$ then either $a \in P$ or $b \in P$. Let A and B be ideals in R with $AB \subseteq P$. Suppose that $A \not\subseteq P$. Choose $a \in A$ with $a \notin P$. Let $b \in B$ be arbitrary. Then $ab \in AB \subseteq P$ and so, because of the property held by P, either $a \in P$ or $b \in P$. Since $a \notin P$ we must have $b \in P$. Thus $B \subseteq P$.

2.4 Theorem: *Let R be a commutative ring with 1. Let P be an ideal in R. Then P is prime if and only if R/P is an integral domain.*

Proof: Suppose that P is prime. Since $P \neq R$ we have $1 \notin P$ (since $\langle 1 \rangle = R$) and so $1 + P \neq 0 + P \in P/R$. Since R is commutative, so is R/P. Finally, note that R/P has no zero divisors because for $a, b \in R$ we have

$$\begin{aligned}(a+P)(b+P) = (0+P) &\Longrightarrow ab + P = 0 + P \Longrightarrow ab \in P \Longrightarrow a \in P \text{ or } b \in P \\ &\Longrightarrow a + P = 0 + P \text{ or } b + P = 0 + P.\end{aligned}$$

Conversely, suppose that R/P is an integral domain. Since $1 + P \neq 0 + P \in R/P$, it follows that $1 \notin P$ and so $P \neq R$. Let $a, b \in R$ with $ab \in P$. Then we have $ab + P = 0 + P$, and so $(a+P)(b+P) = 0 + P$. Since R/P has no zero divisors, this implies that either $a + P = 0 + P$ or $b + P = 0 + P$, and so either $a \in P$ or $b \in P$.

2.5 Example: Let R be a commutative ring with 1. Show that every maximal ideal in R is also prime.

Solution: Let M be a maximal ideal in R. Let $a, b \in R$ with $ab \in M$. Suppose that $a \notin M$. Then we have $M \subsetneqq M + \langle a \rangle$ and so, since M is maximal, we must have $M + \langle a \rangle = R$. In particular $1 \in M + \langle a \rangle$, so we have $1 = m + ar$ for some $r \in R$. Thus

$$b = b \cdot 1 = b(m + ar) = bm + ab\,r \in M\,.$$

We remark that this result also follows from the following theorem.

2.6 Theorem: *Let R be a commutative ring with 1. Let M be an ideal in R. Then M is maximal if and only if R/M is a field.*

Proof: Suppose M is maximal. Since $M \neq R$ we have $1 \notin M$ and so $1+M \neq 0+M \in R/M$. Since R is commutative, so is R/M. Let $a+M$ be a nonzero element in R/M. We must show that $a+M$ is a unit. Since $a+M \neq 0+M$ we have $a \notin M$. Since $a \notin M$ we have $M \subsetneqq M+\langle a \rangle$. Since M is maximal, we must have $M+\langle a \rangle = R$. In particular, $1 \in M+\langle a \rangle$, say $1 = m + ar$ with $r \in R$. Then $1+M = ar+M = (a+M)(r+M)$ and so $r+M$ is the inverse of $a+M$.

Conversely, suppose that R/M is a field. Since $1+M \neq 0+M$ in R/M, we have $1 \notin M$ so $M \neq R$. Let A be an ideal with $M \subseteq A \subseteq R$. Suppose $A \neq M$. Choose $a \in A$ with $a \notin M$. Since $a \notin M$ we have $a+M \neq 0+M$ in R/M. Since R/M is a field, $a+M$ has an inverse, say $(a+M)(b+M) = 1+M$. Then $ab+M = 1+M$ so we have $1-ab \in M$. Since $M \subseteq A$ we have $1-ab \in A$. Since $a \in A$ we have $ab \in A$, so $1 \in A$ and hence $A = R$.

2.7 Example: Find all prime and maximal ideals in $\mathbb{Z}$ (that is redo example 2.2) using Theorems 2.4 and 2.6.

2.8 Example: Since $\mathbb{Q}[x]/\langle x^2-2\rangle \cong \mathbb{Q}[\sqrt{2}]$, which is a field, it follows that $\langle x^2-2\rangle$ is maximal (and prime). In $\mathbb{R}[x]$, however, we have $(x^2-2) = (x-\sqrt{2})(x+\sqrt{2})$, and so the ideal $\langle x^2-2\rangle$ is not maximal because $\langle x^2-2\rangle \subsetneqq \langle x-\sqrt{2}\rangle \subsetneqq \mathbb{R}[x]$ and it is not prime because $(x-\sqrt{2})(x+\sqrt{2}) \in \langle x^2-2\rangle$ but $(x-\sqrt{2}) \notin \langle x^2-2\rangle$ and $(x+\sqrt{2}) \notin \langle x^2-2\rangle$.

2.9 Example: In $\mathbb{Z}[x]$, we have $\langle x\rangle = \{f \in \mathbb{Z}[x] \,|\, f(0)=0\}$. The ideal $\langle x\rangle$ is prime because for $f,g \in \mathbb{Z}[x]$, if $fg \in \langle x\rangle$ then $f(0)g(0)=0$ and so either $f(0)=0$ or $g(0)=0$. But the ideal $\langle x\rangle$ is not maximal since $\langle x\rangle \subsetneqq \langle 2,x\rangle = \{f \in \mathbb{Z}[x] \,|\, f(0) \text{ is even}\} \subsetneqq \mathbb{Z}[x]$.

2.10 Definition: Let R be a commutative ring with 1. Let $a,b \in R$. We say that a **divides** b (or that a is a **divisor** or **factor** of b, or that b is a **multiple** of a), and we write $a|b$, when $b = ar$ for some $r \in R$. We say that a and b are **associates**, and we write $a \sim b$, when $a|b$ and $b|a$. Note that association is an equivalence relation on R.

2.11 Theorem: *Let R be a commutative ring with 1. Let $a,b \in R$. Then*
(1) $a|b$ if and only if $b \in \langle a\rangle$ if and only if $\langle b\rangle \subseteq \langle a\rangle$,
(2) $a \sim b$ if and only if $\langle a\rangle = \langle b\rangle$ if and only if a and b have the same multiples and divisors,
(3) $a \sim 0$ if and only if $a=0$ if and only if $\langle a\rangle = \{0\}$,
(4) $a \sim 1$ if and only if a is a unit if and only if $\langle a\rangle = R$.
(5) if R is an integral domain then $a \sim b$ if and only if $b = au$ for some unit $u \in R$.

