Chapter 1. Complex Numbers

1.1 Definition: A **complex number** is a vector in $\mathbf{R}^2$. The **complex plane** , denoted by $\mathbf{C}$, is the set of complex numbers:

$$\mathbf{C} = \mathbf{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x \in \mathbf{R}, y \in \mathbf{R} \right\} .$$

In $\mathbf{C}$ we usually write $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $iy = yi = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix} .$$

1.2 Definition: If $z = x + iy$ with $x, y \in \mathbf{R}$ then x is called the **real** part of z and y is called the **imaginary** part of z, and we write

$$\operatorname{Re} z = x \text{ , and } \operatorname{Im} z = y \, .$$

1.3 Definition: We define the **sum** of two complex numbers to be the usual vector sum:

$$(a + ib) + (c + id) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix} = (a + c) + i(b + d) \, ,$$

where $a, b \in \mathbf{R}$. We define the **product** of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative and associative and distributive over the sum:

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc) \, .$$

1.4 Example: Let $z = 2 + i$ and $w = 1 + 3i$. Find $z + w$ and zw.

Solution: $z + w = (2 + i) + (1 + 3i) = (2 + 1) + i(1 + 3) = 3 + 4i$, and $zw = (2 + i)(1 + 3i) = 2 + 6i + i - 3 = -1 + 7i$.

1.5 Example: Show that every non-zero complex number has a unique inverse z^{-1} and find a formula for the inverse.

Solution: We let $z = a + ib$, $a, b \in \mathbf{R}$, and we solve $(a + ib)(x + iy) = 1$ to find $z^{-1} = x + iy$: $(a + ib)(x + iy) = 1 \iff (ax - by) + i(ay + bx) = 1 \iff \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \iff x + iy = \frac{1}{a^2 + b^2}(a - ib)$. Thus

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2} \, .$$

1.6 Notation: For $z, w \in \mathbf{C}$ we use the following obvious notation:

$$-z = -1z \; , \quad w - z = w + (-z) \; , \quad \frac{1}{z} = z^{-1} \text{ and } \frac{w}{z} = wz^{-1}.$$

1.7 Example: Find $\dfrac{(4 - i) - (1 - 2i)}{1 + 2i}$.

Solution: $\dfrac{(4-i)-(1-2i)}{1+2i} = \dfrac{3+i}{1+2i} = (3+i)(1+2i)^{-1} = (3+i)(\frac{1}{5} - \frac{2}{5}i) = 1-i.$

1.8 Note: The set of complex numbers is a **field** under the operations of addition and multiplication. This means that for all u, v and w in $\mathbf{C}$ we have

$$\begin{aligned}
&u+v = v+u \\
&(u+v)+w = u+(v+w) \\
&0+u = u \\
&u+(-u) = 0 \\
&uv = vu \\
&(uv)w = u(vw) \\
&1u = u \\
&u\,u^{-1} = 1 \text{ if } u \neq 0 \\
&u(v+w) = uv+uw
\end{aligned}$$

1.9 Definition: If $z = x+iy$ with $x, y \in \mathbf{R}$ then we define the **conjugate** of z to be

$$\overline{z} = x - iy\,.$$

1.10 Definition: If $z = x+iy$ with $x, y \in \mathbf{R}$ then we define the **length** (or **magnitude**) of z to be

$$|z| = \sqrt{x^2+y^2}\,.$$

Given two complex numbers z and w, we define the **distance** between z and w to be

$$d(z,w) = |z-w|\,.$$

1.11 Note: For z and w in $\mathbf{C}$ the following identities are all easy to verify.

$$\begin{aligned}
&\overline{\overline{z}} = z \\
&z+\overline{z} = 2\mathrm{Re}\,z\ , \quad z-\overline{z} = 2i\mathrm{Im}\,z \\
&z\overline{z} = |z|^2\ , \quad |\overline{z}| = |z| \\
&\overline{z+w} = \overline{z}+\overline{w}\ , \quad \overline{zw} = \overline{z}\,\overline{w}\ , \quad |zw| = |z||w|
\end{aligned}$$

1.12 Definition: If $z \neq 0$, we define the **angle** (or **argument**) of z to be the angle $\theta(z)$ from 1 counterclockwise to z. In other words, $\theta(z)$ is the angle such that

$$z = |z|\big(\cos\theta(z) + i\sin\theta(z)\big)\,.$$

1.13 Note: We can think of the angle $\theta(z)$ in several different ways. We can require, for example, that $0 \leq \theta(z) < 2\pi$ so that the angle is uniquely determined. Or we can allow $\theta(z)$ to be any real number, in which case the angle will be unique up to a multiple of 2π. Then again, we can think of $\theta(z)$ as an infinite set of real numbers; $\theta(z) = \{\theta + 2\pi k | k \in z\}$. Perhaps best of all, we can think of $\theta(z)$ as an element of $\mathbf{R}/2\pi$, the set of real numbers modulo 2π (If $\alpha = \beta + 2\pi$ then $\alpha \neq \beta \in \mathbf{R}$ but $\alpha = \beta \in \mathbf{R}/2\pi$).

1.14 Notation: For $\theta \in \mathbf{R}$ (or for $\theta \in \mathbf{R}/2\pi$) we shall write

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

1.15 Note: If $z \neq 0$ and we have $x = \mathrm{Re}\,(z)$, $y = \mathrm{Im}\,(z)$, $r = |z|$ and $\theta = \theta(z)$ then

$$\begin{aligned} &x = r\cos\theta\ , \quad y = r\sin\theta \\ &r = \sqrt{x^2+y^2}\ , \quad \tan\theta = \frac{y}{x},\ \text{if } x \neq 0 \\ &z = re^{i\theta}\ , \quad \overline{z} = r\,e^{-i\,\theta}\ , \quad z^{-1} = \frac{1}{r}e^{-i\,\theta} \end{aligned}$$

We say that $x + i\,y$ is the **cartesian** form of z and $re^{i\,\theta}$ is the **polar** form.

1.16 Example: Let $z = -3 - 4i$. Express z in polar form.

Solution: We have $|z| = 5$ and $\tan\theta(z) = \frac{4}{3}$. Since $\theta(z)$ is in the third quadrant, we have $\theta(z) = \pi + \tan^{-1}\frac{4}{3}$. So $z = 5e^{i(\pi+\tan^{-1}(4/3))}$.

1.17 Example: Let $z = 10e^{i\,\tan^{-1}3}$. Express z in cartesian form.

Solution: $z = 10\left(\cos(\tan^{-1}3) + i\,\sin(\tan^{-1}3)\right) = 10\left(\frac{1}{\sqrt{10}} + i\frac{3}{\sqrt{10}}\right) = \sqrt{10} + 3\sqrt{10}\,i$.

1.18 Example: Find a formula for multiplication in polar coordinates.

Solution: For $z = re^{i\alpha}$ and $w = e^{i\beta}$ we have $zw = rs(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\big) = rs\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big)$ and so we obtain the formula

$$re^{i\alpha}se^{i\beta} = rs\,e^{i(\alpha+\beta)}\,.$$

1.19 Note: An immediate consequence of the above example is that

$$(r\,e^{i\,\theta})^n = r^n e^{i\,n\theta}$$

for $r, \theta \in \mathbf{R}$ and for $n \in \mathbf{Z}$. This result is known as **De Moivre's Law**.

1.20 Example: Find $(1+i)^{10}$.

Solution: This can be done in cartesian coordinates using the binomial theorem (which holds for complex numbers), but it is easier in polar coordinates. We have $1 + i = \sqrt{2}e^{i\,\pi/4}$ so $(1+i)^{10} = (\sqrt{2}e^{i\,\pi/4})^{10} = (\sqrt{2})^{10}e^{i\,10\pi/4} = 32e^{i\,\pi/2} = 32i$.

1.21 Example: Find a formula for the n^{th} roots of a complex number. In other words, given $z = re^{i\theta}$, solve $w^n = z$.

Solution: Let $w = se^{i\alpha}$. We have $w^n = z \iff (se^{i\alpha})^n = re^{i\theta} \iff s^n e^{i\,n\alpha} = re^{i\theta} \iff s^n = r$ and $n\alpha = \theta + 2\pi k$ for some $k \in \mathbf{Z} \iff s = \sqrt[n]{r}$ and $\alpha = \dfrac{\theta + 2\pi k}{n}$ for some $k \in \mathbf{Z}$. Notice that when $z \neq 0$ there are exactly n solutions obtained by taking $0 \leq k < n$. So we obtain the formula

$$(r\,e^{i\,\theta})^{1/n} = \sqrt[n]{r}\,e^{i\,(\theta+2\pi k)/n} \quad , \qquad k \in \{0, 1, \ldots, n-1\}\,.$$

In particular, $(r\,e^{i\,\theta})^{1/2} = \pm\sqrt{r}\,e^{i\,\theta/2}$. For $0 < a \in \mathbf{R}$ we have $z^2 = a \iff z = \pm\sqrt{a}$, and for $0 > a \in \mathbf{R}$ we have $z^2 = a \iff z = \pm\sqrt{|a|}\,i$.

1.22 Note: For $0 \neq w \in \mathbf{C}$, we can think of $w^{1/n}$ as any one of the n solutions to $z^n = w$, or we can think of it as the set of all n solutions. Consider the following "proof" that 1=-1:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i^2 = -1\,.$$

1.23 Note: It is not hard to show that the quadratic formula works for complex numbers; indeed, for $a, b, c \in \mathbf{C}$ with $a \neq 0$ we have $az^2 + bz + c = 0 \iff z^2 + \frac{b}{a}\,z + \frac{c}{a} = 0 \iff (z - \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$. Thus $az^2 + bz + c = 0$ if and only if

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,,$$

where $\sqrt{b^2 - 4ac}$ denotes either one of the two square roots when $b^2 \neq 4ac$.

1.24 Example: Let $f(z) = z^4 + 2z^2 + 4$. Factor f over the complex numbers.

Solution: By the quadratic formula, $f(z) = 0$ when $z^2 = -1 \pm \sqrt{3}\,i$ or in polar coordinates $z = 2e^{\pm i\,2\pi/3}$. Thus the roots of f are $z = \pm\sqrt{2}e^{\pm i\,\pi/3}$, and so f factors as

$$z^4 + 2z^2 + 4 = \left(z - \sqrt{2}e^{i\,\pi/3}\right)\left(z - \sqrt{2}e^{-i\,\pi/3}\right)\left(z + \sqrt{2}e^{i\,\pi/3}\right)\left(z + \sqrt{2}e^{-i\,\pi/3}\right).$$

1.25 Note: We do *not* have inequalities between complex numbers. We can *only* write $a < b$ or $a \leq b$ in the case that a and b are both *real* numbers. But there are several inequalities between *real* numbers which concern complex numbers. For $z \in \mathbf{C}$ and $w \in C$,

$$|\mathrm{Re}\,(z)| \leq |z|\ , \quad |\mathrm{Im}\,(z)| \leq |z|$$
$$|z + w| \leq |z| + |w|\ , \quad \text{this is called the } \textbf{triangle inequality}$$
$$|z + w| \geq \big||z| - |w|\big|$$

The first two inequalities follow from the fact that $|z|^2 = |\mathrm{Re}\,(z)|^2 + |\mathrm{Im}\,(z)|^2$. We can then prove the triangle inequality as follows: $|z+w|^2 = (z+w)(\overline{z}+\overline{w}) = |z|^2+|w|^2+(w\overline{z}+z\overline{w}) = |z|^2 + |w|^2 + 2\mathrm{Re}\,(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$. The last inequality follows from the triangle inequality since $|z| = |z + w - w| \leq |z + w| + |w|$ and $|w| = |z + w - z| \leq |z + w| + |z|$. (Alternatively, the last two inequalities can be proven using the Law of Cosines).

1.26 Example: Given complex numbers a and b, describe the set $\left\{z \in \mathbf{C}\big||z-a| < |z-b|\right\}$.

Solution: Geometrically, this is the set of all z such that z is closer to a than to b, so it is the **half-plane** which contains a and lies on one side of the perpendicular bisector of the line segment ab.

1.27 Example: Given a complex number a, describe the set $\left\{z \in \mathbf{C}\big|1 < |z - a| < 2\right\}$.

Solution: $\left\{z\big||z - a| = 1\right\}$ is the circle centred at a of radius 1 and $\left\{z\big||z - a| = 2\right\}$ is the circle centred at a of radius 2, and $\left\{z \in \mathbf{C}\big|1 < |z - a| < 2\right\}$ is the region between these two circles. Such a region is called an **annulus**.

1.28 Note: Historically, complex numbers arose in the study of cubic equations. An equation of the form $ax^3+bx^2+cx+d=0$, where $a,b,c,d\in\mathbf{C}$ with $a\neq 0$ can be solved as follows. First, divide by a to obtain an equation of the form $x^3+Bx^2+Cx+D=0$. Next, make the substitution $y=x+\frac{B}{3}$ and rewrite the equation in the form $y^3+py+q=0$. To solve this, look for a solution of the form $y=z+r\,z^{-1}$. We need $0=y^3+py+q=(z^3+3rz+3r^2z^{-1}+r^3z^{-3})+p(z+rz^{-1})+q=z^3+(3r+p)z+q+r(3r+p)z^{-1}+r^3z^{-3}$. Choose $r=-p/3$ so that this simplifies to $z^3+q-\frac{p^3}{27}z^{-3}=0$. Finally, we multiply by z^3 to obtain $z^6+qz^3-\frac{p^3}{27}$, which we can solve for z^3 using the quadratic formula.

1.29 Example: Let $f(x)=x^3+3x^2+4x+1$. Note that $f'(x)=3x^2+6x+4=3(x+1)^2+1>0$, so f is increasing and hence has exactly one real root. Find the real root of f.

Solution: Let $y=x+1$. Then $x^3+3x^2+4x+1=(y-1)^3+3(y-1)^2+4(y-1)+1=y^3+y-1$. Try $y=z+rz^{-1}$ with $r=-\frac{1}{3}$, so we have $y^3+y-1=(z-\frac{1}{3}z^{-1})^3+(z-\frac{1}{3}z^{-1})-1=z^3-1-\frac{1}{27}z^{-3}$. We solve $z^6-z^3-\frac{1}{27}=0$ using the quadratic formula, and obtain $z^3=\frac{1\pm\sqrt{\frac{31}{27}}}{2}$. If $z=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}$ then $r\,z^{-1}=-\frac{1}{3}\sqrt[3]{\frac{2}{1+\sqrt{\frac{31}{27}}}}=-\frac{1}{3}\sqrt[3]{\frac{2(1-\sqrt{\frac{31}{27}})}{1-\frac{31}{27}}}=\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$. Similarly, if $z=\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$ then $r\,z^{-1}=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}$. In either case we have $y=z+rz^{-1}=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}+\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$, and $x=y-1=\sqrt[3]{\frac{\sqrt{\frac{31}{27}}+1}{2}}-\sqrt[3]{\frac{\sqrt{\frac{31}{27}}-1}{2}}-1$. (Notice that we didn't use complex numbers in this example).

1.30 Example: Find the three real roots of $f(x)=x^3-3x+1$.

Solution: Try $x=z+rz^{-1}$ with $r=1$ so that $f(x)=(z+z^{-1})^3-3(z+z^{-1})+1=z^3+1+z^{-3}$. Multiply by z^3 and solve $z^6+z^3+1=0$ to get $z^3=\frac{-1\pm\sqrt{3}\,i}{2}=e^{\pm i\,2\pi/3}$. If $z^3=e^{i\,2\pi/3}$ then $z=e^{i\,2\pi/9}$, $e^{i\,8\pi/9}$ or $e^{i\,14\pi/9}$ and so $x=z+z^{-1}=z+\overline{z}=2\mathrm{Re}\,(z)=2\cos(\frac{2\pi}{9})$, $2\cos(\frac{8\pi}{9})$ or $2\cos(\frac{14\pi}{9})$. If $z^3=e^{-i\,2\pi/3}$ then we obtain the same values for x. Thus the three real roots are $2\cos(40^\circ)$, $-2\cos(20^\circ)$ and $2\cos(80^\circ)$. (Notice that in this example we used complex numbers to solve a problem involving real variables!)

Chapter 2. Complex Functions

2.1 Note: A map $f : \mathbf{R} \to \mathbf{R}$ (or $f : I \to \mathbf{R}$ where I is an interval in $\mathbf{R}$) may be visualized by drawing a picture of its **graph**, which is a curve in $\mathbf{R}^2$:

$$\text{Graph}\,(f) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| y = f(x) \right\} .$$

2.2 Note: A map $f : \mathbf{R} \to \mathbf{C}$ (or a map $f : I \to \mathbf{C}$ where I is an interval in $\mathbf{R}$) may be visualized by drawing its **image**, which is a curve in $\mathbf{C}$:

$$\text{Image}\,(f) = \{ f(t) \in \mathbf{C} | t \in \mathbf{R} \} .$$

2.3 Example: The **line segment** from $a \in \mathbf{C}$ to $b \in \mathbf{C}$ is the image of the map

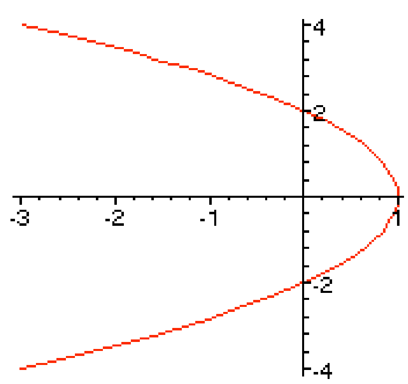
$$z(t) = a + t(b - a) \ , \quad 0 \le t < 1 .$$

2.4 Example: The **circle** centred at $a \in \mathbf{C}$ with radius $r > 0$ is the image of the map

$$z(t) = a + r \cos t + i\, r \sin t = a + r\, e^{i\,t} .$$

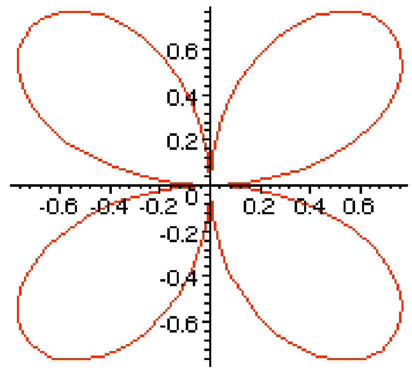
2.5 Example: Describe the image of the map $z(t) = (1 + i\,t)^2$.

Solution: We can sketch the image of *any* map $z(t)$ simply by plotting points. Try plotting the points $z(t)$ for $t = -2, -1, 0, 1, 2$. For this particular map, we can eliminate the parameter t to describe the image: $z(t) = (1 + i\,t)^2 = (1 - t^2) + i\,(2t)$ so we have $x = 1 - t^2$ and $y = 2t$, and so $x = 1 - \frac{1}{4}y^2$. This shows that the image is the parabola $x = 1 - \frac{1}{4}y^2$.



2.6 Example: Describe the image of the map $z(t) = \sin(2t)e^{i\,t}$.

Solution: Since $z(t)$ is given in polar coordinates, it is easier to sketch this curve on a polar grid (the cartesian grid consists of vertical lines $x =$ const. and horizontal lines $y =$ const., while the polar grid consists of cirles $r =$ const. and rays $\theta =$ const.) Sketch the curve on a polar grid which includes the rays $\theta = \frac{\pi}{12}k$, and you will see that the curve is a four-leafed rose: it consists of one loop in each of the four quadrants.



2.7 Note: To visualize a map $f : \mathbf{C} \to \mathbf{R}$ (or a map $f : U \to \mathbf{R}$ where $U \subset \mathbf{C}$) we can draw the **level curves** (also called **contour lines**). These are the inverse images of constant values of $u \in \mathbf{R}$, and they are curves in $\mathbf{C}$:

$$f^{-1}(u) = \{z \in \mathbf{C} | f(z) = u\} .$$

We can use the level curves of f to help draw its **graph**, which is a surface in $\mathbf{R}^3$:

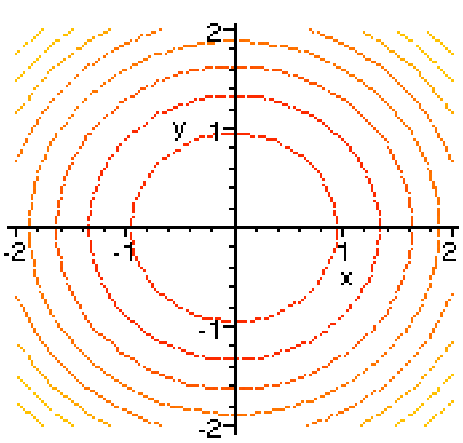
$$\text{Graph}\,(f) = \left\{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbf{R}^3 \middle| u = f(x+iy) \right\}$$

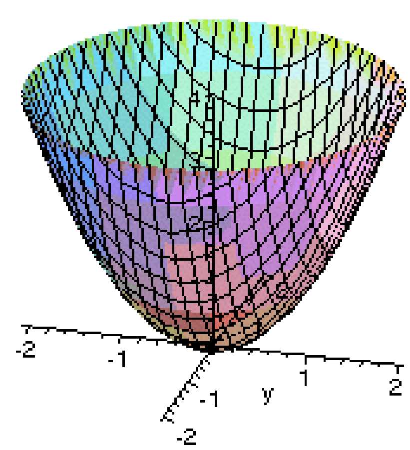
2.8 Example: Describe the level curves and the graph of the map $u = f(z) = \text{Re}\,(z)$.

Solution: We have $f^{-1}(u) = \{u + i\,y \,|\, y \in \mathbf{R}\}$, which is the line $x = u$. And we have $\text{Graph}\,(f) = \left\{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} \middle| u = x \right\}$, which is the plane through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ perpendicular to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

2.9 Example: Describe the level curves and the graph of $u = f(z) = |z|^2$.

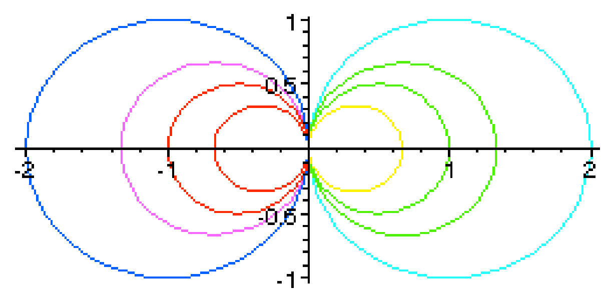
Solution: We have $f^{-1}(u) = \{x + i\,y \,|\, x^2 + y^2 = u\}$ which, for $u > 0$, is the circle about the origin of radius $\sqrt{u}$. Also, $\text{Graph}\,(f) = \left\{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} \middle| u = x^2 + y^2 \right\}$, which is a paraboloid.





2.10 Example: Sketch the level curves of $u = f(z) = \mathrm{Re}\,(1/z)$.

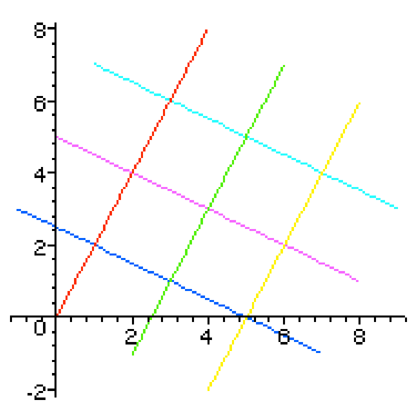
Solution: We have $u(x+i\,y) = \dfrac{x}{x^2+y^2}$. When $u=0$ we have $x=0$, and when $u \neq 0$ we have $\dfrac{x}{x^2+y^2} = u \iff x = u\,x^2 + u\,y^2 \iff x^2 - \frac{x}{u} + y^2 = 0 \iff (x - \frac{1}{2u})^2 + y^2 = \frac{1}{4u^2}$ so the level curve u =constant is the circle centred at $(\frac{1}{2u}, 0)$ with radius $\frac{1}{2u}$. These circles all go through the origin. If you sketch several of them you will see that they form the pattern which is made by the electric field of a dipole (a small bar magnet).



2.11 Note: To visualize a map $f : \mathbf{C} \to \mathbf{C}$ (or a map $f : U \to \mathbf{C}$ where $U \subset \mathbf{C}$) we can sketch the **images** of various curves in the domain (if $z = x + i\,y$ then we usually draw the images of the lines $x =$ const. and $y =$ const. while if $z = r\,e^{i\,\theta}$ then we draw the images of the circles $r =$ const. and the rays $\theta =$ const.). Alternatively, we could draw the **inverse images** of various curves in the range (if $w = f(z)$ with $w = u + i\,v$ then we might draw the inverse images of the lines $u =$ const. and $v =$ const.)

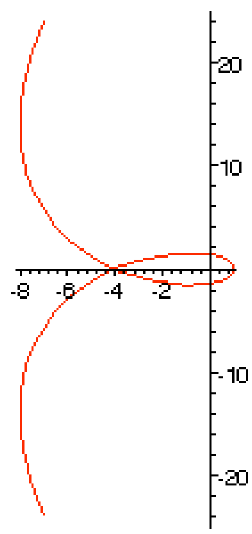
2.12 Example: Give a geometric description of the map $w(z) = a\,z + b$ where $a \in \mathbf{C}$ and $b \in \mathbf{C}$. Sketch the images of the lines $x = -1, 0, 1$ and $y = -1, 0, 1$ when $z = x + i\,y$ and $a = 1 + 2i$ and $b = 4 + 3i$.

Solution: If $a = re^{i\,\alpha}$ and $z = s\,e^{i\,\beta}$ then $a\,z = (r\,s)e^{i(\alpha+\beta)}$, so multiplying z by a has the effect of scaling z by a factor of $r = |a|$ and rotating the result about the origin by the angle $\alpha = \theta(a)$. Adding b is the same as translating by b. This geometric description shows that the three vertical lines $x = -1, 0, 1$ will be sent to the three lines which are parallel to $a\,i = -2 + i$ and which pass through the points $w(-1) = 3 + i$, $w(0) = 4 + 3i$ and $w(1) = 5 + 5i$, respectively, and the three horizontal lines $y = -1, 0, 1$ are sent to the three lines parallel to $a = 1 + 2i$ through $w(-i) = 6 + 2i$, $w(0) = 4 + 3i$ and $w(i) = 2 + 4i$, respectively. This can also be shown algebraically. For example, the vertical line $x = c$ is given parametrically by $z(t) = c + i\,t$, $t \in \mathbf{R}$, and it is sent to $w(z(t)) = a(c + i\,t) + b = ac + b + i\,at = w(c) + at$, which is the line through $w(c)$ parallel to $i\,a$.



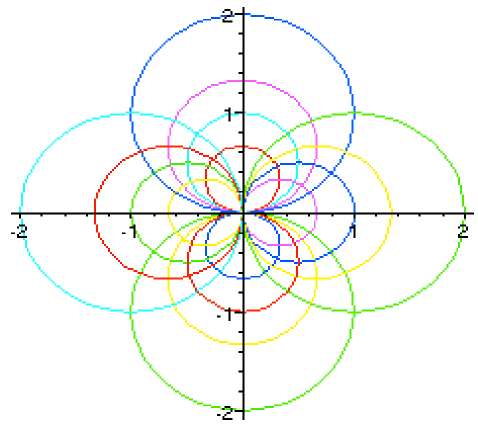
2.13 Example: Let $w(z) = z^4$. Describe the images of the circles $r = \text{const.}$ and the rays $\theta = \text{const.}$ where $z = r\,e^{i\,\theta}$. Also, sketch the image of the line $x = 1$, where $z = x + i\,y$.

Solution: We have $w = (r\,e^{i\,\theta})^4 = r^4 e^{i\,4\theta}$, so if $w = s\,e^{i\,\phi}$ then we have $s = r^4$ and $\phi = 4\theta$. Thus the circle $r = c$ is mapped to the circle $s = c^4$ and the ray $\theta = \alpha$ is mapped to the ray $\phi = 4\alpha$. The line $x = 1$ is given parametrically by $z = 1 + i\,t$ and it is mapped to the curve $w(t) = (1 + i\,t)^4 = 1 + 4t\,i - 6t^2 - 4t^3\,i + t^4 = (1 - 6t^2 + t^4) + i\,(4t - 4t^3)$, so its image is the curve given parametrically by $u(t) = 1 - 6t^2 + t^4$ and $v(t) = 4t - 4t^3$. The u-intercepts occur when $v = 0$, that is when $t = 0, \pm 1$ and the v-intercepts occur when $u = 0$, that is when $t^2 = 3 \pm 2\sqrt{2}$. Also, We have $u'(t) = -12t + 4t^3 = 4t(t^2 - 3)$ and $v'(t) = 4 - 12t^2 = 4(1 - 3t^2)$, and so the curve is vertical when $u'(t) = 0$, that is when $t = 0, \pm\sqrt{3}$ and it is horizontal when $v'(t) = 0$, that is when $t = \pm 1/\sqrt{3}$. To sketch the curve, plot the points when $t = 0, \pm 1/\sqrt{3}, \pm 1, \pm\sqrt{3}, \pm 2$, and perhaps also when $t = \pm\sqrt{3 \pm 2\sqrt{2}}$.



2.14 Example: Let $w(z) = \dfrac{1}{z}$. Describe the images of the circles $r = \text{const.}$ and the rays $\theta = \text{const.}$, and then describe the images of the lines $x = \text{const.}$ and $y = \text{const.}$

Solution: If $z = r\,e^{i\,\theta}$ and $w = s\,e^{i\,\phi}$ then we have $w = \dfrac{1}{r\,e^{-i\,\theta}} = \dfrac{1}{r} e^{i\,\theta}$ so that $s = \dfrac{1}{r}$ and $\phi = \theta$. This map is known as the **inversion** in the unit circle: the circle $r = c$ is mapped to the circle $s = 1/c$ while the ray $\theta = \alpha$ is mapped to itself. If $z = x + i\,y$ and $w = u + i\,v$ then the vertical line $x = c$ is given parametrically by $z(t) = c + i\,t$ and it is sent to $w(z(t)) = \dfrac{c + i\,t}{c^2 + t^2}$, so its image is the curve we have $u(t) = \dfrac{c}{c^2 + t^2}$ and $v(t) = \dfrac{t}{c^2 + t^2}$. When $c = 0$ we have $u = 0$ and $v = t/t^2 = 1/t$, so the line $x = 0$ (excluding the origin) is mapped to the line $u = 0$ (excludind the origin). When $c \neq 0$, we can use the expression for $u(t)$ to solve for t to get $t^2 = (c - u\,c^2)/u$ and then we can substitute this into the expression $v^2(t) = t^2/(c^2 + t^2)^2$ and simplify to get $v^2 = \frac{1}{c}u - u^2$ or equivalently $(u - \frac{1}{2c})^2 + v^2 = (\frac{1}{2c})^2$. Thus the image of the line $x = c$, $c \neq 0$ is the circle centred at $\frac{1}{2c}$ with radius $\frac{1}{2c}$, excluding the origin. Similarly, the image of the horizontal line $y = c$ is the circle centred at $\frac{1}{2c}\,i$ with radius $\frac{1}{2c}$, excluding the origin.