Proof: We prove Part (5) and leave the other proofs as an exercise. Suppose that $b = au$ where $u \in R$ is a unit. Since $b = au$ we have $a|b$ and since $a = bu^{-1}$ we have $b|a$. Since $a|b$ and $b|a$ we have $a \sim b$ (we did not need to assume that R is an integral domain for this direction). Now suppose that R is an integral domain and that $a \sim b$, say $a = br$ and $b = as$ with $r,s \in R$. Then we have $b = as = brs$ so that $b(1-rs) = 0$. Since R is an integral domain, either $b=0$ or $1-rs=0$. If $b=0$ then $a = br = 0$, so we have $b = a \cdot u$ for any unit u (for example $u=1$). If $1-rs=0$ then $rs=1$ so that r and s are units, so we have $b = au$ where $u = s$ (which is a unit).

2.12 Example: In the ring $\mathbb{Z}$, we have $k \sim \ell \iff k = \pm\ell$. Verify that in $\mathbb{Z}_{12}$ the association classes are $\{0\}$, $\{1,5,7,11\}$, $\{2,10\}$, $\{3,9\}$, $\{4,8\}$, $\{6\}$.

2.13 Definition: Let R be a commutative ring with 1. Let $a \in R$ be a non-zero non-unit. We say that a is **reducible** when $a = bc$ for some non-units $b, c \in R$, and otherwise we say that a is **irreducible**. We say that a is **prime** when for all $b, c \in R$, if $a \big| bc$ then either $a \big| b$ or $a \big| c$.

2.14 Theorem: *Let R be a commutative ring with 1. Let $a, b \in R$ with $a \sim b$. Then*
(1) $a = 0$ if and only if $b = 0$,
(2) a is a unit if and only if b is a unit,
(3) a is reducible if and only if b is reducible,
(4) a is irreducible if and only if b is irreducible,
(5) a is prime if and only if b is prime.

Proof: The proof is left as an exercise.

2.15 Example: In the ring $\mathbb{Z}$, for $k \in \mathbb{Z}$, k is irreducible if and only if k is prime if and only if $k = \pm p$ for some (positive) prime number p.

2.16 Example: As an exercise, verify that in the ring $\mathbb{Z}_{12}$, the irreducible elements are 2 and 10 and the prime elements are 2, 3, 9 and 10.

2.17 Example: Use the method of the Sieve of Eratosthenes to find several irreducible elements in $\mathbb{Z}[\sqrt{3}\, i]$ and also some irreducible elements which are not prime.

2.18 Theorem: *Let R be a commutative ring with 1. Let $a \in R$. Then*
(1) If a is irreducible then the divisors of a are the units in R and the associates of a in R.
(2) a is prime if and only if $\langle a \rangle$ is a non-zero prime ideal.

Proof: The proof is left as an exercise.

2.19 Theorem: *Let R be an integral domain and let $a \in R$. Then*
(1) if a is prime then a is irreducible,
(2) a is irreducible if and only if $\langle a \rangle$ is maximal amongst non-zero proper principal ideals,
(3) if R is a PID and a is irreducible, then a is prime.

Proof: To Prove Part (1), suppose that a is prime. Suppose that $a = bc$ with $b, c \in R$. Since $a = bc$ we have $a \big| bc$ and hence, since a is prime, either $a|b$ or $a|c$. Suppose that $a|b$, say $b = ar$. Then $a = bc = arc$ so that $a(1 - rc) = 0$. Since R is an integral domain and $a \neq 0$ it follows that $rc = 1$ so that c is a unit. A similar argument shows that if $a|c$ then b is a unit, and so a is irreducible, as required.

To prove Part (2), suppose that a is irreducible. Since $a \neq 0$ we have $\langle a \rangle \neq 0$ and since a is not a unit we have $\langle a \rangle \neq R$. Let $b \in R$ and suppose that $\langle a \rangle \subseteq \langle b \rangle \subseteq R$. Since $\langle a \rangle \subseteq \langle b \rangle$ we have $a \in \langle b \rangle$, say $a = bc$ with $c \in \mathbb{R}$. Since a is irreducible, either b is a unit, in which case $\langle b \rangle = R$, or c is a unit in which case $b \sim a$ so that $\langle b \rangle = \langle a \rangle$.

Suppose, conversely, that $\langle a \rangle$ is maximal amongst nonzero proper principal ideals in R. Since $\langle a \rangle \neq \{0\}$ we have $a \neq 0$ and since $\langle a \rangle \neq R$ it follows that a is not a unit. Suppose that $a = bc$ where $b, c \in R$. Since $a = bc$ we have $a \in \langle b \rangle$ so that $\langle a \rangle \subseteq \langle b \rangle$. By the maximality of $\langle a \rangle$, either $\langle b \rangle = \langle a \rangle$ or $\langle b \rangle = R$. If $\langle b \rangle = R$ then b is a unit. Suppose that $\langle b \rangle = \langle a \rangle$, say $b = ar$ with $r \in R$. Then $a = bc = arc$ so that $a(1 - rc) = 0$. Since $a(1 - rc) = 0$ and $a \neq 0$ and R is an integral domain, it follows that $rc = 1$ so that c is a unit. This completes the proof of Part (2).

Finally note that if a is irreducible and R is a PID then, by Part (2), $\langle a \rangle$ is a maximal ideal, hence $\langle a \rangle$ is a prime ideal, hence a is prime. This proves Part (3).

2.20 Definition: A **Euclidean domain** (or ED) is an integral domain R together with a function $N : R \setminus \{0\} \to \mathbb{N}$, called a **norm**, with the property that for all $a, b \in R$ with $a \neq 0$ there exist $q, r \in R$ such that $b = qa + r$ and either $r = 0$ or $N(r) < N(a)$.