2.15 Definition: We define the **exponential** function by

$$e^{x+i\,y} = e^x e^{i\,y} = e^x \cos y + i\, e^x \sin y\,.$$

We also write $\exp(z) = e^z$.

2.16 Note: It is not hard to check that the exponential function has the following properties for all complex numbers z and w:

$$\begin{aligned}
&e^0 = 1\\
&e^{-z} = 1/e^z\ , \quad e^{nz} = (e^z)^n, n \in \mathbf{Z}\\
&e^{z+w} = e^z e^w\ , \quad e^{z-w} = e^z/e^w\\
&e^z = e^w \iff w = z + i\,2\pi k \text{ for some } k \in \mathbf{Z}
\end{aligned}$$

2.17 Example: Let $w(z) = e^z$. Describe the images of the lines $x =$ const. and $y =$ const. where $z = x + i\,y$.

Solution: We have $w = e^x e^i\,y$, so if $w = r\,e^{i\,\theta}$ then we have $r = e^x$ and $\theta = y$. So the vertical line $x = c$ is mapped to the circle $r = e^c$, and the horizontal line $y = c$ is mapped to the ray $\theta = c$. Notice that the domain of e^z is all of $\mathbf{C}$ while the range is $\mathbf{C} \setminus \{0\}$. Also notice that if the domain of e^z is restricted to the horizontal strip $\alpha < y < \alpha + 2\pi$, then it is 1:1 and its range is the plane $\mathbf{C}$ with the ray $\theta = \alpha$ removed.

2.18 Definition: We define the **trigonometric** functions by

$$\sin z = \frac{e^{i\,z} - e^{-i\,z}}{2i}\ , \quad \cos z = \frac{e^{i\,z} + e^{-i\,z}}{2}\ , \quad \tan z = \frac{\sin z}{\cos z}$$

and $\sec z = 1/\cos z$, $\csc z = 1/\sin z$ and $\cot z = \cos z/\sin z$. We define the **hyperbolic** functions by

$$\sinh z = \frac{e^z - e^{-z}}{2}\ , \quad \cosh z = \frac{e^z + e^{-z}}{2}\ , \quad \tanh z = \frac{\sinh z}{\cosh z}$$

and $\coth z = \cosh z/\sinh z$.

2.19 Note: It is not hard to verify the following properties, where $z, w \in \mathbf{C}$:

$$\begin{aligned}
&\sin(z+2\pi) = \sin z\,, \quad \cos(z+2\pi) = \cos z \\
&\sin(-z) = -\sin z\,, \quad \cos(-z) = \cos z \\
&\sin^2 z + \cos^2 z = 1 \\
&\sin(z+w) = \sin z\cos w + \cos z\sin w\,, \quad \sin(2z) = 2\sin z\cos z \\
&\cos(z+w) = \cos z\cos w - \sin z\sin w\,, \quad \cos(2z) = \cos^2 z - \sin^2 z \\
&\sinh(-z) = -\sinh z\,, \quad \cosh(-z) = \cosh z \\
&\cosh^2 z - \sinh^2 z = 1 \\
&\sinh(z+w) = \sinh z\cosh w + \cosh z\sinh w\,, \quad \sinh(2z) = 2\sinh z\cosh z \\
&\cosh(z+w) = \cosh z\cosh w + \sinh z\sinh w\,, \quad \cosh(2z) = \cosh^2 z + \sinh^2 z
\end{aligned}$$

In fact *all* of the trigonometric identities and hyperbolic identies which hold for real numbers also hold for complex numbers. Here are some more properties:

$$\begin{aligned}
&\sinh(z+i\,2\pi) = \sinh z\,, \quad \cosh(z+i\,2\pi) = \cosh z \\
&\sinh(i\,z) = i\,\sin z\,, \quad \cosh(i\,z) = \cos z \\
&\sin(i\,z) = i\,\sinh z\,, \quad \cos(i\,z) = \cosh z \\
&\sin(x+i\,y) = \sin x\cosh y + i\,\cos x\sinh y\,, \quad |\sin(x+i\,y)|^2 = \sin^2 x + \sinh^2 y \\
&\cos(x+i\,y) = \cos x\cosh y - i\,\sin x\sinh y\,, \quad |\cos(x+i\,y)|^2 = \cos^2 x + \sinh^2 y \\
&\sinh(x+i\,y) = \sinh x\cos y + i\,\cosh x\sin y\,, \quad |\sinh(x+i\,y)|^2 = \sinh^2 x + \sin^2 y \\
&\cosh(x+i\,y) = \cosh x\cos y + i\,\sinh x\sin y\,, \quad |\cosh(x+i\,y)|^2 = \sinh^2 x + \cos^2 y
\end{aligned}$$

2.20 Example: Find $\sin(\frac{\pi}{6} + i\,\ln 2)$.

Solution: $\sin(\frac{\pi}{6}+i\,\ln 2) = \sin(\frac{\pi}{6})\cosh(\ln 2)+i\,\cos(\frac{\pi}{6})\sinh(\ln 2) = \dfrac{1}{2}\,\dfrac{5}{4}+i\,\dfrac{\sqrt{3}}{2}\,\dfrac{3}{4} = \dfrac{5+3\sqrt{3}\,i}{8}$

2.21 Example: Solve $\cosh z = -2$.

Solution: If $z = x + i\,y$ then we have $\cosh z = \cosh x\cos y + i\,\sinh x\sin y$, so we have $\cosh z = -2$ when $\cosh x\cos y = -2$ and $\sinh x\sin y = 0$. We cannot have $\sinh x = 0$, since if $\sinh x = 0$ then $x = 0$ so $\cosh x\cos y = \cos y \neq -2$. So we must have $\sin y = 0$ and so $y = k\pi$ for some $k \in \mathbf{Z}$ and we have $\cos y = \pm 1$. To have $\cosh x\cos y = -2$, we must have $\cos y = -1$ and $\cosh x = 2$ (since $\cosh x$ is always positive). We can solve $\cosh x = 2$ as follows: $\cosh x = 2 \iff e^x + e^{-x} = 4 \iff (e^x)^2 - 4e^x + 1 = 0 \iff e^x = 2 \pm \sqrt{3}$ so we have $x = \ln(2 \pm \sqrt{3})$ or equivalently $x = \pm\ln(2+\sqrt{3})$. Thus $z = \pm\ln(2+\sqrt{3}) + i\,(\pi + 2\pi k)$ for some $k \in \mathbf{Z}$.

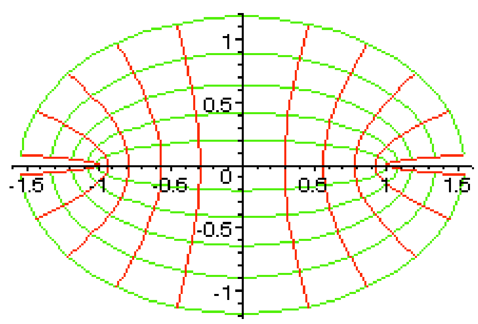
2.22 Example: Let $w(z) = \sin z$. Describe the images of the lines $x = \text{const.}$ and $y = \text{const.}$ where $z = x + i\,y$.

Solution: The vertical line $x = c$ is given parametrically by $z(t) = c + i\,t$ and it is mapped to the curve $w(t) = \sin(c + i\,t) = \sin c\cosh t + i\,\cos c\sinh t$. If $w = u + i\,v$ then we have $u(t) = \sin c\cosh t$ and $v(t) = \cos c\sinh t$. Using the identity $\cosh^2 t - \sinh^2 t = 1$ we obtain

$\dfrac{u^2}{\sin^2 c} - \dfrac{v^2}{\cos^2 c} = 1$, provided that $t \neq \frac{\pi}{2}k, k \in \mathbf{Z}$. This is the equation of a hyperbola. The image of the line $x = c$ will be one of the two branches of this hyperbola; when $\sin c$ is positive $u(t)$ is also positive and the image is the branch on the right; when $\sin c$ is negative, the image is the branch on the left. When $\sin c = 0$ (so that $c = \pi k$), the image is the line $u = 0$, that is, the v-axis. When $\cos c = 0$, the image lies on the line $v = 0$ (the u-axis) and it is either the interval $[1, \infty)$ (when $\sin c = 1$) or else the interval $(-\infty, -1]$ (when $\sin c = -1$).

The horizontal line $y = c$ is given parametrically by $z(t) = t + i\,c$ and it is mapped to $w(t) = \sin t \cosh c + i \,\cos t \sinh c$ so we have $u(t) = \sin t \cosh c$ and $v(t) = \cos t \sinh t$. Since $\sin^2 t + \cos^2 t = 1$ we have $\dfrac{u^2}{\cosh^2 c} + \dfrac{v^2}{\sinh^2 c} = 1$. The line $y = c$ is mapped to this ellipse, unless $c = 2\pi k\, i$ in which case the image can be seen to be the line segment $[-1, 1]$ on the u-axis.

If you sketch a few of these hyperbolas and ellipses, you will get a nice picture of two orthogonal families of curves. You will see that the domain and the range of $\sin z$ are both $\mathbf{C}$. When the domain of $\sin z$ is restricted to the vertical strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$, it becomes 1:1 and its image is the plane $\mathbf{C}$ with the two intervals $(-\infty, -1]$ and $[1, \infty)$ removed.



2.23 Note: If a map $f : U \to f(U)$ is 1:1 then it has an **inverse function**, f^{-1}, given by

$$f(z) = w \iff f^{-1}(w) = z$$

or equivalently by

$$f(f^{-1}(w)) = w \;, \quad f^{-1}(f(z)) = z$$

If a map f is *not* 1:1, then sometimes we can restrict its domain so that it *becomes* 1:1. An alternate approach is to allow f^{-1} to take on more that a single value and to define $f^{-1}(w) = \{z \in U | f(z) = w\}$. (If the inverse function is not single-valued, then it is not really a function at all, but rather a **multi-function**). We shall be using both of these approaches, and we shall not always specify which approach we are taking.

2.24 Example: In real variable calculus, to define $\sin^{-1} x$ it is customary to restrict the domain of $\sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ so that it becomes 1:1. If we thought instead of $\sin^{-1} x$ as a multi-function then for example we would have $\sin^{-1}(\frac{1}{2}) = \{\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k, k \in \mathbf{Z}\}$.

2.25 Example: The change-of-coordinate map $f(r, \theta) = (r \cos\theta, r \sin\theta)$ is not 1:1. We can make it 1:1 by restricting the domain to $\{(r, \theta) | r > 0, 0 < \theta < 2\pi\}$. If we make this restriction then the inverse function is given by $f^{-1}(x, y) = (r, \theta)$ where $r = |x + i\,y|$ and $\theta = \theta(x + i\,y)$ where $0 < \theta(x + i\,y) < 2\pi$. Alternatively, if we think of f^{-1} as a multi-function, then $f^{-1}(x, y) = (r, \theta)$ where $r = |x + i\,y|$ and $\theta = \theta(x + i\,y)$ where this time $\theta(x + i\,y)$ denotes the set $\{\theta \in \mathbf{R} | r\, e^{i\,\theta} = x + i\,y\}$.

2.26 Definition: The inverse of the exponential function e^z is the **logarithmic** function, denoted by $\log z$.

2.27 Example: Find a formula for $\log z$.

Solution: Let $z = r\,e^{i\,\theta}$ and $w = u + i\,v$. Then $w = \log z \iff e^w = z \iff e^u e^{i\,v} = r\,e^{i\,\theta}$, which happens whn $e^u = r$ and $v = \theta + 2\pi k$ for some $k \in \mathbf{Z}$. Thus

$$\log\left(r\,e^{i\,\theta}\right) = \ln r + i(\theta + 2\pi k),\ \ k \in \mathbf{Z}$$

This is the formula for the multi-valued logarithm. If we pick one particular value of k, the resulting function is called a **branch** of the logarithm. If we restrict the domain of e^z to make it 1:1, then the inverse function is the branch with $k = 0$, that is $\log(r\,e^{i\,\theta}) = \ln r + i\,\theta$. This is called the **principal branch** of the logarithm.

2.28 Example: Find $\log(1 - i)$

Solution: $\log(1 - i) = \log(\sqrt{2}e^{-i\,\pi/4}) = \ln\sqrt{2} + i\,(-\frac{\pi}{4} + 2\pi k)$, $k \in \mathbf{Z}$.

2.29 Note: For the multi-valued logarithm, you should convince yourself that the following formulas make sense and they all hold:

$$\begin{aligned} e^{\log z} &= z \\ \log(z\,w) &= \log z + \log w \\ \log(z/w) &= \log z - \log w \end{aligned}$$

2.30 Definition: We can use the logarithm to define **complex exponents**: given $a \in \mathbf{C}$ we define

$$z^a = \exp(a \log z)\,.$$

2.31 Example: Find i^{-2i}.

Solution: $i^{-2i} = \exp(-2i\,\log i) = \exp(-2i\,(i\,(\frac{\pi}{2} + 2\pi k)) = \exp(\pi + 4\pi k)$, $k \in \mathbf{Z}$.

2.32 Example: Find the principal branch of $z^{2/3}$.

Solution: Write $z = r\,e^{i\,\theta}$. Then $z^{2/3} = \exp(\frac{2}{3}\log z) = \exp(\frac{2}{3}(\ln r + i\,\theta)) = r^{2/3}\exp(i\,2\theta/3)$.

2.33 Note: Check that

$$\begin{aligned} &z^n = \exp(n \log z)\ \ \text{is single valued for } n \in \mathbf{Z} \\ &z^{1/n} = \exp(\frac{1}{n}\log z)\ \ \text{takes n values for } n \in \mathbf{Z} \\ &z^{-a} = (z^a)^{-1} \end{aligned}$$

2.34 Definition: The **inverse trigonometric functions** are denoted by $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$ and so on. The **inverse hyperbolic** functions are denoted by $\sinh^{-1} z$, $\cosh^{-1} z$, $\tanh^{-1} z$ and so on.

2.35 Note: Since the trigonometric and the hyperbolic functions are defined using the exponential function, their inverses can be expressed in terms of the logarithmic function:

$$\begin{aligned}
\sin^{-1} z &= -i \log\left(i\,z + (1-z^2)^{1/2}\right) \\
\cos^{-1} z &= -i \log\left(z + (z^2-1)^{1/2}\right) \\
\tan^{-1} z &= \frac{i}{2}\log\frac{i+z}{i-z} \\
\sinh^{-1} z &= \log\left(z + (z^2+1)^{1/2}\right) \\
\cosh^{-1} z &= \log\left(z + (z^2-1)^{1/2}\right) \\
\tanh^{-1} z &= \frac{1}{2}\log\frac{1+z}{1-z}
\end{aligned}$$

where the square roots are double valued. Let us derive the formula for $\sin^{-1} z$. We have $w = \sin^{-1} z \iff z = \sin w \iff z = (e^{iw} - e^{-iw})/2i \iff (e^{iw})^2 - 2iz(e^{iw}) - 1 = 0 \iff e^{iw} = iz \pm \sqrt{1-z^2}$ so we obtain $iw = \log(iz \pm \sqrt{1-z^2})$, as required.

2.36 Example: Find $\cosh^{-1}(-2)$.

Solution: Actually, we already did this in example 2.21, but we'll do it again using the above logarithmic formula: we have $\cosh^{-1}(-2) = \log(-2 \pm \sqrt{3}) = \log((2 \pm \sqrt{3})e^{i\,\pi}) = \ln(2 \pm \sqrt{3}) + i\,(\pi + 2\pi k)$, $k \in \mathbf{Z}$.

Chapter 3. Sets, Limits and Continuity

3.1 Notation: Given $a, b \in \mathbf{C}$ and $r \in \mathbf{R}$, let $[a, b]$ denote the line segment from a to b

$$[a, b] = \{a + t(b - a) | t \in \mathbf{R}\},$$

let $S(a, r)$ denote the circle about a of radius r

$$S(a, r) = \big\{z \in \mathbf{C} \big| |z - a| = r\big\},$$

and let $D(a, r)$, $\overline{D}(a, r)$ and $D^*(a, r)$ denote the disc, the closed disc, and the punctured disc, centred at a of radius r

$$\begin{aligned} D(a, r) &= \big\{z \in \mathbf{C} \big| |z - a| < r\big\} \\ \overline{D}(a, r) &= \big\{z \in \mathbf{C} \big| |z - a| \leq r\big\} \\ D^*(a, r) &= \big\{z \in \mathbf{C} \big| 0 < |z - a| < r\big\} \end{aligned}$$

We use the same notation for subsets of $\mathbf{R}^n$. In $\mathbf{R}^3$, for example, $S(a, r)$ is a sphere and $D(a, r)$ is a ball.

3.2 Definition: Let $E \subset \mathbf{C}$ (or more generally $E \subset \mathbf{R}^n$).
a) A point $a \in E$ is an **interior point** of E if $\exists r > 0\ D(a, r) \subset E$.
b) The **interior** of E, denoted by E^0, is the set of all interior points of E.
c) E is **open** if every point of E is an interior point, in other words if $E = E^0$.
d) A point $a \in \mathbf{C}$ is a **limit point** of E if $\forall r > 0\ D^*(a, r) \cap E \neq \emptyset$.
e) The **closure** of E, denoted by $\overline{E}$, is the union of E with the set of all its limit points.
f) E is **closed** if every limit point of E lies in E, in other words if $E = \overline{E}$.
g) The **boundary** of E is the set $\partial E = \overline{E} \setminus E^0$.
h) The **complement** of E is the set $E^c = \{z \in \mathbf{C} | z \notin E\}$.

3.3 Example: For $a, b \in \mathbf{R}$, the interval $[a, b]$ is closed in $\mathbf{R}$, $[a, b]^0 = (a, b)$, $\overline{[a, b]} = [a, b]$ and $\partial[a, b] = \{a, b\}$. For $a, b \in \mathbf{C}$, the segment $[a, b]$ is closed in $\mathbf{C}$, $[a, b]^0 = \emptyset$, $\overline{[a, b]} = [a, b]$ and $\partial[a, b] = [a, b]$.

3.4 Example: In $\mathbf{C}$, he disc $D(a, r)$ and the punctured disc $D^*(a, r)$ are both open, while the closed disc $\overline{D}(a, r)$ is closed. Their interiors are $D(a, r)^0 = \overline{D}(a, r)^0 = D(a, r)$ and $D^*(a, r)^0 = D^*(a, r)$. Their closures are all equal: $\overline{D(a, r)} = \overline{\overline{D}(a, r)} = \overline{D^*(a, r)} = \overline{D}(a, r)$. Their boundaries are $\partial D(a, r) = \partial\overline{D}(a, r) = S(a, r)$ and $\partial D^*(a, r) = S(a, r) \cup \{0\}$.

3.5 Example: The annulus $A = \big\{z \in \mathbf{C} \big| r < |z - a| \leq R\big\}$ is neither open nor closed. Its interior is the open annulus $A^0 = \big\{z \in \mathbf{C} \big| r < |z - a| < R\big\}$, its closure is the closed annulus $\overline{A} = \big\{z \in \mathbf{C} \big| r \leq |z - a| \leq R\big\}$, and its boundary is the union of the two circles $\partial A = S(a, r) \cup S(a, R)$.

3.6 Example: Let $E = \{1/n | n \in \mathbf{Z}^*\}$ (where $\mathbf{Z}^* = \mathbf{Z} \setminus \{0\}$). Then E is neither open nor closed: its interior is emplty $E^0 = \emptyset$, its only limit point is 0, its closure is $\overline{E} = E \cup \{0\}$, and its boundary is equal to its closure $\partial E = \overline{E}$.

3.7 Theorem: *Let $E \subset \mathbf{C}$ (or let $E \subset \mathbf{R}^n$). Then*
a) E is open iff E^c is closed and E is closed iff E^c is open.
b) E^0 is open and if $U \subset E$ is open then $U \subset E^0$.
c) $\overline{E}$ is closed and if $E \subset K$ with K closed then $\overline{E} \subset K$.

Proof: (I may include some proofs)

3.8 Theorem: *Let $\{E_\alpha | \alpha \in A\}$ be a (possibly infinite) family of sets in $\mathbf{C}$ (or in $\mathbf{R}^n$) and let $\{E_{\alpha_i} | i = 1, 2, \cdots n\}$ be a finite sub-family. Then*
a) $\Big(\bigcap_{\alpha \in A} E_\alpha\Big)^c = \bigcup_{\alpha \in A} ({E_\alpha}^c)$ *and* $\Big(\bigcup_{\alpha \in A} E_\alpha\Big)^c = \bigcap_{\alpha \in A} ({E_\alpha}^c)$.
b) If the sets E_α are open then $\bigcup_\alpha E_\alpha$ and $\bigcap_i^n E_{\alpha_i}$ are open.
If the sets E_α are closed then $\bigcup_{i=1}^n E_\alpha$ and $\bigcap_\alpha E_\alpha$ are closed.

Proof: (I may include some proofs here later).

3.9 Definition: Let E, A and B be subsets of $\mathbf{C}$ (or of $\mathbf{R}^n$).
a) E is **bounded** if $\exists r \; E \subset D(0, r)$.
b) E is **convex** if $a, b \in E \Rightarrow [a, b] \subset E$.
c) We say that E is **disconnected** if it is possible to find disjoint open sets U and V with $E \cap U \neq \emptyset$ and $E \cap V \neq \emptyset$ and $E \subset U \cup V$. In this case we say that the sets U and V separate E. If E is not disconnected then we say it is **connected**.
d) E is **compact** if every open cover of E admits a finite subcover, in other words, if $E \subset \bigcup_{\alpha \in A} U_\alpha$, where the U_α are open sets in $\mathbf{C}$, then we have $E \subset \bigcup_{i=1}^n U_{\alpha_i}$ for some $\alpha_i \in A$.

3.10 Example: The segment $[a, b]$ and the disc $D(a, r)$ are both bounded. The line $\big\{z \in \mathbf{C}\big| |z - a| = |z - b|\big\}$ and the half-plane $\big\{z \in \mathbf{C}\big| |z - a| < |z - b|\big\}$ are both unbounded.

3.11 Example: The sets $[a, b]$, $D(a, r)$ and $\overline{D}(a, r)$ are all convex. The punctured disc $D^*(a, r)$ is *not* convex; for example if $u = a - \frac{r}{2}$ and $v = a + \frac{r}{2}$ then $u, v \in D^*(a, r)$ but $[u, v] \not\subset D^*(a, r)$. Also, the half-annulus $E = \big\{z \in \mathbf{C}\big| 1 < |z| < 2, \mathrm{Re}\,(z) > 0\big\}$ is *not* convex; for example, take a and v to be $\frac{1}{2} \pm i$.

3.12 Example: Each of the sets $[a, b]$, $D(a, r)$, $\overline{D}(a, r)$ and $D^*(a, r)$ is a connected set. The union $[0, 1] \cup [i, i+1]$ is *not* connected, because the two segments $[0, 1]$ and $[i, 1+i]$ are separated by $U = \{z | \mathrm{Im}\,(z) < \frac{1}{2}\}$ and $V = \{z | \mathrm{Im}\,(z) > \frac{1}{2}\}$. The union $D(-1, 1) \cup D(1, 1)$ is *not* connected since the two discs $D(-1, 1)$ and $D(1, 1)$ are separated. On the other hand, the union $D(-1, 1) \cup \overline{D}(1, 1)$ *is* connected.

3.13 Theorem: *(The Heine-Borel Theorem) Let $E \subset \mathbf{C}$ (or $E \subset \mathbf{R}^n$). Then the following are equivalent:*
a) E is compact.
b) Every infinite subset of E has a limit point in E.
c) E is closed and bounded.

Proof: I may include the proof later.

3.14 Example: This theorem makes it easy to recognize compact sets. The segment $[a, b]$ and the closed disc $\overline{D}(a, r)$ are both compact (since they are closed and bounded). The open disc $D(a, r)$ is not compact, since it is not closed. The line $\big\{z \in \mathbf{C}\big| |z - a| = |z - b|\big\}$ is not compact, since it is not bounded.

3.15 Example: The set $E = \{1/n | n \in \mathbf{Z}^*\}$ is not closed (since $E \neq \overline{E}$) so it is not compact. This means that we must be able to find an open cover with no finite subcover. Indeed, notice that the distance between two neighbouring points in E is $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, so if we let $U_n = D\big(\frac{1}{n}, \frac{1}{n(n+1)}\big)$ then each point of E lies in exactly one of the sets U_n. Thus $\{U_n | n \in \mathbf{Z}^*\}$ covers E, that is $E \subset \bigcup_{n \in \mathbf{Z}^*} U_n$, but if we remove even one of these discs then the remaining discs will not cover E.

However, the set $K = \overline{E} = E \cup \{0\}$ *is* closed and bounded and hence compact, so any open cover of K ought to have a finite subcover. Indeed, consider the cover $\{U_n | n \in \mathbf{Z}\}$ where $U_n = D\big(\frac{1}{n}, \frac{1}{n(n+1)}\big)$ for $n \in \mathbf{Z}^*$ (as above) and $U_0 = D(0, \frac{1}{N})$ where N is some positive integer. Since $\frac{1}{n} \in U_0$ whenever $|n| > N$, we see that $\big\{U_n \big| |n| < N\big\}$ is a finite subcover, that is, $E \subset \bigcup_{n=-N}^{N} U_n$

3.16 Definition: Let $f : U \to \mathbf{C}$ where U is an open set in $\mathbf{C}$ (or $\mathbf{R}^n$), and let a be a limit point of U. We write

$$\lim_{z \to a} f(z) = b \qquad \text{or} \qquad f(z) \to b \text{ as } z \to a$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $z \in U$, $z \in D^*(a, \delta) \Rightarrow f(z) \in D(b, \epsilon)$ or, in other words, such that $f(U \cap D^*(a, \epsilon)) \subset D(b, \epsilon)$. We write

$$\lim_{z \to a} f(z) = \infty \qquad \text{or} \qquad f(z) \to \infty \text{ as } z \to a$$

if for all $R > 0$ there exists $\delta > 0$ such that $f(U \cap D^*(a, \delta)) \subset \overline{D}(0, R)^c$.

3.17 Theorem: *Let f and g be maps from U to $\mathbf{C}$ where U is open in $\mathbf{C}$, and let a be a limit point of U.*

a) If we write $f(z) = u(z) + i\, v(z)$, where $u, v : U \to \mathbf{R}$, then $\lim_{z \to a} f(z)$ exists if and only if $\lim_{z \to a} u(z)$ and $\lim_{z \to a} v(z)$ both exist, and in this case, $\lim_{z \to a} f(z) = \lim_{z \to a} u(z) + i \lim_{z \to a} v(z)$.

b) Suppose $\lim_{z \to a} f(z) = p$ and $\lim_{z \to a} g(z) = q$, and let $c \in \mathbf{C}$. Then

i) $\lim_{z \to a} c\, f(z) = c\, p$

ii) $\lim_{z \to a} f(z) \pm g(z) = p \pm q$

iii) $\lim_{z \to a} f(z) g(z) = p\, q$

iv) $\lim_{z \to a} f(z)/g(z) = p/q$ provided that $q \neq 0$

v) $\lim_{z \to a} |f(z)| = |p|$.

The theorem also holds for functions with values in $\mathbf{R}^n$, except for parts b) iii) and iv).

Proof: To prove part a), suppose first that $\lim_{z \to a} f(z)$ exists, say $\lim_{z \to a} f(z) = b = s + i\,t$ with $s, t \in \mathbf{R}$. Note that $u(z) - s = \text{Re}\,(f(z) - b)$ so we have $|u(z) - s| \leq |f(z) - b|$. So, given $\epsilon > 0$ choose $\delta > 0$ such that $0 < |z - a| < \delta \Longrightarrow |f(z) - b| < \epsilon$ and then we have $|u(z) - s| \leq |f(z) - b| < \epsilon$. This shows that $\lim_{z \to a} u(z) = s$. Similarly $\lim_{z \to a} v(z) = t$.

Next, we suppose that $\lim_{z \to a} u(z)$ and $\lim_{z \to a} v(z)$ exist, say $\lim_{z \to a} u(z) = s$ and $\lim_{z \to a} v(z) = t$ and let $b = s + i\,t$. By the triangle inequality we have $|f(z) - b| \leq |u(z) - s| + |v(z) + t|$. So, given $\epsilon > 0$ we choose $\delta > 0$ such that $0 < |z - a| < \delta \Longrightarrow \big(|u(z) - s| < \frac{\epsilon}{2}$ and $|v(z) - t| < \frac{\epsilon}{2}\big)$. Then we have $|f(z) - b| \leq |u(z) - s| + |v(z) - t| < \epsilon$. This shows that $\lim_{z \to a} f(z) = b$.

Part b) i)-iv) can be proven in the same way as the analogous results for real-valued functions. For example, to prove part b) iii), we can use the equality $f(z)g(z) - pq = (f(z) - p)(g(z) - q) + (f(z) - p)\,q + (g(z) - q)\,p$. Given $\epsilon > 0$ we choose $\delta > 0$ so that $0 < |z-a| < \delta \Longrightarrow \left(|f(z)-p| < \sqrt{\frac{\epsilon}{3}},\, |g(z)-q| < \sqrt{\frac{\epsilon}{3}},\, |f(z)-p|\,|q| < \frac{\epsilon}{3} \text{ and } |g(z)-q|\,|p| < \frac{\epsilon}{3}\right)$. Then we have $|f(z)g(z) - pq| \leq |f(z) - p|\,|g(z) - q| + |f(z) - p|\,|q| + |g(z) - q|\,|p| < \sqrt{\frac{\epsilon}{3}}\sqrt{\frac{\epsilon}{3}} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

The proof of part b) v) is left as an exercise.

3.18 Example: Let $f(z) = \dfrac{z^2 - 2z + 5}{z - 1 - 2i}$. Find $\lim\limits_{z\to 1+2i} f(z)$.

Solution: $\lim\limits_{z\to 1+2i} f(z) = \lim\limits_{z\to 1+2i} \dfrac{(z-(1+2i))(z-(1-2i))}{(z-(1+2i))} = \lim\limits_{z\to 1+2i} (z-(1-2i)) = 4i$. (We can prove the last equality from the definition of limit: given $\epsilon > 0$ choose $\delta = \epsilon$ so that for $0 < |z-(1+2i)| < \delta$ we have $|f(z)-4i| = |(z-(1-2i)-4i| = |z-(1+2i)| < \delta = \epsilon$).