2.21 Definition: A **principal ideal domain** (or PID) is an integral domain R such that every ideal in R is principal.

2.22 Definition: A **unique factorization domain** (or UFD) is an integral domain R with the property that for every nonzero non-unit $a \in R$ we have
(1) $a = a_1 a_2 \cdots a_l$ for some $l \in \mathbb{Z}^+$ and some irreducible elements $a_i \in R$, and
(2) if $a = a_1 a_2 \cdots a_l = b_1 b_2 \cdots b_m$ where $l, m \in \mathbb{Z}^+$ and each a_i and b_j is irreducible, then $m = l$ and for some permutation $\sigma \in S_m$ we have $a_i \sim b_{\sigma(i)}$ for all i.

2.23 Example: The ring $\mathbb{Z}$ is a Euclidean domain with norm given by $N(k) = |k|$.

2.24 Example: Every field F is a Euclidean domain, using any function $N : F \setminus \{0\} \to \mathbb{N}$ as a norm. Indeed, given $a, b \in F$ with $a \neq 0$ we can choose $q = \frac{b}{a}$ and $r = 0$ to get $b = aq + r$.

2.25 Example: If F is a field then $F[x]$ is a Euclidean domain with norm $N(f) = \deg(f)$.

2.26 Example: Show that in the ring $\mathbb{Z}[\sqrt{3}\, i]$, the elements 2 and $1 \pm \sqrt{3}\, i$ are irreducible and $2 \not\sim 1 \pm \sqrt{3}\, i$. It follows that $\mathbb{Z}[\sqrt{3}\, i]$ not a unique factorization domain because $4 = 2 \cdot 2 = (1 + \sqrt{3}\, i)(1 - \sqrt{3}\, i)$.

2.27 Theorem: *Every Euclidean domain is a principal ideal domain.*

Proof: Let R be a Euclidean domain with norm N. Let A be an ideal in R. If $A = \{0\}$ then A is principal with $A = \langle 0 \rangle$. Suppose that $A \neq \{0\}$. Choose a nonzero element $0 \neq a \in A$ of smallest possible norm. We claim that $A = \langle a \rangle$. Since $a \in A$ we have $\langle a \rangle \subseteq A$. Let $b \in A$ be arbitrary. Choose $q, r \in R$ such that $b = qa + r$ and either $r = 0$ or $N(r) < N(a)$. Note that $r = b - qa \in A$ so we must have $r = 0$ by the choice of a. Thus $b = qa \in \langle a \rangle$.

2.28 Definition: A ring R is called **Noetherian** when it satisfies the following condition, which is called the **ascending chain condition**: for every ascending chain of ideals $A_1 \subseteq A_2, \subseteq A_3 \subseteq \cdots$ in R, there exists $n \in \mathbb{Z}^+$ such that $A_k = A_n$ for all $k \geq n$.

2.29 Theorem: *Every principal ideal domain is Noetherian.*

Proof: Let R be a principal ideal domain. Let $a_1, a_2, a_3, \cdots \in R$ with

$$\langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \langle a_3 \rangle \subseteq \cdots.$$

Let $A = \bigcup\limits_{k=1}^{\infty} \langle a_k \rangle$. Verify that A is an ideal. Choose $a \in R$ so that $A = \langle a \rangle$. Since $a \in A$, we can choose $n \in \mathbb{Z}^+$ so that $a \in \langle a_n \rangle$. For all $k \geq n$, we have $\langle a_k \rangle \subseteq A = \langle a \rangle \subseteq \langle a_n \rangle \subseteq \langle a_k \rangle$ and so $\langle a_k \rangle = \langle a_n \rangle$.

2.30 Theorem: *Every principal ideal domain is a unique factorization domain.*

Proof: Let R be a principal ideal domain. Let $a \in R$ be a non-zero non-unit. We claim that a has an irreducible factor. If a is irreducible then we are done. Suppose that a is reducible, say $a = a_1 b_1$ where a_1 and b_1 are non-units. Note that $\langle a \rangle \subsetneqq \langle a_1 \rangle$. If a_1 is irreducible then we are done. Suppose that a_1 is reducible, say $a_1 = a_2 b_2$ where a_2 and b_2 are non-units. Then $a = a_1 b_1 = a_2 b_2 b_1$ and $\langle a \rangle \subsetneqq \langle a_1 \rangle \subsetneqq \langle a_2 \rangle$. If a_2 is irreducible then we are done, and otherwise we continue this procedure. Eventually, the procedure must end giving us an irreducible factor a_n of a, otherwise we would obtain an infinite chain of ideals $\langle a \rangle \subsetneqq \langle a_1 \rangle \subsetneqq \langle a_2 \rangle \subsetneqq \cdots$, contradicting the fact that R is Noetherian.

Next we claim that $a = a_1 a_2 \cdots a_l$ for some $l \in \mathbb{Z}^+$ and some irreducible $a_i \in R$. If a is irreducible then we are done. Suppose that a is reducible. Let a_1 be an irreducible factor of a, and say $a = a_1 b_1$. Note that b_1 is not a unit since, if it was then we would have $a \sim a_1$, but a is reducible and a_1 is not. If b_1 is irreducible then we are done. Suppose b_1 is reducible. Let a_2 be an irreducible factor of b_1 and say $b_1 = a_2 b_2$. As above, note that b_2 is not a unit. If b_2 is irreducible then we are done, and otherwise we continue the procedure. Eventually, the procedure must end giving us $a = a_1 a_2 \cdots a_n b_n$ with each a_i and n_n irreducible, otherwise we would obtain an infinite chain $\langle a \rangle \subsetneqq \langle b_1 \rangle \subsetneqq \langle b_2 \rangle \subsetneqq \cdots$.

Finally, we claim that if $a = a_1 a_2 \cdots a_l = b_1 b_2 \cdots b_l$ with $l, m \in \mathbb{Z}^+$ and each a_i and b_j irreducible, then $m = l$ and for some permutation $\sigma \in S_m$ we have $a_i \sim b_{\sigma(i)}$ for all i. Suppose that $a = a_1 a_2 \cdots a_l = b_1 b_2 \cdots b_m$ where $l, m \in \mathbb{Z}^+$ and the a_i and b_j are irreducible. Since $a_1 \big| a_1 a_2 \cdots a_l$, we have $a_1 \big| b_1 b_2 \cdots b_m$. Since a_1 is irreducible and R is a principal ideal domain, it follows that a_1 is prime by Part 3 of Theorem 2.19. Since a_1 is prime and $a_1 \big| b_1 b_2 \cdots b_m$, it follows that $a_1 \big| b_k$ for some k. After permuting the elements b_i we can assume $a_1 \big| b_1$. Since b_1 is irreducible, its divisors are units and associates and, since a_1 is not a unit, we have $a_1 \sim b_1$. Since $a_1 \sim b_1$ we have $b_1 = a_1 u$ for some unit u. Thus we have $a_1 a_2 \cdots a_l = b_1 b_2 \cdots b_m = a_1 u b_2 b_3 \cdots b_m$, and by cancellation, $a_2 a_3 \cdots a_l = u b_2 b_3 \cdots b_m$. A suitable induction argument gives $l = m$ and $a_i \sim b_i$ for all i.

2.31 Example: Show that $\mathbb{Z}[i]$ is a ED.

2.32 Example: Since $\mathbb{Z}[\sqrt{3}\, i]$ is not aUFD, it cannot be a PID. Find an ideal in $\mathbb{Z}[\sqrt{3}\, i]$ which is not principal.

2.33 Example: Show that $\mathbb{Z}\left[\frac{1+\sqrt{19}\, i}{2}\right]$ is a PID, but not a ED (under any norm).

Polynomial Rings

2.34 Note: Here are a few remarks about polynomials. Recall that $R[x]$ denotes the ring of polynomials with coefficients in the ring R, and R^R denotes the ring of all functions $f: R \to R$.

(1) A polynomial $f \in R[x]$ determines a function $f \in R^R$. Given $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ we obtain the function $f: R \to R$ given by $f(x) = \sum_{i=0}^{n} a_i x^i$.

(2) Although we do not usually distinguish notationally between the polynomial $f \in R[x]$ and its corresponding function $f \in R^R$, they are not always identical. If the ring R is not commutative then multiplication of polynomials does not agree with multiplication of functions. For $f, g \in R[x]$ given by $f(x) = a + bx$ and $g(x) = c + dx$, in the ring $R[x]$ we have $(fg)(x) = (a+bx)(c+dx) = (ac) + (ad+bc)x + (bd)x^2$, but in the ring R^R we have $(fg)(x) = (a+bx)(c+dx) = ac + adx + bxc + bxdx$.

(3) Equality of polynomials may not agree with equality of functions. For $f, g \in R[x]$ given by $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ we have $f = g \in R[x]$ if and only if $a_i = b_i$ for all i (and if say $n < m$ then $b_i = a_i = 0$ for $i > n$), but $f = g \in R^R$ if and only if $f(x) = g(x)$ for all $x \in R$. These two notions of equality do not always agree. For example if R is finite then the ring $R[x]$ is infinite but the ring R^R is finite. Indeed if $|R| = n$ then $R[x]$ is countably infinite but $\left|R^R\right| = n^n$. For a more specific example, if $f(x) = x^p - x$ then we have $f \neq 0 \in \mathbb{Z}_p[x]$ (because its coefficients are not equal to zero) but $f = 0 \in {\mathbb{Z}_p}^{\mathbb{Z}_p}$ because, by Fermat's Little Theorem, we have $f(x) = 0$ for all $x \in \mathbb{Z}_p$.

(4) Recall that for $f(x) = \sum_{i=0}^{n} a_i x^i$ with each $a_i \in R$ and $a_n \neq 0$, the element $a_n \in R$ is called the leading coefficient of f, and the non-negative integer n is called the degree of $f(x)$, and we write $\deg(f) = n$. For convenience, we also define $\deg(0) = -1$. When R is an integral domain, it is easy to see that for $0 \neq f, g \in R[x]$ we have $\deg(fg) = \deg(f) + \deg(g)$. When R is not an integral domain, however, we only have $\deg(fg) \leq \deg(f) + \deg(g)$ because the product of the two leading coefficients can be equal to zero.

(5) When R is an integral domain, because we have $\deg(fg) = \deg(f) + \deg(g)$ for all $0 \neq f, g \in R[x]$, it is easy to see that the units in $R[x]$ are the constant polynomials $f(x) = c$ where c is a unit in R. In particular, when F is a field, the units in $F[x]$ are the elements $f \in F[x]$ with $\deg(f) = 0$. In the ring $\mathbb{Z}_4[x]$ (which is not an integral domain) we have $(1+2x)^2 = 1 + 4x + 4x^2 = 1$, so $f(x) = (1+2x)$ is a unit in $\mathbb{Z}_4[x]$.

2.35 Theorem: (*Division Algorithm*) *Let R be a ring with 1. Let $f,g\in R[x]$ and suppose that the leading coefficient of g is a unit in R. Then there exist unique polynomials $q,r\in R$ such that $f=qg+r$ and $\deg(r)<\deg(g)$.*

Proof: First we prove existence. If $\deg(f)<\deg(g)$ then we can take $q=0$ and $r=f$. Suppose that $\deg(f\geq\deg(g)$, Say $f(x)=\sum_{i=0}^{n}a_ix^i$ with $a_i\in R$ and $a_n\neq 0$ and $g(x)=\sum_{i=0}^{m}b_ix^i$ with $b_i\in R$ and b_m is a unit. Note that the polynomial $a_n{b_m}^{-1}x^{n-m}g(x)$ has degree n and leading coefficient a_n. It follows that the polynomial $f(x)-a_n{b_m}^{-1}x^{n-m}g(x)$ has degree smaller than n (because the leading coefficients cancel). We can suppose, inductively, that there exist polynomials $p,r\in R[x]$ such that $f(x)-a_n{b_m}^{-1}x^{n-m}g(x)=p(x)g(x)+r(x)$ and $\deg(r)<\deg(g)$. Then we have $f=qg+r$ by taking $q(x)=a_n{b_m}^{-1}x^{n-m}-p(x)$.

Next we prove uniqueness. Suppose that $f=qg+r=pg+s$ where $q,p,r,s\in R[x]$ with $\deg(r)<\deg(g)$ and $\deg(s)<\deg(g)$. Then we have $(q-p)g=s-r$ and so $\deg\big((q-p)g\big)=\deg(s-r)$. Since the leading coefficient of g is a unit (hence not a zero divisor), it follows that $\deg\big((q-p)g\big)=\deg(q-p)+\deg(g)$. If we had $q-p\neq 0$ then we would have $\deg\big((q-p)g\big)\geq\deg(g)$ but $\deg(s-r)<\deg(g)$, giving a contradiction. Thus we must have $q-p=0$. Since $q-p=0$ we have $s-r=(q-p)g=0$. Since $q-p=0$ and $s-r=0$ we have $q=p$ and $r=s$, proving uniqueness.