3.19 Example: For each $z \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$, let $\theta(z)$ be the angle of z chosen so that $0 \leq \theta(z) < 2\pi$. Then $\theta : \mathbf{C}^* \to [0, 2\pi)$. Show that if $a \geq 0$ the $\lim\limits_{z\to a} \theta(z)$ does not exist.

Solution: Suppose (for a contradiction) that $\lim\limits_{z\to a} \theta(z)$ does exist, say $\lim\limits_{z\to a} \theta(z) = b$. Let $\epsilon = \frac{\pi}{2}$ and choose $\delta > 0$ so that $z \in D^*(a,\delta) \Longrightarrow |\theta(z) - b| < \epsilon = \frac{\pi}{2}$. Since $a \pm i\,\frac{\delta}{2} \in D^*(a,\delta)$, we have $|\theta(a \pm i\,\frac{\delta}{2}) - b| < \epsilon = \frac{\pi}{2}$. Since $a + i\,\frac{\delta}{2}$ is in the first quadrant and $a - i\,\frac{\delta}{2}$ is in the second quadrant, we have $0 < \theta(a + i\,\frac{\delta}{2}) \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq \theta(a - i\,\frac{\delta}{2}) < 2\pi$. Thus $\pi = \frac{3\pi}{2} - \frac{\pi}{2} \leq \left|\theta(a + i\,\frac{\delta}{2}) - \theta(a - i\,\frac{\delta}{2})\right| \leq \left|\theta(a + i\,\frac{\delta}{2}) - b\right| + \left|\theta(a - i\,\frac{\delta}{2}) - b\right| < \frac{\pi}{2} + \frac{\pi}{2} = \pi$. We have thus obtained a contradiction.

3.20 Definition: Let $f : U \to \mathbf{C}$ where U is an open set in $\mathbf{C}$ (or $\mathbf{R}^n$), and let $a \in U$. We say that f is **continuous** at a if $\lim\limits_{z\to a} f(z) = f(a)$ or, in other words, f is continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that $f(D(a,\delta)) \subset D(f(a), \epsilon)$. We say that f is **continous** in U if f is continuous at every point in U.

3.21 Theorem: *Let f and g be maps from U to $\mathbf{C}$ with U open in $\mathbf{C}$ and let $c \in \mathbf{C}$.*
a) If $f(z) = u(z) + i\,v(z)$ with $u, v : U \to \mathbf{R}$ then f is continuous at a if and only if both u and v are continuous at a.
b) If f and g are both continuous at a then
i) cf is continuous at a
ii) $f \pm g$ is continuous at a
iii) fg is continuous at a
iv) f/g is continuous at a, provided that $g(a) \neq 0$.
v) $|f|$ is continuous at a.
c) If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.
The theorem also holds for functions with values in $\mathbf{R}^n$ except for parts b)iii) and iv).

Proof: Parts a) and b) are proved as in theorem 3.17. For part c), suppose that f is continuous at a and that g is continuous at b where $b = f(a)$. Given $\epsilon > 0$ choose δ_0 so that $z \in D(b, \delta_0) \Rightarrow g(z) \in D(g(b), \epsilon)$. Then choose $\delta > 0$ so that $z \in D(a, \delta) \Rightarrow f(z) \in D(b, \delta_0)$. We have $z \in D(a, \delta) \Rightarrow f(z) \in D(b, \delta_0) \Rightarrow g(f(z)) \in D(g(b), \epsilon) = D(g(f(a), \epsilon)$. This shows that $g(f(z))$ is continuous at a.

3.22 Example: Let $U = \mathbf{C} \setminus \{x \in \mathbf{R} | x \geq 0\} = \{r\,e^{i\,\theta} | r > 0, 0 < \theta < 2\pi\}$. Let $\theta : U \to (0, 2\pi)$ be the angle function. Show that θ is continuous in U.

Solution: Write $z = x + i\,y$ with $x, y \in \mathbf{R}$. For $\mathrm{Im}\,(z) > 0$, the angle function is given by the formula $\theta(x + i\,y) = \cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$. This formula expresses $\theta(x + i\,y)$ using sums, products, quotients and composites of known continuous functions, and so it must be continuous, by parts b) and c) of the above theorem. Thus $\theta(z)$ is continuous at all points z with $\mathrm{Im}\,(z) > 0$.

Similarly, for $\mathrm{Re}\,(z) < 0$, $\theta(z)$ is given by the formula $\theta(x+i\,y) = \pi + \tan^{-1}\left(y/x\right)$, and for $\mathrm{Im}\,(z) < 0$ we have $\theta(x + i\,y) = 2\pi - \cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$. These are both continuous and so $\theta(z)$ is continuous for all $z \in U$.

3.23 Note: If we choose $\theta(z) \in [0, 2\pi)$ for all $z \in \mathbf{C}^*$, then we have seen (in example 3.19) that for $a > 0$, $\lim\limits_{z \to a} \theta(z)$ does not exist, so $\theta : \mathbf{C}^* \to [0, 2\pi)$ is *not* continuous for all $z \in \mathbf{C}^*$. In fact it is impossible to choose $\theta(z) \in \mathbf{R}$ so that $\theta : \mathbf{C}^* \to \mathbf{R}$ is continuous. As in the above example, we must restrict the domain to make the angle function continuous. Indeed, for any $\alpha \in \mathbf{R}$, if we restrict the domain to $U_\alpha = \{r\,e^{i\,\theta} | r > 0, \alpha < \theta < \alpha + 2\pi\}$ and choose $\theta(z)$ with $\alpha < \theta(z) < \alpha + 2\pi$ then $\theta : U_\alpha \to (\alpha, \alpha + 2\pi)$ will be continuous.

3.24 Note: We have found formulas for the real and imaginary parts of the identity $f(z) = z$, the exponential $f(z) = e^z$, the trigonometric functions, and the hyperbolic functions. These formulas reveal that they are all continuous in their domains. Also, any branch of the logarithm $\log z = \ln |z| + i\,\theta(z)$ is continuous provided that $\theta(z)$ is chosen to be continuous. The inverse trigonometric and inverse hyperbolic functions can all be expressed in terms of the logarithm, and so they are also continuous provided that $\theta(z)$ is chosen to be continuous. Any complex function which can be expressed using sums, products, quotients and composites of the above functions will be continuous in its domain.

3.25 Theorem: *Let $f : U \to \mathbf{C}$ where U is open in $\mathbf{C}$ (or in $\mathbf{R}^n$).*
a) f is continuous if and only if $f^{-1}(V)$ is open for every open set $V \subset \mathbf{C}$.
b) If f is continuous and $E \subset U$ is connected then $f(E)$ is connected.
c) If f is continuous and $K \subset U$ is compact then $f(K)$ is compact.
d) If f is continuous and $K \subset U$ is compact then $|f(x)|$ attains its extreme values on K.

Proof: We prove part a). Suppose first that f is continuous and let $V \subset \mathbf{C}$ be open. We must show that $f^{-1}(V)$ is open, so given $a \in U$ we need to show there exists $\delta > 0$ such that $D(a, \delta) \subset U$. Let $a \in f^{-1}(V)$, which means that $f(a) \in V$. Choose $\epsilon > 0$ so that $D(f(a), \epsilon) \subset V$. Then choose $\delta > 0$ so that $f(D(a, \delta)) \subset D(f(a), \epsilon) \subset V$ so that we have $D(a, \delta) \in f^{-1}(V)$.

Next we suppose that $f^{-1}(V)$ is open for every open set $V \subset \mathbf{C}$. We want to show that f is continuous at every point $a \in U$. Let $a \in U$. Given $\epsilon > 0$, let $V = D(f(a), \epsilon)$. Then $a \in f^{-1}(V)$ and since V is open, $f^{-1}(V)$ is open, so we can choose $\delta > 0$ such that $D(a, \delta) \subset f^{-1}(V)$. Then we will have $f(D(a, \delta)) \subset D(f(a), \epsilon)$. This shows that f is continuous.

To prove part b) we suppose that f is continuous and that E is connected, and (looking for a contradiction) we shall suppose that $f(E)$ is not connected. Say $f(E) \subset V \cup W$ where V and W are disjoint open sets which separate E. You may check that the open sets $f^{-1}(V)$ and $f^{-1}(W)$ separate E giving a contradiction.

To prove part c), suppose that f is continuous and that $K \subset U$ is compact. We wish to show that $f(K)$ is compact. Let $\{V_\alpha\}$ be an open cover of $f(K)$. Since f is continuous, each of the sets $f^{-1}(V_\alpha)$ will be open, and you can check that they cover K. Since K is compact, we can find a finite subcover, say $K \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$. You may check that this implies that we have $f(K) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$, so $\{V_\alpha\}$ has a finite subcover.

Part d) follows from part c), because if f is continuous and $K \subset U$ is compact then $|f|$ will also be continuous and so $|f|(K) = \big\{|f(z)|\big|z \in K\big\}$ is a compact set in $\mathbf{R}$. Any closed and bounded set in $\mathbf{R}$ includes its extreme values.

Chapter 4. Derivatives

4.1 Note: From now on, we shall always use the letter U to denote an open set.

4.2 Definition: Recall that for a function $f : U \subset \mathbf{R} \to \mathbf{R}$ we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists, and we say that f is **differentiable** at $x = a$ and $f'(a)$ is called the (real) **derivative** of f at a. Equivalently, we see that f is differentiable at $x = a$ if there exists a real number $f'(a)$ such that $\lim\limits_{x \to a} \left| \dfrac{f(x) - f(a)}{x - a} - f'(a) \right| = 0$. This last condition can be rewritten as $\lim\limits_{x \to a} \dfrac{|R(x)|}{|x - a|} = 0$, where $R(x) = f(x) - \big(f(a) + f'(a)(x - a)\big)$. In this way we obtain a definition which applies to functions $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$.

A function $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is **differentiable** at $x = a$ if there exists an $m \times n$ matrix $f'(a)$ such that $\lim\limits_{x \to a} \dfrac{|R(x)|}{|x - a|} = 0$, where $R(x) = f(x) - \big(f(a) + f'(a)(x - a)\big)$. The matrix $f'(a)$ is called the (real) **derivative** of f at $x = a$. We also write $Df(a) = f'(a)$. In the case that $f : U \subset \mathbf{C} \to \mathbf{C}$, we shall use the notation $Df(a)$ for the real derivative of f, and we shall reserve the notation $f'(a)$ for the complex derivative which will be defined soon. We say that $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is **differentiable** (in U) if it is differentiable at every point $a \in U$.

For a map $f : U \subset \mathbf{R}^n \to \mathbf{R}$, the j^{th} **partial derivative** of f is given by

$$f_{x_j}(a) = \frac{\partial f}{\partial x_j}(a) = g'(0)\,,$$

if it exists, where g is the map from $\mathbf{R}$ to $\mathbf{R}$ given by $g(t) = f(a + t\, e_j)$, with e_j denoting the j^{th} standard basis vector in $\mathbf{R}^n$. Notice that if $f : \mathbf{R} \to \mathbf{R}$ then $\frac{\partial f}{\partial x}(a) = f'(a)$.

We now recall (without proof) some theorems from vector calculus.

4.3 Theorem: *Let $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$, and let f_i be the components of f so that $f(x) = \big(f_1(x), \cdots f_m(x)\big)$. Then*
a) If f is differentiable at $x = a$ then the partial derivatives $\frac{\partial f_i}{\partial x_j}$ all exist and

$$Df(a) = f'(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

b) If f is $\mathcal{C}^1$ in U, which means that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ all exist and are continuous in U, then f is differentiable in U.

4.4 Theorem: *a) If $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a then it is continuous at a.*
b) If $f, g : U \subset \mathbf{R}^n \to \mathbf{R}^m$ are both differentiable at $x = a$, and if $c \in \mathbf{R}$ then
i) $(c\,f)'(a) = c\,f'(a)$ and
ii) $(f \pm g)'(a) = f'(a) \pm g'(a)$.
iii) (The Product Rule) If $m = 1$ then $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.
iv) (The Quotient Rule) If $m = 1$ then $(f/g)'(a) = \big(f'(a)g(a) - f(a)g'(a)\big)/g^2(a)$.
c) (The Chain Rule) If $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a, and $g : V \subset \mathbf{R}^m \to \mathbf{R}^l$ is differentiable at $f(a)$ then $h(x) = g(f(x))$ is differentiable at a and $h'(a) = g'(f(a))f'(a)$.
c) (The Inverse Function Theorem) If $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is $\mathcal{C}^1$ in U and if the matrix $f'(a)$ is invertible, then we can make f invertible by restricting its domain, and if $g = f^{-1}$ then g is also $\mathcal{C}^1$ with $g'(f(x)) = f'(x)^{-1}$.

4.5 Example: If $f : \mathbf{R} \to \mathbf{R}^n$ is given by $f(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ then $f'(t) = \begin{pmatrix} {x_1}'(t) \\ \vdots \\ {x_n}'(t) \end{pmatrix}$. This is called the **tangent vector** to the curve $f(t)$. In particular, if $z(t) = x(t) + i\,y(t)$ is a map from $\mathbf{R}$ to $\mathbf{C}$ then $z'(t) = x'(t) + i\,y'(t)$.

4.6 Example: If $f : U \subset \mathbf{R}^n \to \mathbf{R}$ then $f'(x) = \left(\frac{\partial f}{\partial x_1}(a) \quad \frac{\partial f}{\partial x_2}(a) \quad \ldots \quad \frac{\partial f}{\partial x_n}(a) \right)$. We define the **gradient** of f at a to be the transpose of $f'(a)$, and we write $\nabla f = f'(x)^T$. Given a point $a \in U$ and a vector $v \in \mathbf{R}^n$, choose any curve $\alpha : \mathbf{R} \to U$ with $\alpha(0) = a$ and $\alpha'(0) = v$, and then set $g(t) = f(\alpha(t))$. By the chain rule, we have $g'(t) = f'(\alpha(t))\alpha'(t)$ and so $g'(0) = f'(a)v = \nabla f(a) \cdot v$. We call this the **directional derivative** $D_v f(a)$ of f at a in the direction of v, so we have $D_v f(a) = f'(a)v = \nabla f(a) \cdot v$.

Notice that the gradient $\nabla f(a)$ is perpendicular to the level set $f(x) = f(a)$. To see this, choose any curve $x(t)$ with $x(0) = a$ and with $f(x(t)) = f(a)$ (so that $x(t)$ lies in the level set). Then by the chain rule we have $f'(x(t))x'(t) = 0$, and setting $t = 0$ gives $f'(a)x'(0) = 0$ or equivalently $\nabla f(a) \cdot x'(a) = 0$. Thus $\nabla f(a)$ is perpendicular to $x'(0)$.

4.7 Example: Given a differentiable map $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$, notice that the i^{th} row of the matrix $f'(a)$ is equal to ${f_i}'(a) = \nabla f(a)^T$, where f_i is the i^{th} component of f. So the i^{th} row is perpendicular to the level set $f_i(x) = f_i(a)$.

We use the notation

$$f_{x_j}(a) = \frac{\partial f}{\partial x_j}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{pmatrix}$$

for the j^{th} column of the matrix $f'(a)$. Notice that this is equal to the tangent vector to the curve $g(t) = f(a + t\,e_j)$, where e_j is the j^{th} standard basis vector.