2.36 Corollary: (*The Remainder Theorem*) *Let R be a ring with 1, let $f\in R[x]$, and let $a\in R$. When we divide $f(x)$ by $(x-a)$ to obtain the quotient $q(x)$ and remainder $r(x)$, the remainder is the constant polynomial $r(x)=f(a)$.*

Proof: Use the division algorithm to obtain $q,r\in R[x]$ such that $f=q(x)(x-a)+r(x)$ and $\deg(r)<\deg(x-a)$. Since $\deg(x-a)=1$ we have $\deg(r)\in\{-1,0\}$, and so r is a constant polynomial, say $r(x)=c$ with $c\in R$. Then we have $f(x)=q(x)(x-a)+c$. Put in $x=a$ to get $f(a)=q(a)(a-a)+c=q(a)\cdot 0+c=c$.

2.37 Corollary: (*The Factor Theorem*) *Let R be a commutative ring with 1, let $f\in R[x]$ and let $a\in R$. Then $f(a)=0$ if and only if $(x-a)\big|f(x)$.*

Proof: Suppose that $f(a)=0$. Choose $q,r\in R[x]$ such that $f(x)=q(x)(x-a)+r(x)$ and $\deg(r)<\deg(x-a)$. Then $r(x)$ is the constant polynomial $r(x)=f(a)=0$ and so we have $f(x)=q(x)(x-a)$. Since $f(x)=(x-a)q(x)$ we have $(x-a)\big|f(x)$. Conversely, suppose that $(x-a)\big|f(a)$ and choose $p\in R[x]$ so that $f(x)=(x-a)p(x)$. Then $f(a)=(a-a)p(a)=0\cdot p(a)=0$.

2.38 Definition: Let R be a commutative ring with 1, let $f\in R[x]$, and let $a\in R$. We say that a is a **root** of f when $f(a)=0$. When $f\neq 0$, we define the **multiplicity** of a as a root of f to be the largest $m=m(f,a)\in\mathbb{N}$ such that $(x-a)^m\big|f(x)$ (where we use the convention that $(x-a)^0=1$). Note that a is a root of f if and only if $m(f,a)\geq 1$.

2.39 Example: Let $f(x)=x^3-3x-2\in\mathbb{Q}[x]$. Since $f(x)=(x+1)^2(x-2)\in\mathbb{Q}[x]$, we have $m(f,2)=1$ and $m(f,-1)=2$.

2.40 Example: Let p be an odd prime and let $f(x)=x^p-a\in\mathbb{Z}_p[x]$. Find $m(f,a)$.

2.41 Theorem: (*The Roots Theorem*) *Let R be an integral domain, let $0 \neq f \in R[x]$ and let $n = \deg(f)$. Then*

(1) f has at most n distinct roots in R, and
(2) if $a_1, a_2, \cdots, a_\ell$ are all of the distinct roots of f in R and $m_i = m(f, a_i)$ for $1 \leq i \leq \ell$, then $(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_\ell)^{m_\ell}\big|f(x)$ and so $\sum_{i=1}^{\ell} m(f,a) \leq n$.

Proof: We prove Part (1) and leave the proof of Part (2) as an exercise. If $\deg(f) = 0$, then $f(x) = c$ for some $0 \neq c \in R$, and so $f(x)$ has no roots. Let f be a polynomial with $\deg(f) = n \geq 1$ and suppose, inductively, that every polynomial $g \in R[x]$ with $\deg(g) = n-1$ has at most $n-1$ distinct roots. Suppose that a is a root of f in R. By the Factor Theorem, $(x-a)\big|f(x)$ so we can choose a polynomial $g \in R[x]$ so that $f(x) = (x-a)g(x)$. Note that $\deg(g) = n-1$ so, by the induction hypothesis, g has at most $n-1$ distinct roots. Let $b \in R$ be any root of f with $b \neq a$. Since $f(x) = (x-a)g(x)$ and $f(b) = 0$ we have $0 = f(b) = (b-a)g(b)$. Since $(b-a)g(b) = 0$ and $(b-a) \neq 0$ and R has no zero divisors, it follows that $g(b) = 0$. Thus b must be one of the roots of g. Since every root b of f with $b \neq a$ is equal to one of the roots of g, and since g has at most $n-1$ distinct roots, it follows that f has at most n distinct roots, as required.

2.42 Example: When R is not an integral domain, a polynomial $f \in R[x]$ of degree n can have more than n roots. For example, in the ring $\mathbb{Z}_6[x]$ the polynomial $f(x) = x^2 + x$ has roots 0, 2, 3 and 5.

2.43 Theorem: (*The Rational Roots Theorem*) *Let $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{Z}[x]$ where $n \in \mathbb{Z}^+$ and $c_n \neq 0$. Let $r, s \in \mathbb{Z}$ with $s \neq 0$ and $\gcd(r,s) = 1$. Then if $f\left(\frac{r}{s}\right) = 0$ then $r|c_0$ and $s|c_n$.*

Proof: Suppose that $f\left(\frac{r}{s}\right) = 0$, that is $c_0 + c_1\frac{r}{s} + c_2\frac{r^2}{s^2} + \cdots + c_n\frac{r^n}{s^n} = 0$. Multiply by s^n to get

$$0 = c_0 s^n + c_1 s^{n-1} r^1 + \cdots + c_{n-1} s^1 r^{n-1} + c_n r^n.$$

Thus we have

$$\begin{aligned} c_0 s^n &= -r\left(c_1 s^{n-1} + \cdots + c_{n-1} s^1 r^{n-2} + c_n r^{n-1}\right) \text{ and} \\ c_n r^n &= -s\big(c_0 s^{n-1} + c_1 s^{n-2} r^1 + \cdots + c_{n-1} r^{n-1}\big) \end{aligned}$$

and it follows that $r\big|c_0 s^n$ and that $s\big|c_n r^n$. Since $\gcd(r,s) = 1$ we also have $\gcd(r, s^n) = 1$, and since $r\big|c_0 s^n$ it follows that $r|c_0$. Since $\gcd(s,r) = 1$ we also have $\gcd(s, r^n) = 1$, and since $s\big|c_n r^n$ it follows that $s|c_n$.

2.44 Example: Show that $\sqrt{1+\sqrt{2}} \notin \mathbb{Q}$.

2.45 Note: Here are a few remarks about irreducible polynomials.

(1) When F is a field, we know that $F[x]$ is a unique factorization domain. For $f \in F[x]$ we know that $f = 0$ if and only if $\deg(f) = -1$, and f is a unit if and only if $\deg(f) = 0$, and for $0 \neq f, g \in F[x]$ we know that $\deg(fg) = \deg(f) + \deg(g)$. It follows that for $f \in F[x]$, if $\deg(f) = 1$ then f is irreducible. It also follows that for $f \in F[x]$, if $\deg(f) = 2$ or 3 then f is reducible in $F[x]$ if and only if f has a f has a root in F.

(2) For $f \in \mathbb{C}[x]$, we know (from the Fundamental Theorem of Algebra) that f is irreducible if and only if $\deg(f) = 1$. For $f \in \mathbb{R}[x]$, we know that f is irreducible polynomial if and only if either $\deg(f) = 1$ or $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$ and $b^2 - 4ac < 0$.

(3) When p is a fairly small prime number and n is a fairly small positive integer, it is easy to list all reducible and irreducible polynomials $f \in \mathbb{Z}_p[x]$ with $\deg(f) \leq n$. Note that it suffices to list monic polynomials (since for $f \in \mathbb{Z}_p[x]$ and $0 \neq c \in \mathbb{Z}_p[x]$ we have $f \sim cf$). We start by listing all monic polynomials of degree 1, that is all polynomials of the form $f(x) = x + a$ with $a \in \mathbb{Z}_p$, and noting that they are all irreducible. Having constructed all reducible and irreducible monic polynomials of all degrees less than n, we can construct all of the reducible monic polynomials of degree n by forming products of the reducible monic polynomials of smaller degree in all possible ways, and then all the remaining monic polynomials of degree n must be irreducible.

2.46 Example: Note that $f(x) = x^3 - 3x + 1$ is irreducible in $\mathbb{Q}[x]$ because it is cubic and has no roots in $\mathbb{Q}$ by the Rational Roots Theorem. The same polynomial is reducible in $\mathbb{R}[x]$ and in $\mathbb{C}[x]$ because it is cubic.

2.47 Example: List all monic reducible and irreducible polynomials in $\mathbb{Z}_2[x]$ of degree less than 4, then determine the number of irreducible polynomials in $\mathbb{Z}_2[x]$ of degree 4.

2.48 Definition: Let R be an integral domain. Define a binary relation on the set $R \times (R \setminus \{0\})$ by stipulating that

$$(a, b) \sim (b, d) \iff ad = bc.$$

It is easy to check that this is an equivalence relation. Let

$$F = Q(R) = \big(R \times (R \setminus \{0\})\big) \big/ \sim \; = \Big\{ [(a, b)] \Big| a, b \in R, b \neq 0 \Big\}.$$

Define addition an multiplication operations on F by

$$\big[(a, b)\big] + \big[(c, d)\big] = \big[(ad + bc \, , \, bd)\big],$$
$$\big[(a, b)\big] \, \big[(c, d)\big] = \big[(ac \, , \, bd)\big].$$

It is not hard to verify that these operations are well-defined (noting that when $b \neq 0$ and $d \neq 0$ we also have $bd \neq 0$ because R is an integral domain) and that they make F into a field with zero element $[(0, 1)]$ and identity element $[(1, 1)]$. This field $F = Q(R)$ is called the **quotient field** of the integral domain R. For $a, b \in R$ with $b \neq 0$ we use the following notation:

$$\tfrac{a}{b} = [(a, b)] \; , \; a = [(a, 1)] \; , \; \tfrac{1}{b} = [(1, b)].$$

The use of the notation $a = [(a, 1)]$, for $a \in R$, allows to consider R as a subring of its quotient field F.

2.49 Example: The quotient field of $\mathbb{Z}$ is equal to $\mathbb{Q}$, and the quotient field of $\mathbb{Z}[\sqrt{2}]$ is equal to $\mathbb{Q}[\sqrt{2}]$.

2.50 Example: When R is an integral domain, the quotient field of the polynomial ring $R[x]$ is the **field of rational functions** $R(x) = \left\{ \frac{f}{g} \,\middle|\, f, g \in R[x], g \neq 0 \right\}$. More generally, the quotient field of $R[x_1, \cdots, x_n]$ is the field of rational functions $R(x_1, \cdots, x_n)$.

2.51 Definition: Let R be a unique factorization domain. For a polynomial $f \in R[x]$, the **content** of f, written as $c(f)$, is a greatest common divisor of the coefficients of f. Note that the greatest common divisor is unique up to association and so $c(f)$ is unique up to association, that is up to multiplication by a unit. We often abuse notation by writing $c(f) = a$ when in fact $c(f) \sim a$. We say that f is **primitive** when $c(f) = 1$ (that is when $c(f)$ is a unit). Note that $f = 0$ if and only if $c(f) = 0$. Note that when $f \in R[x]$ and $a \in R$ we have $c(af) = a\, c(f)$. In particular, we have $f = c(f)\, g$ for a primitive polynomial $g \in R[x]$.

2.52 Example: For $f(x) = 6x + 30 \in \mathbb{Z}[x]$ we have $c(f) = 6$. Since $\deg(f) = 1$, it follows that f is irreducible in $\mathbb{Q}[x]$. But since $c(f) = 6$, it follows that f is reducible in $\mathbb{Z}[x]$, indeed in $\mathbb{Z}[x]$ we have $f(x) = 2 \cdot 3 \cdot (x + 5)$.

2.53 Theorem: (*Gauss' Lemma*) *Let R be a UFD with quotient field F.*
(1) For all $f, g \in R[x]$ we have $c(fg) = c(f)c(g)$.
(2) Let $0 \neq f \in R[x]$ and let $g(x) = \frac{1}{c(f)} f(x) \in R[x]$. Then f is irreducible in $F[x]$ if and only if g is irreducible in $R[x]$.
(3) Let $0 \neq f \in R[x]$. Then f is reducible in $F[x]$ if and only if f can be factored as a product of two nonconstant polynomials in $R[x]$.