In particular, if $f : U \subset \mathbf{C} \to \mathbf{C}$ is given by $w(z) = u(z) + i\,v(z)$ with $z = x + i\,y$, then

$$Df(a) = f'(a) = \begin{pmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{pmatrix} .$$

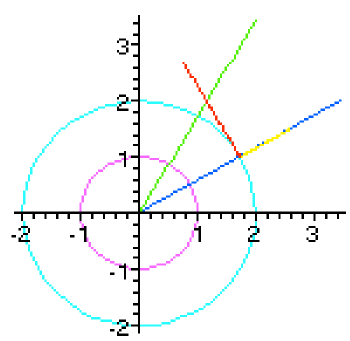
The columns $f_x = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$ and $f_y = \begin{pmatrix} u_y \\ v_y \end{pmatrix}$ are the tangent vectors to the curves $f(a+t)$ and $f(a + i\,t)$ respectively, and the rows $u' = (\,u_x \quad u_y\,)$ and $v' = (\,v_x \quad v_y\,)$ are perpendicular to the level curves $u = u(a)$ and $v = v(b)$ respectively.

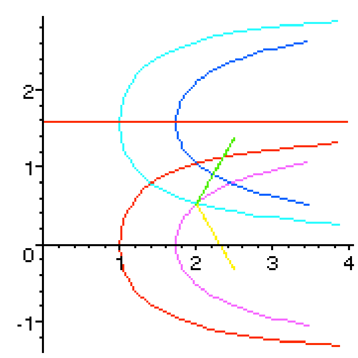
4.8 Example: Let f be the change of coordinate map $(x,y)=f(r,\theta)=(r\cos\theta,r\sin\theta)$. Then

$$Df(r,\theta)=f'(r,\theta)=\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

At $(r,\theta)=(2,\frac{\pi}{6})$, we have $(x,y)=f(2,\frac{\pi}{6})=(\sqrt{3},1)$ and $f'(2,\frac{\pi}{6})=\begin{pmatrix} \sqrt{3}/2 & -1 \\ 1/2 & \sqrt{3} \end{pmatrix}$. You should draw a picture for yourself showing the images of the lines $r=0,1,2$ and $\theta=0,\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ (the images are circles and rays), and at the point $(x,y)=(\sqrt{3},1)$ you should draw the tangent vectors $f_r=\begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$ and $f_\theta=\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$. Also, draw a picture showing the level curves $x=0,1,\sqrt{3}$ and $y=0,1,\sqrt{3}$ (they are multiples of $r=\sec\theta$ and $r=\csc\theta$) and at the point $(r,\theta)=(2,\frac{\pi}{6})$ draw the gradient vectors $x'=\begin{pmatrix} \frac{\sqrt{3}}{2} & -1 \end{pmatrix}$ and $y'=\begin{pmatrix} \frac{1}{2} & \sqrt{3} \end{pmatrix}$.

The map f is not $1:1$ so it does not have an inverse, but since the matrix $f'(2,\frac{\pi}{6})$ is invertible, we know that we can make f invertible by restricting its domain. If $g=f^{-1}$ near the point $(r,\theta)=(2,\frac{\pi}{3})$, then we have $g'(\sqrt{3},1)=\begin{pmatrix} \sqrt{3}/2 & -1 \\ 1/2 & \sqrt{3} \end{pmatrix}^{-1}=\frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$. This can also be verified by finding a formula for g, for example if we restrict the domain of f to $r>0$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ then $(r,\theta)=g(x,y)=\left(\sqrt{x^2+y^2}\,,\tan^{-1}(y/x)\right)$.





4.9 Note: We now wish to interpret the real derivative Df of a map $f:U\subset\mathbf{C}\to\mathbf{C}$, which is a 2×2 matrix, in terms of complex numbers. Indeed any real 2×2 matrix A corresponds to two complex numbers in the following two ways. Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and write $z=x+i\,y$ with $x,y\in\mathbf{R}$. Then

$$\begin{aligned} A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a \\ c \end{pmatrix}x+\begin{pmatrix} b \\ d \end{pmatrix}y \\ &= (a+i\,c)x+(b+i\,d)y \\ &= (a+ic)\frac{z+\overline{z}}{2}+(b+i\,d)\frac{z-\overline{z}}{2i} \\ &= \frac{1}{2}\big((a+d)+i\,(c-b)\big)\,z=\frac{1}{2}\big((a-d)+i\,(c+b)\big)\,\overline{z} \end{aligned}.$$

Thus we have $A\begin{pmatrix} x \\ y \end{pmatrix}=px+qy=uz+v\overline{z}$ where p and q are the columns of A, that is $p=a+i\,c$ and $q=b+i\,d$, and u and v are given by $u=\frac{1}{2}\big((a+d)+i\,(c-b)\big)=\frac{1}{2}(p-i\,q)$ and $v=\frac{1}{2}\big((a-d)+i\,(c+b)\big)=\frac{1}{2}(p+i\,q)$. Note also that $p=u+v$ and $q=i\,(u-v)$.

Conversely, given $u=\alpha+i\,\beta$ and $v=\gamma+i\,\delta$, with $\alpha,\beta,\gamma,\delta\in\mathbf{R}$, we have

$$u\,z=(\alpha+i\,\beta)(x+i\,y)=(\alpha x-\beta y)+i\,(\beta x+\alpha y)=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$
$$v\overline{z}=(\gamma+i\,\delta)(x-i\,y)=(\gamma x+\delta y)+i\,(\delta x-\gamma y)=\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

so if $A\begin{pmatrix}x\\ y\end{pmatrix}=u\,z+v\,\overline{z}$ then A is given by $A=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}+\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}$.

4.10 Definition: Note 4.9 allows us to express the definition of differentiability in terms of complex numbers. Indeed, if $f:U\subset\mathbf{C}\to\mathbf{C}$ and $a\in U$ then f is differentiable at a $\Longleftrightarrow$ there exists a 2×2 matrix $Df(a)$ with real entries such that $\lim\limits_{z\to a}\dfrac{|R(z)|}{|z-a|}=0$, where $R(z)=f(z)-\big(f(a)+Df(a)(z-a)\big)$ $\Longleftrightarrow$ there exist complex numbers $f_x(a)$ and $f_y(a)$ such that $\lim\limits_{z\to a}\dfrac{|R(z)|}{|z-a|}=0$, where $R(z)=f(z)-\big(f(a)+f_x(a)\text{Re}\,(z-a)+f_y(a)\text{Im}\,(z-a)\big)$ $\Longleftrightarrow$ there exist complex numbers $f_z(a)$ and $f_{\overline{z}}(a)$ such that $\lim\limits_{z\to a}\dfrac{|R(z)|}{|z-a|}=0$, where $R(z)=f(z)-\big(f(a)+f_z(a)(z-a)+f_{\overline{z}}(\overline{z}-\overline{a})\big)$.

In this case we have

$$\begin{aligned}
Df&=\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}\\
f_x&=u_x+i\,v_x=f_z+f_{\overline{z}}\\
f_y&=u_y+i\,v_y=i\,(f_z-f_{\overline{z}})\\
f_z&=\frac{1}{2}(f_x-i\,f_y)=\frac{1}{2}\big((u_x+v_y)+i\,(v_x-u_y)\big)\\
f_{\overline{z}}&=\frac{1}{2}(f_x+i\,f_y)=\frac{1}{2}\big((u_x-v_y)+i\,(u_y-v_x)\big)
\end{aligned}.$$

Also, if $f_z=\alpha+i\,\beta$ and $f_{\overline{z}}=\gamma+i\,\delta$ with $\alpha,\beta,\gamma,\delta\in\mathbf{R}$ then

$$Df=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}+\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}.$$

If $w=f(z)$ then other notations for these include $Df=Dw$, $f_x=\frac{\partial f}{\partial x}=w_x=\frac{\partial w}{\partial x}$, $f_y=\frac{\partial f}{\partial y}=w_y=\frac{\partial w}{\partial y}$, $f_z=\frac{\partial f}{\partial z}=\partial f=w_z=\frac{\partial w}{\partial z}=\partial w$, $f_{\overline{z}}=\frac{\partial f}{\partial \overline{z}}=\overline{\partial}f=w_{\overline{z}}=\frac{\partial w}{\partial z}=\overline{\partial}w$.

4.11 Example: Show that $\frac{\partial z}{\partial z}=1$, $\frac{\partial z}{\partial \overline{z}}$, $\frac{\partial \overline{z}}{\partial z}=0$, $\frac{\partial \overline{z}}{\partial \overline{z}}=1$, and $\frac{\partial a}{\partial z}=\frac{\partial a}{\partial \overline{z}}=0$, where $a\in\mathbf{C}$.

Solution: If $f(z)=z$, then we have $f(x+i\,y)=u(x,y)+i\,v(x,y)$, where $u(x,y)=x$ and $v(x,y)=y$. So $Df=\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $f_x=u_x+i\,v_x=1$, $f_y=u_y+i\,v_y=i$, $f_z=\frac{1}{2}(f+x-i\,f_y)=1$ and $f_{\overline{z}}=\frac{1}{2}(f_x-i\,f_y)=0$.

If $f(z)=\overline{z}$, then we have $u(x,y)=x$ and $v(x,y)=-y$. So $Df=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $f_x=1$, $f_y=-i$, $f_z=0$ and $f_{\overline{z}}=1$.

If $f(z)=a\in\mathbf{C}$ then $u(x,y)=\text{Re}\,(a)$ and $v(x,y)=\text{Im}\,(a)$. So $Df=0$ and hence $f_x=f_y=f_zz=f_{\overline{z}}=0$.

4.12 Theorem: *Let $f : U \subset \mathbf{C} \to \mathbf{C}$ be differentiable.*
a) For $\alpha = x, y, z$ or $\overline{z}$ we have
i) $(cf)_\alpha = c\, f_\alpha$
ii) $(f \pm g)_\alpha = f_\alpha \pm f_\alpha$
iii) (The Product Rule) $(fg)_\alpha = f_\alpha g + f g_\alpha$
iv) (The Quotient Rule) $(f/g)_\alpha = (f_\alpha g - f g_\alpha)/g^2$, when $g \neq 0$
b) Define $\overline{f} : U \to \mathbf{C}$ by $\overline{f}(z) = \overline{f(z)}$. Then $\overline{f}_z = \overline{f_{\overline{z}}}$ and $\overline{f}_{\overline{z}} = \overline{f_z}$.
c) (The Chain Rule) Suppose $f : U \to V \subset \mathbf{C}$ and $g : V \to \mathbf{C}$ are both differentiable, and let $h(z) = g(f(z))$. Then h is differentiable, and if we write $w = f(z)$ and $q = g(w)$, then

$$\begin{pmatrix} q_z & q_{\overline{z}} \\ \overline{q}_z & \overline{q}_{\overline{z}} \end{pmatrix} = \begin{pmatrix} q_w & q_{\overline{w}} \\ \overline{q}_w & \overline{q}_{\overline{w}} \end{pmatrix} \begin{pmatrix} w_z & w_{\overline{z}} \\ \overline{w}_z & \overline{w}_{\overline{z}} \end{pmatrix}.$$

Equivalently, we have $\dfrac{\partial q}{\partial z} = \dfrac{\partial q}{\partial w}\dfrac{\partial w}{\partial z} + \dfrac{\partial q}{\partial \overline{w}}\dfrac{\partial \overline{w}}{\partial z}$ *and* $\dfrac{\partial q}{\partial \overline{z}} = \dfrac{\partial q}{\partial w}\dfrac{\partial w}{\partial \overline{z}} + \dfrac{\partial q}{\partial \overline{w}}\dfrac{\partial \overline{w}}{\partial \overline{z}}$.

Proof: We prove the product rule, and leave the rest of part a) as an exercise. We write $f = u + i\,v$ and $g = s + i\,t$ where u, v, s and t are real-valued. Then $fg = (us - vt) + i\,(ut + vs)$. The product rule in theorem 4.4 applies to the functions u, v, s and t, so we have

$$\begin{aligned} (fg)_x &= (us - vt)_x + i\,(ut + vs)_x \\ &= (u_x s + u s_x - v_x t - v t_x) + i\,(u_x t + u t_x + v_x s + v s_x) \\ &= (u_x + i\,v_x)(s + i\,t) + (u + i\,v)(s_x + i\,t_x) \\ &= f_x g + f g_x \end{aligned}$$

Similarly, $(fg)_y = f_y g + f g_y$. Then, using this result, we have

$$\begin{aligned} (fg)_z &= \tfrac{1}{2}\big((fg)_x - i\,(fg)_y\big) \\ &= \tfrac{1}{2}\big((f_x g + f g_x) - i\,(f_y g + f g_y)\big) \\ &= \tfrac{1}{2}(f_x - i\,f_y)\,g + f\,\tfrac{1}{2}(g_x - i\,g_y) \\ &= f_z g + f g_z\,. \end{aligned}$$

To prove part b), write $f = u + i\,v$ with u and v real-valued. Then $\overline{f} = u - i\,v$ so $D\overline{f} = \begin{pmatrix} u_x & u_y \\ -v_x & -v_y \end{pmatrix}$ and hence $\overline{f}_x = u_x - i\,v_x = \overline{f_x}$ and $\overline{f}_y = u_y - i\,v_y = \overline{f_y}$. So we have $\overline{f}_z = \frac{1}{2}(\overline{f}_x - i\,\overline{f}_y) = \frac{1}{2}(\overline{f_x} - i\,\overline{f_y}) = \overline{\frac{1}{2}(f_x + i\,f_y)} = \overline{f_{\overline{z}}}$, and similarly, $\overline{f}_{\overline{z}} = \overline{f_z}$.

To prove part c), write $z = x + i\,y$, $f(z) = w = u + i\,v$ and $g(w) = q = s + i\,t$. Then

$$\begin{aligned} q_z &= \tfrac{1}{2}\big((s_x + t_y) + i\,(t_x - s_y)\big) \\ &= \frac{1}{2}\big((s_u u_x + s_v v_x + t_u u_y + t_v v_y) + i\,(t_u u_x + t_v v_x - s_u u_y - s_v v_y)\big)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} q_w w_z + q_{\overline{w}} \overline{w}_z &= q_w w_z + q_{\overline{w}} \overline{w_{\overline{z}}} \\ &= \tfrac{1}{2}\big((s_u + t_v) + i\,(t_u - s_v)\big)\,\tfrac{1}{2}\big((u_x + v_y) + i\,(v_x - u_y)\big) \\ &\quad + \tfrac{1}{2}\big((s_u - t_v) + i\,(t_u + s_v)\big)\,\tfrac{1}{2}\big((u_x - v_y) - i\,(v_x + u_y)\big)\,. \end{aligned}$$

Expanding and simplifying this last expression shows that $q_z = q_w w_z + q_{\overline{w}} \overline{w}_z$. Similarly, we can show that $q_{\overline{z}} = q_w w_{\overline{z}} + q_{\overline{w}} \overline{w}_{\overline{z}}$.

4.13 Example: Let $f(z) = z^2 + 3z\overline{z}$. Find f_z and $f_{\overline{z}}$.

Solution: We solve this using two methods. First, by example 4.11 and theorem 4.12, we can calculate $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ using all the same rules that we use to find partial derivatives of real functions of two real variables. We have $f_z = 2z + 3\overline{z}$ and $f_{\overline{z}} = 3z$.