Proof: Let $f, g \in R[x]$. If $f = 0$ or $g = 0$ then we have $c(fg) = 0 = c(f)c(g)$. Suppose that $f \neq 0$ and $g \neq 0$. Let $h(x) = \frac{1}{c(f)} f(x)$ and $k(x) = \frac{1}{c(g)} g(x)$. Then we have $h, k \in R[x]$ with $c(h) = c(k) = 1$ and $fg = c(f)c(g)hk$ so that $c(fg) = c(f)c(g)c(hk)$. Thus to prove Part (1) it suffices to show that $c(hk) = 1$. Let $h(x) = \sum_{i=0}^{n} a_i x^i$ and $k(x) = \sum_{i=0}^{m} b_i x^i$ with $a_n \neq 0$ and $b_m \neq 0$. Suppose, for a contradiction, that $c(hk) \neq 1$. Let p be a prime factor of $c(hk)$. Then p divides all of the coefficients of $(hk)(x) = (a_0 b_0) + (a_1 b_0 + a_0 b_1)x + \cdots + (a_n b_m)x^{n+m}$. Since $c(h) = 1$, p does not divide all the coefficients of $h(x)$, so we can choose an index $r \geq 0$ so that $p | a_i$ for all $i < r$ and $p \nmid a_r$. Since $c(k) = 1$ we can choose an index $s \geq 0$ so that $p | b_i$ for all $i < s$ and $p \nmid b_s$. Since p divides every coefficient of $(hk)(x)$, it follows that in particular p divides the coefficient

$$c_{r+s} = a_0 b_{r+s} + a_1 b_{r+s-1} + \cdots + a_r b_s + \cdots + a_{r+s-1} b_1 + a_{r+s}.$$

Since $p | c_{r+s}$ and $p | a_i$ for all $i < r$ and $p | b_i$ for all $i < s$ it follows that $p | a_r b_s$. Since p is prime and $p | a_r b_s$ it follows that $p | a_r$ or $p | b_s$. But r and s were chosen so that $p \nmid a_r$ and $p \nmid b_s$ so we have obtained the desired contradiction. This proves Part (1).

To prove Parts (2) and (3), let $0 \neq f(x) \in R[x]$ and let $g(x) = \frac{1}{c(f)} f(x)$, and note that $g \in R[x]$ with $c(g) = 1$. Suppose that g is reducible in $R[x]$, say $g(x) = h(x)k(x)$ where $h(x)$ and $k(x)$ are non-units in $R[x]$. Since $c(h)c(k) = c(hk) = c(g) = 1$ it follows that $c(h) = c(k) = 1$. Note that $h(x)$ cannot be a constant polynomial since if we had $h(x) = r$ with $r \in R$, then we would have $c(h) = r$ and also $c(h) = 1$ so that r is a unit in R, but then h would be a unit in $R[x]$. Similarly $k(x)$ cannot be a constant polynomial. Since $h(x)$ and $k(x)$ are nonconstant polynomials in $R[x]$, they are also nonconstant polynomials in $F[x]$. Since $f(x) = c(f)g(x) = c(f)h(x)k(x)$ and since $c(f)h(x)$ and $k(x)$ are both nonconstant polynomials (hence nonunits) in $F[x]$, it follows that $f(x)$ is reducible in $F[x]$.

Conversely, suppose that $f(x)$ is reducible in $F[x]$, say $f(x) = h(x)k(x)$ where h and k are nonzero, nonunits in $F[x]$. Since h and k are nonzero nonunits in $F[x]$, they are nonconstant polynomials. Let a be a least common multiple of the denominators of the coefficients of $h(x)$ and let b be a least common multiple of denominators of the coefficients of $k(x)$, and note that $ah(x) \in R[x]$ and $bk(x) \in R[x]$. Let $p(x) = \frac{1}{c(ah)}ah(x)$ and let $q(x) = \frac{1}{c(bk)}bk(x)$ and note that $p(x), q(x) \in R[x]$ with $c(p) = c(q) = 1$ and that $\deg(p) = \deg(h)$ and $\deg(q) = \deg(k)$. Since $f(x) = ah(x)\,bk(x) = c(ah)c(bk)p(x)q(x)$ we have $c(f) = c(ah)c(bk)c(pq) = c(ah)c(bk)$ so $g(x) = \frac{1}{c(f)}f(x) = \frac{1}{c(ah)c(bk)}ah(x)\,bk(x) = p(x)q(x)$. Since $g(x) = p(x)q(x)$ where $p(x)$ and $q(x)$ are nonconstant polynomials in $R[x]$, we see that $g(x)$ is reducible in $R[x]$.

2.54 Theorem: (*Modular Reduction*) *Let* $f(x) = \sum_{i=0}^{n} c_i x^i$ *with* $n \in \mathbb{Z}^+$, $c_i \in \mathbb{Z}$ *and* $c_n \neq 0$. *Let* p *be a prime number with* $p \not| c_n$. *Let* $\overline{f}(x) = \sum_{i=0}^{n} \overline{c_i}\, x^i \in \mathbb{Z}_p[x]$ *where* $\overline{c_i} = [c_i] \in \mathbb{Z}_p$. *If* $\overline{f}$ *is irreducible in* $\mathbb{Z}_p[x]$ *then* f *is irreducible in* $\mathbb{Q}[x]$.