Our second solution is to express f in terms of real variables, and then use definition 4.10. We have $f(z) = f(x+i\,y) = (x+i\,y)^2 + 3(x+i\,y)(x-i\,y) = (4x^2+2y^2) + i\,(2xy)$, and so $Df = \begin{pmatrix} 8x & 4y \\ 2y & 2x \end{pmatrix}$. Thus we have $f_x = 8x + i\,2y$ and $f_y = 4y + i\,2x$, and so $f_z = \frac{1}{2}(f_x - i\,f_y) = \frac{1}{2}(8x + i\,2y - i\,4y + 2x) = 5x - i\,y = 5\frac{z+\overline{z}}{2} - i\,\frac{z-\overline{z}}{2i} = 2z + 3\overline{z}$, and $f_{\overline{z}} = \frac{1}{2}(f_x + i\,f_y) = \frac{1}{2}(8x + i\,2y + i\,4y - 2x) = 3x + i\,3y = 3z$.

4.14 Example: Let $f(z) = \dfrac{(z+\overline{z})z}{2+z\overline{z}}$. Find $f_z(1+i)$ and $f_{\overline{z}}(1+i)$.

Solution: By the product and quotient rules, $\dfrac{\partial f}{\partial z} = \dfrac{(z+(z+\overline{z}))(2+z\overline{z}) - (z+\overline{z})z\overline{z}}{(2+z\overline{z})^2}$, so $\dfrac{\partial f}{\partial z}(1+i) = \dfrac{(3+i)(4)-(2)(2)}{(4)^2} = \dfrac{2+i}{4}$. Also, we have $\dfrac{\partial f}{\partial \overline{z}} = \dfrac{(z)(2+z\overline{z}) - (z+\overline{z})z^2}{(2+z\overline{z})^2}$ and so $\dfrac{\partial f}{\partial \overline{z}}(1+i) = \dfrac{(1+i)(4)-(2)(2i)}{(4)^2} = \dfrac{1}{4}$.

4.15 Example: Let $w = f(z) = iz + \overline{z}$, let $q = g(w) = w^2 - \overline{w}$, and let $h(z) = g(f(z))$. Find $h_z(1+2i)$ and $h_{\overline{z}}(1+2i)$.

Solution: We provide three solutions to this problem. Our first solution uses the chain rule in theorem 4.12. We have

$$\begin{pmatrix} h_z & h_{\overline{z}} \end{pmatrix} = \begin{pmatrix} g_w & g_{\overline{w}} \end{pmatrix} \begin{pmatrix} f_z & f_{\overline{z}} \\ \overline{f}_z & \overline{f}_{\overline{z}} \end{pmatrix} = \begin{pmatrix} 2w & -1 \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 2wi-1 & 2w+i \end{pmatrix}.$$

When $z = 1+2i$ we have $w = f(z) = i(1+2i) + (1-2i) = -1-i$ and so we obtain $h_z = 2wi - 1 = 2(-1-i)\,i - 1 = 1-2i$ and $h_{\overline{z}} = 2w + i = 2(-1-i) + i = -2-i$.

Our second solution is to expand the composite $g(f(z))$ so that we can avoid using the chain rule. We have $h(z) = g(f(z)) = (iz+\overline{z})^2 - (-i\overline{z}+z) = -z^2 + 2iz\overline{z} + \overline{z}^2 + i\,\overline{z} - z$. Thus we have $h_z = -2z + 2i\overline{z} - 1$ so $h_z(1+2i) = -2(1+2i) + 2i(1-2i) - 1 = 1-2i$ and we have $h_{\overline{z}} = 2iz + 2\overline{z} + i$ so $h_{\overline{z}} = 2i(1+2i) + 2(1-2i) + i = -2-i$.

The third solution is to express f, g and h in terms of real variables. Write $z = x+i\,y$, $w = f(z) = u + i\,v$ and $q = h(z) = s + i\,t$. Then $f(x+i\,y) = i(x+i\,y) + (x-i\,y) = (x-y) + i\,(x-y)$ so $u = x-y$ and $v = x-y$, and $g(u+i\,v) = (u+i\,v)^2 - (u-i\,v) = (u^2-v^2-u) + i\,(2uv-v)$ so $s = u^2 - v^2 - u$ and $t = 2uv + v$. By the chain rule for real variables,

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 2u-1 & -2v \\ 2v & 2u+1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2u-2v-1 & -2u+2v+1 \\ 2u+2v+1 & -2u-2v-1 \end{pmatrix}$$

so $h_z = \frac{1}{2}\big((2u-2v-1)+(-2u-2v-1)\big) + \frac{i}{2}\big((2u+2v+1)-(-2u+2v+1)\big) = (-2v-1) + i\,(2u)$ and $h_{\overline{z}} = \frac{1}{2}\big((2u-2v-1)-(-2u-2v-1)\big) + \frac{i}{2}\big((2u+2v+1)+(-2u+2v+1)\big) = 2u + i\,(2v+1)$. When $z = 1+2i$, we have $w = f(z) = i(1+2i) + (1-2i) = -1-i$, so $u = v = -1$ and hence $h_z(1+2i) = (-2v-1) + i\,(2u) = 1-2i$ and $h_{\overline{z}}(1+2i) = 2u + i\,(2v+1) = -2-i$.

4.16 Definition: Let $f : U \subset \mathbf{C} \to \mathbf{C}$. We define

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

provided that the limit exists, we say that f is **holomorphic** at $z = a$ and thet $f'(a)$ is the **derivative** of f at a. Equivalently, we say that f is holomorphic at $z = a$ if there exists a complex number $f'(a)$ such that

$$\lim_{z \to a} \frac{|S(z)|}{|z - a|} = 0 \, ,$$

where $S(z) = f(z) - \big(f(a) + f'(a)(z - a)\big)$. We say that f is **holomorphic** in U if it is holomorphic at every point in U.

4.17 Definition: For $f : U \subset \mathbf{C} \to \mathbf{C}$ we define

$$f^{\times}(a) = \lim_{z \to a} \frac{f(z) - f(a)}{\overline{z} - \overline{a}}$$

provided the limit exists, and if so we say that f is **conjugate-holomorphic** at $z = a$. Equivalently, f is conjugate-holomorphic at a if there exists a complex number $f^{\times}(a)$ such that $\lim_{z \to a} \dfrac{|T(z)|}{|z - a|} = 0$ where $T(z) = f(z) - \big(f(a) + f^{\times}(a)(\overline{z} - \overline{a})\big)$.

4.18 Note: We have now used the notation $f'(a)$ for two apparently different objects. The *real* derivative $f'(a)$ is a 2×2 matrix, while the *complex* derivative $f'(a)$ is an element of $\mathbf{C}$, that is, is a 2×1 matrix. From now on we wish to make a distinction between these two different derivatives, so we shall use the following convention: for $f : U \subset \mathbf{C} \to \mathbf{C}$

$Df(a)$ denotes the real derivative of f (if it exists)
$f'(a)$ denotes the complex derivative of f (if it exists)

4.19 Theorem: *a) Let $f : U \subset \mathbf{C} \to \mathbf{C}$.*
i) $f'(a)$ exists $\iff$ $Df(a)$ exists and $f_{\overline{z}}(a) = 0$. In this case $f'(a) = f_z(a)$.
ii) $f^{\times}(a)$ exists $\iff$ $Df(a)$ exists and $f_z(a) = 0$. In this case $f^{\times}(a) = f_{\overline{z}}(a)$.
b) Suppose that $f : U \subset \mathbf{C} \to \mathbf{C}$ is differentiable. Then
i) f is holomorphic $\iff$ $f_{\overline{z}} = 0$
$\iff u_x = v_y$ *and* $u_y = -v_x$
$\iff Df$ *is of the form* $Df = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ *for some* $\alpha, \beta \in \mathbf{R}$.
In this case, $f' = f_z = \alpha + i\,\beta = u_x + i\,v_x = u_x - i\,u_y = v_y - i\,u_y = v_y + i\,v_x$.
ii) f is conjugate-holomorphic $\iff$ $f_z = 0$
$\iff u_x = -v_y$ *and* $u_y = v_x$
$\iff Df$ *is of the form* $Df = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}$ *for some* $\gamma, \delta \in \mathbf{R}$.
In this case, $f^{\times} = f_{\overline{z}} = \gamma + i\,\delta = u_x + i\,v_x = u_x + i\,u_y = -v_y + i\,u_y = -v_y + i\,v_x$.

Proof: Compare definitions 4.16 and 4.17 to definition 4.10.

4.20 Definition: The two differential equations $u_x = v_y$ and $u_y = -v_x$ are called the **Cauchy-Riemann** equations.

4.21 Example: Let $f(z) = z^2 + 2|z|^2$. Determine where f is holomorphic and where it is conjugate-holomorphic.

Solution: We have $f(z) = z^2 + 2z\overline{z}$, so $f_z = 2z + 2\overline{z} = 4\text{Re}\,(z)$, and $f_{\overline{z}} = 2z$. Thus f is conjugate-holomorphic when $f_z = 4\text{Re}\,(z) = 0$, that is along the y-axis, and f is holomorphic when $f_{\overline{z}} = z = 0$, that is at the origin.

4.22 Theorem: *a) If $f : U \subset \mathbf{C} \to \mathbf{C}$ is holomorphic (or conjugate-holomorphic) at a then f is continuous at a.*
b) If $f, g : U \subset \mathbf{C} \to \mathbf{C}$ are both be holomorphic at a, then
i) $(cf)'(a) = c\, f'(a)$
ii) $(f \pm g)'(a) = f'(a) \pm g'(a)$
iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
iv) $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$, provided $g(a) \neq 0$.
Similar results hold when f and g are both conjugate-holomorphic.
c) (The Chain Rule) Let f, g, h and k be maps from open sets in $\mathbf{C}$ to $\mathbf{C}$ with f and g holomorphic and h and k conjugate-holomorphic. Then
i) $g \circ f$ is holomorphic with $(g \circ f)'(z) = g'(f(z))f'(z)$.
ii) $h \circ f$ is conjugate-holomorphic with $(h \circ f)^\times(z) = h^\times(f(z))\overline{f'(z)}$.
iii) $f \circ h$ is conjugate-homorphic with $(f \circ h)^\times(z) = \overline{f'(h(z))}h^\times(z)$.
iv) $k \circ h$ is holomorphic with $(k \circ h)'(z) = k^\times(h(z))\overline{h^\times(z)}$.
d) (The Inverse Function Theorem) If f is holomorphic in U and $f'(a) \neq 0$ then we can make f invertible by restricting its domain, and then the inverse function $g = f^{-1}$ will be holomorphic near $f(a)$ with $g'(f(z)) = 1/f'(z)$.
A similar result holds when f is conjugate-holomorphic.

Proof: Part a) holds since $\lim\limits_{z \to a} \big(f(z) - f(a)\big) = \lim\limits_{z \to a} \left(\dfrac{f(z) - f(a)}{z - a}\,(z - a)\right) = f'(a) \cdot 0 = 0.$

Part b) follows immediately from part a) of theorem 4.12.

Part c) follows almost immediately from part c) of theorem 4.12. For example, we prove part (i). Write $w = f(z)$ and $q = g(w)$. Since f and g are holomorphic, we have $\frac{\partial w}{\partial \overline{z}} = 0$ and $\frac{\partial q}{\partial \overline{w}} = 0$. So by the chain rule of theorem 4.12 , we have $\frac{\partial q}{\partial \overline{z}} = \frac{\partial q}{\partial w}\frac{\partial w}{\partial \overline{z}} + \frac{\partial q}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial \overline{z}} = 0$, which shows that $g \circ f$ is holomorphic, and $\frac{\partial q}{\partial z} = \frac{\partial q}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial q}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial z} = \frac{\partial q}{\partial w}\frac{\partial w}{\partial z}$.

Part d) is not easy to prove as it is stated. It is hard to show that if $f'(a) \neq 0$ then we can make f invertible by restricting its domain and it is hard to show that its inverse will be holomorphic. (We may show this later). However, if we asume that $g = f^{-1}$ exists and is holomorphic then we have $g(f(z)) = z$, and so by the chain rule, $g'(f(z))f'(z) = 1$.

4.23 Theorem: *The maps z^n, $n \in \mathbf{Z}$, the exponential map e^z, the trigonometric fuctions and the hyperbolic functions are all holomorphic in their domains. Also, any continuous branch of the logarithm $\log z$ (with an open domain) is holomorphic. We have*
i) $(z^n)' = n\, z^{n-1}$, where $n \in \mathbf{Z}$.
ii) $(e^z)' = e^z$
iii) $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$, $(\tan z)' = \sec^2 z$.
iv) $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$, $(\tanh z)' = \text{sech}^2 z$.
v) $(\log z)' = \dfrac{1}{z}$ for any branch of $\log z$.

Proof: Part i) can be shown from the definition of the derivative. Let $f(z) = z^n$, $n \in \mathbf{Z}$. Then $f'(z) = \lim_{w\to z} \frac{w^n - z^n}{w - z} = \lim_{w\to z} \big(w^{n-1} + w^{n-2}z + \cdots + w\, z^{n-2} + z^{n-1}\big) = n\, z^{n-1}$. Alternatively, part (i) could be proven by induction using the product rule from the base case $z' = 1$.

To prove part (ii), let $w = f(z) = e^z$ and write $z = x + i\,y$ and $w = u + i\,v$. Then $f(z) = e^{x+i\,y} = e^x \cos y + i\, e^x \sin y$, so $Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$, and we see that f is holomorphic in $\mathbf{C}$ with $f'(z) = e^x \cos y + i\, e^x \sin y = e^z$.

We derive the formula for the derivative of $\sin z$ in two ways. One way is to let $f(z) = \sin z$ and write $z = x + i\,y$ and $f(z) = u(z) + i\,v(z)$. Then we have $f(z) = \sin x \cosh y + i\, \cos x \sinh y$, and $Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \cos x \cosh y & \sin x \sinh y \\ -\sin x \sinh y & \cos x \cosh y \end{pmatrix}$ and so f is holomorphic in $\mathbf{C}$ and $f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos(z)$.

An easier way is to apply part (ii) and the differentiation rules in theorem 4.22 b) to the definition of $\sin z$. Indeed $(\sin z)' = \frac{1}{2i}(e^{iz} - e^{-iz})' = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$.

The rest of the derivative formulas are left as an exercise.

4.24 Example: The two above theorems show that elementary complex functions can be differentiated much like the real elementary functions. For example, let $f(z) = (z^3 e^{\sin z})^5$, then $f'(z) = 5(z^3 e^{\sin z})^4 (3z^2 e^{\sin z} + z^3 e^{\sin z} \cos z)$.

4.25 Example: Let $f(z) = \overline{\sin\big(z^2 + (1+i)\overline{z}\big)}$. Find f_z and $f_{\overline{z}}$.

Solution: We have $f(z) = q(w(u(z)))$, where $u(z) = z^2 + (1+i)\overline{z}$, $w(u) = \sin u$ and $q(w) = \overline{w}$. Note that $u_z = 2z$, $u_{\overline{z}} = (1+i)$, $w_u = \cos u$, $w_{\overline{u}} = 0$, $q_w = 0$ and $q_{\overline{w}} = 1$. By the chain rule, $w_z = w_u u_z + w_{\overline{u}} \overline{u}_z = 2z \cos u$ and also $w_{\overline{z}} = w_u u_{\overline{z}} + w_{\overline{u}} \overline{u}_{\overline{z}} = (1+i)\cos u$. Using the chain rule again, we have $q_z = q_w w_z + q_{\overline{w}} \overline{w}_z = \overline{w}_z = \overline{w_{\overline{z}}} = (1-i)\overline{\cos u}$ and also $q_{\overline{z}} = q_w w_{\overline{z}} + q_{\overline{w}} \overline{w}_{\overline{z}} = \overline{w}_{\overline{z}} = \overline{w_z} = \overline{2z \cos u}$. Thus $f_z = (1-i)\overline{\cos u} = (1-i)\overline{\cos\big(z^2 + (1+i)\overline{z}\big)}$ and $f_{\overline{z}} = \overline{2z \cos u} = \overline{2z \cos\big(z^2 + (1+i)\overline{z}\big)}$.