Proof: Suppose that $f(x)$ is reducible in $\mathbb{Q}[x]$. By Gauss' Lemma, we can choose two nonconstant polynomials $g, h \in \mathbb{Z}[x]$ such that $f = gh \in \mathbb{Z}[x]$. Write $g(x) = \sum_{i=0}^{k} a_i x^k \in \mathbb{Z}[x]$ and $h(x) = \sum_{i=0}^{\ell} b_i x^i \in \mathbb{Z}[x]$ with $a_k \neq 0$, $b_\ell \neq 0$ and $k, \ell \geq 1$. Let $\overline{g} = \sum_{i=0}^{k} \overline{a}_i x^i \in \mathbb{Z}_p[x]$ and $\overline{h}(x) = \sum_{i=0}^{\ell} \overline{b}_i x^i \in \mathbb{Z}_p[x]$, and note that $\overline{f} = \overline{g}\,\overline{h} \in \mathbb{Z}_p[x]$. Since $c_n = a_k b_\ell$ and $p \not| c_n$ it follows that $p \not| a_k$ and $p \not| b_\ell$ in $\mathbb{Z}$ so $\overline{a}_k \neq 0$ and $\overline{b}_\ell \neq 0$ in $\mathbb{Z}_p$. Thus $\deg(\overline{g}) = \deg(g) = k$ and $\deg(\overline{h}) = \deg(h) = \ell$ so that $\overline{g}$ and $\overline{h}$ are nonconstant polynomials in $\mathbb{Z}_p[x]$, and so the polynomial $\overline{f} = \overline{g}\overline{h}$ is reducible in $\mathbb{Z}_p[x]$.

2.55 Exercise: Prove that $f(x) = x^5 + 2x + 4$ is irreducible in $\mathbb{Q}[x]$ by working in $\mathbb{Z}_3[x]$.

2.56 Exercise: Show that $f(x) = x^4 + 1$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{Z}_p[x]$ for every prime number $p \in \mathbb{Z}^+$.

2.57 Theorem: (*Eisenstein's Criterion*) *Let* $f(x) = \sum_{i=0}^{n} c_i x^i$ *with* $n \in \mathbb{Z}^+$, $c_i \in \mathbb{Z}$ *and* $c_n \neq 0$. *Let* p *be a prime number such that* $p_i | c_i$ *for* $0 \leq i < n$ *and* $p \not| c_n$ *and* $p^2 \not| c_0$. *Then* f *is irreducible in* $\mathbb{Q}[x]$.

Proof: Suppose, for a contradiction, that $f(x)$ is reducible in $\mathbb{Q}[x]$. By Gauss' Lemma, we can choose two nonconstant polynomials $g, h \in \mathbb{Z}[x]$ such that $f = gh \in \mathbb{Z}[x]$. Write $g(x) = \sum_{i=0}^{k} a_i x^k \in \mathbb{Z}[x]$ and $h(x) = \sum_{i=0}^{\ell} b_i x^i \in \mathbb{Z}[x]$ with $k, \ell \geq 1$ and $a_k \neq 0$, $b_\ell \neq 0$. Since $c_0 = a_0 b_0$ and $p | c_0$ but $p^2 \not| c_0$, it follows that p divides exactly one of the two numbers a_0 and b_0. Suppose that p divides a_0 but not b_0 (the case that p divides b_0 but not a_0 is similar). Since $p | c_1$, that is $p \big| (a_0 b_1 + a_1 b_0)$, and $p | a_0$ it follows that $p | a_1 b_0$, and since $p \not| b_0$ it follows that $p | a_1$. Since $p | c_2$, that is $p \big| (a_0 b_2 + a_1 b_1 + a_2 b_0)$ and $p | a_0$ and $p | a_1$, it follows that $p | a_2 b_0$, and since $p \not| b_0$ it then follows that $p | a_2$. Repeating this argument we find, inductively, that $p | a_i$ for all $i \geq 0$, and in particular we have $p | a_k$. Since $c_n = a_k b_\ell$ and $p | a_k$ it follows that $p | c_n$, giving the desired contradiction.

2.58 Example: Note that $f(x) = 5x^5 + 3x^4 - 18x^3 + 12x + 6$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion using $p = 3$.

2.59 Exercise: Let p be a prime number. Show that $f(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is irreducible in $\mathbb{Q}[x]$,

2.60 Theorem: *If R is a UFD then so is $R[x]$.*

Proof: Suppose that R is a UFD and let F be the quotient field of R. Note that the units in $R[x]$ are the constant polynomials which are also units in R. Let $f \in R[x]$ be a non-zero non-unit. If f is a constant polynomial, then the factorization of f in $R[x]$ is the same as the factorization of f in R. Suppose that $\deg(f) \geq 1$. Let $g = \frac{1}{c(f)} f$ so that $g \in R[x]$ with $c(g) = 1$. The factorization of $c(f)$ in $R[x]$ is the same as the factorization in R, so it suffices to show that the polynomial g factors uniquely into irreducibles in $R[x]$. Since $F[x]$ is a ED, hence a UFD, we know that g factors into irreducibles in $F[x]$. By Gauss' Lemma, we can multiply each of the irreducible factors in $F[x]$ by an element of F to write g as a product of irreducible factors in $R[x]$, say $g = f_1 f_2 \cdots f_\ell$ where each f_j is irreducible in $R[x]$. Since $c(g) = 1$ we must have $c(f_j) = 1$ for each index j.

Suppose that $g = f_1 f_2 \cdots f_\ell = g_1 g_2 \cdots g_m$ where f_j and g_k are irreducible in $R[x]$ with $c(f_j) = c(g_k) = 1$ for all j, k. Note that each f_j must be non-constant since if we had $f_j(x) = r \in R$ then we would have $c(f_j) = r$ and $c(f_j) = 1$ so that r is a unit in R, but then f_j would be a unit in $R[x]$. Similarly each g_k is non-constant. It follows that the polynomials f_j and g_k are also irreducible in $F[x]$. By unique factorization in $F[x]$, we must have $m = \ell$ and, after possibly reordering the polynomials g_k, we have $f_j \sim g_j$ in $F[x]$ for all indices j. Since $f_j \sim g_j$ in $F[x]$, we have $g_j = u f_j$ for some $0 \neq u \in F$. Say $u = \frac{a}{b}$ where $a, b \in R$ with $\gcd(a, b) = 1$. Then we have $a\, f_j = b\, g_j$ in $R[x]$. Since $c(f_j) = c(g_j) = 1$ we have $c(a f_j) = a$ and $c(b g_j) = b$ and it follows that $a \sim b$ in R, hence $a = bv$ for some unit $v \in R$. Thus we have $g_j = u f_j = \frac{a}{b} f_j = v f_j$ and so $f_j \sim g_j$ in $R[x]$.

2.61 Corollary: *If R is a UFD then so is the polynomial ring $R[x_1, x_2, \cdots, x_n]$.*

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