An alternate solution is to note that for $z = x + i\,y$ we have $e^{\overline{z}} = e^{x - i\,y} = e^x(\cos y - i \sin y) = \overline{e^z}$, and so from the definition of $\sin z$ we also have $\sin(\overline{z}) = \overline{\sin z}$. Thus $f(z) = \sin\big(\overline{z}^2 + (1-i)z\big)$ and so $f_z(z) = (1-i)\cos\big(\overline{z}^2 + (1-i)z\big)$ and $f_{\overline{z}}(z) = 2\overline{z}\cos\big(\overline{z}^2 + (1-i)z\big)$.

4.26 Example: Let $f(z) = z^2 - 2z + 3$. Then $f'(z) = 2z - 2$ and we have $f(2) = 3$ and $f'(2) = 2$. Since $f'(2) \neq 0$, we can restrict the domain of f so that it is invertible. Let g be the inverse function. Find $g'(3)$.

Solution: By the inverse function theorem, we have $g'(3) = \frac{1}{f'(2)} = \frac{1}{2}$.

4.27 Note: To find the derivative of a branch of a multifunction, first express it in terms of one (or more) branches of the logarithm, then take the derivative.

4.28 Example: Find a formula for the derivative of one branch of z^w, where $w \in \mathbf{C}$.

Solution: Let $z^w = \exp(w \log z)$, where $\log z$ is a branch of the logarithm. Then

$$(z^w)' = \exp(w \log z)\frac{w}{z} = \frac{w\, z^w}{z}\,.$$

(Notice that this is similar to the familiar formula $(z^w) = w\, z^{w-1}$; the familiar formula has the disadvantage that it does not specify which branch of z^{w-1} we should use).

Chapter 5. Conformal Maps

5.1 Note: Later on we shall see that every holomorphic function is $\mathcal{C}^\infty$, which means that all partial derivatives of all orders exist (and are continuous). For this chapter we shall assume that all functions are $\mathcal{C}^2$, which means that all the second order partial derivatives of f (namely u_{xx}, u_{xy}, u_{yx}, u_{yy}, v_{xx}, v_{xy}, v_{yx} and v_{yy}) exist and are continuous. We shall also use the fact that for $\mathcal{C}^2$ functions, we always have $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

5.2 Definition: A map $f : U \subset \mathbf{R}^n \to \mathbf{R}^n$ is said to **preserve orientation** at $x = a$ if $|Df(a)| > 0$, and it **reverses orientation** at a if $|Df(a)| < 0$.

5.3 Note: Let $f : U \subset \mathbf{C} \to \mathbf{C}$. If f is holomorphic at $z = a$ and $f'(a) \neq 0$ then f preserves orientation at a, since $|Df(a)| = \det \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha^2 + \beta^2 > 0$. On the other hand, if f is conjugate-holomorphic at a with $f^\times(a) \neq 0$ then f reverses orientation at a since $|Df(a)| = \det \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix} = -(\gamma^2 + \delta^2) < 0$.

5.4 Definition: A map $f : U \subset \mathbf{R}^n \to \mathbf{R}^n$ is called an **isometry** if it preserves distance, that is if $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbf{R}^n$. Using some linear algebra, one can show that f is an isometry if and only if f is of the form $f(x) = Ax + b$ for some vector $b \in \mathbf{R}^n$ and some **orthogonal** $n \times n$ matrix A (A is orthogonal means that $A^T A = I$).

5.5 Note: Since the 2×2 orthogonal matrices are the matrices either of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, we see that the isometries in $\mathbf{R}^2$ are the maps f of the form $f(z) = \begin{cases} az + b \\ a\overline{z} + b \end{cases}$ for some $a, b \in \mathbf{C}$ with $|a| = 1$.

5.6 Definition: A map $f : U \subset \mathbf{R}^n \to \mathbf{R}^n$ is called a **similarity** of **scaling factor** $k > 0$ if it scales distances by a factor of k, that is if $|f(x) - f(y)| = k|x - y|$ for all $x, y \in \mathbf{R}^n$. It is not hard to see that f is a similarity of scaling factor k if and only if $\frac{1}{k} f$ is an isometry.

5.7 Note: A map $f : U \subset \mathbf{C} \to \mathbf{C}$ is a similarity of scaling factor $k > 0$ if and only if f is of the form $f(z) = \begin{cases} az + b \\ a\overline{z} + b \end{cases}$ for some $a, b \in \mathbf{C}$ with $|a| = k$.

5.8 Note: Let $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ be differentiable at a. Given a vector $v \in \mathbf{R}^n$, choose a curve $\alpha(t)$ with $\alpha(0) = a$ and $\alpha'(0) = v$. Then $(f \circ \alpha)'(0) = Df(\alpha(0))\alpha'(0) = Df(a)v$. So we say that f sends the vector v at a to the vector $w = Df(a)v$ at $f(a)$.

5.9 Definition: A map $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is called **conformal** at a if it preserves angles between curves at a, or to be precise, f is conformal at a if

$$\frac{(Df\, v) \cdot (Df\, w)}{|Df\, v|\, |Df\, v|} = \frac{v \cdot w}{|v|\, |w|}$$

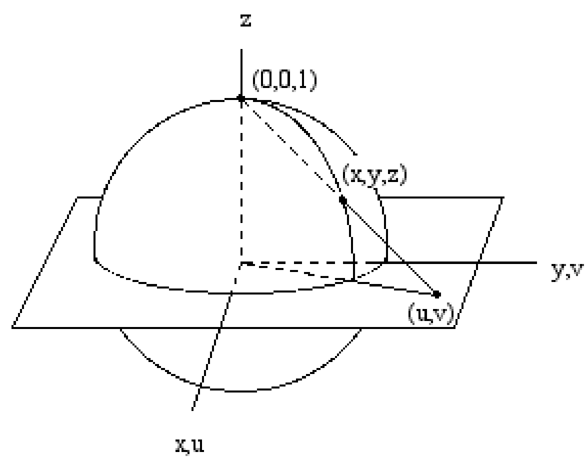
for all vectors $v, w \in \mathbf{R}^n$. We say f is conformal in U if it is conformal at every $a \in U$.

5.10 Note: Using linear algebra, one can show that f is conformal at a if and only if $Df(a)^T Df(a) = k\,I$ for some $k > 0$. We shall show only that the latter implies the former; suppose that $Df^T Df = k\,I$ with $k > 0$. Then

$$(Df\,v)\cdot(Df\,w) = (Df\,v)^T(Df\,w) = v^T Df^T Df\,w = v^T k\,I\,w = k\,v^T w = k\,v\cdot w$$

and in particular $|Df\,v| = \sqrt{(Df\,v)\cdot(Df\,v)} = \sqrt{k}\,|v|$, and similarly $|Df\,w| = \sqrt{k}\,|w|$. It follows that f is conformal; f behaves locally like a similarity of scaling factor $\sqrt{k}$.

5.11 Example: The **steriographic projection** from the unit sphere, with the north pole removed, to the complex plane is the map ϕ which sends the point (x,y,z) on the sphere to the point of intersection (u,v) of the line through (x,y,z) and the plane $z=0$. Find a formula for ϕ and ϕ^{-1}, and show that stereographic projection is conformal.



Solution: The line through $(0,0,1)$ and (x,y,z) is given by $(0,0,1)+t(x,y,z-1)$, $t\in\mathbf{R}$. The point of intersection of this line with the plane $z=0$ occurs when $1+t(z-1)=0$, that is when $t=1/(1-z)$. The point of intersection is $(0,0,1)+\frac{1}{1-z}(x,y,z-1)=(\frac{x}{1-z},\frac{y}{1-z},0)$, so we have

$$(u,v)=\phi(x,y,z)=\left(\frac{x}{1-z},\frac{y}{1-z}\right).$$

Given (u,v) on the other hand, the line through $(0,0,1)$ and (u,v) is given by $\alpha(t)=(0,0,1)+t(u,v,-1)=(tu,tv,1-t)$. The point of intersection with the unit sphere occurs when $|\alpha(t)|=1$, so we need $(tu)^2+(tv)^2+(1-t)^2=1$, that is $t^2u^2+t^2v^2-2t+t^2=0$, or $t\,(tu^2+tv^2+t-2)=0$. The point of intersection occurs when $t=\frac{2}{u^2+v^2+1}$, so we obtain the formula

$$(x,y,z)=\phi^{-1}(u,v)=\left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Now, we show that ϕ^{-1} is conformal. We have

$$D\phi^{-1}=\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}=\frac{2}{(u^2+v^2+1)^2}\begin{pmatrix} -u^2+v^2+1 & -2uv \\ -2uv & u^2-v^2+1 \\ 2u & 2v \end{pmatrix}$$

and a quick calculation yields

$$(D\phi^{-1})^T(D\phi^{-1})=\frac{4}{(u^2+v^2+1)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that near the point (u,v), ϕ^{-1} behaves like a similarity of scaling factor $2/(u^2+v^2+1)$.

5.12 Theorem: *Let $f : U \subset \mathbf{C} \to \mathbf{C}$.*

a) f is conformal at a if and only if either

f is holomorphic at a with $f'(a) \neq 0$, in which case f preserves orientation, or

f is conjugate-holomorhic at a with $f^\times(a) \neq 0$, in which case f reverses orientation.

b) If U is connected, then f is conformal in U if and only if either

f is homorphic in U with $f'(z) \neq 0$ for all z, so f preserves orientation, or

f is conjugate-holomorphic in U with $f^\times(z) \neq 0$ for all z, so f reverses orientation.

Proof: To prove part a), note that f is conformal at a if and ony if $Df(a)$ is a positive scalar multiple of an orthogonal matrix. Since the 2×2 orthogonal matrices are the matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, we see that f is conformal if and only if $Df = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ or $Df = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}$ for some α, β or $\gamma, \delta \in \mathbf{R}$ not both equal to zero.

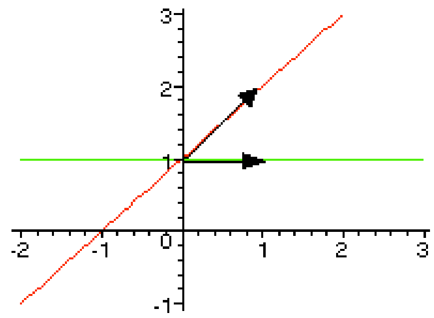
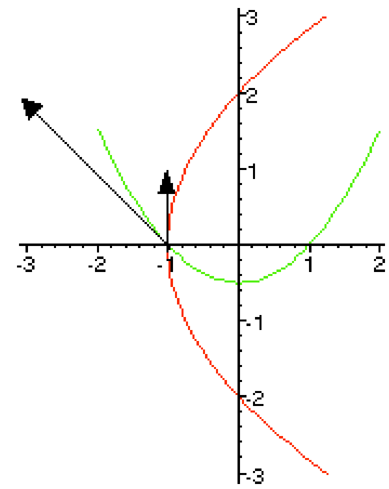
Part b) involves a subtle point: if f is conformal in U then how do we know that f cannot be holomorphic at some points $a \in U$ and conjugate-holomorphic at other points? It is for this reason that we must assume that U is connected. Since we have assumed that all functions in this chapter are $\mathcal{C}^2$ we know that u_x, u_y, v_x and v_y are all continuous and so $|Df| = u_x v_y - u_y v_x$ is also continuous. At each point $a \in U$ we have $|Df(a)| \neq 0$, so $|Df|$ is a continuous map from U to $\mathbf{R}^*$. Since U is connected, we know that $|Df|(U)$ is also connected and lies in $\mathbf{R}^*$. This implies that either $|Df(a)| > 0$ for all a or $|Df(a)| < 0$ for all a.

5.13 Note: If $f : U \subset \mathbf{C} \to \mathbf{C}$ is holomorphic at a with $f(a) = b$ and $f'(a) = r\, e^{i\,\theta}$, where $r > 0$, then by the definition of the (complex) derivative, for z near a we have $f(z) \cong f(a) + f'(a)(z-a) = b + re^{i\theta}(z-a)$. This shows that locally, f behaves like the following similarity: translate by $-a$, rotate by θ, scale by r, then translate by b.

5.14 Example: Let $f(z) = z^2$. Then f is holomorphic in $\mathbf{C}$ and $f'(z) = 2z$ so $f'(z) \neq 0$ in $\mathbf{C}^*$. Hence $f(z) = z^2$ is conformal in $\mathbf{C}^*$ and preserves orientation. Verify directly that f preserves the oriented angle from $\alpha(t) = i + t$ to $\beta(t) = i + (1+i)\,t$.

Solution: We have $\alpha(0) = \beta(0) = i$, $\alpha'(0) = 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta'(0) = 1 + i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the angle from $\alpha'(0)$ to $\beta'(0)$ is $\frac{\pi}{4}$. The images are $\gamma(t) = f(\alpha(t)) = (i+t)^2 = (t^2-1) + i\,2t$ (this is the parabola $u = \frac{1}{4}v^2 - 1$) and $\delta(t) = f(\beta(t)) = (i + (1+i)t)^2 = -(1+2t) + i\,(2t + 2t^2)$ (check that this is the parabola $v = \frac{1}{2}u^2 - \frac{1}{2}$). Note that $\gamma(0) = \delta(0) = -1$, so the two parabolas intersect at -1. We have $\gamma'(t) = 2t + 2i$ so $\gamma'(0) = 2i = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and we have $\delta'(t) = -2 + i\,(4t+2)$ so $\delta'(0) = -2 + 2i = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. So the angle from $\gamma'(0)$ to $\delta'(0)$ is $\frac{\pi}{4}$.

Notice also that α and β meet at i, and we have $f(i) = -1$ and $f'(i) = 2i = 2e^{i\,\pi/2}$. So near $z = i$, f can be approximated as follows: translate by $-i$, rotate by $\frac{\pi}{2}$, scale by 2, then translate by -1. Indeed, this is precisely what happens to the tangent vectors.

5.15 Definition: Let $u : U \subset \mathbf{C} \to \mathbf{C}$. The (2 dimensional) **Laplacian** is the differential operator ∇^2 given by

$$\nabla^2 u = u_{xx} + u_{yy}\,.$$

The map u is called **harmonic** in U if it is $\mathcal{C}^2$ and satisfies **Laplace's equation**

$$\nabla^2 u = 0\,.$$

5.16 Note: There are several functions from physics which satisfy Laplace's equation. Steady state temperature (in a homogeneous medium), electrostatic potential (in a vaccume) and the velocity potential for a steady flow of fluid (irrotational and indecompressible) al satisfy Laplace's equation.

5.17 Example: As an exercise, you should check that the map $u(x,y,z) = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$ satisfies the 3 dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$, but that the map $u(x,y) = -\dfrac{1}{\sqrt{x^2+y^2}}$ does not satisfy the 2 dimensional Laplace equation. The first map u represents the electic potential surrounding a point charge in $\mathbf{R}^3$, but the second map u does not represent the potential which surrounds a long straight wire. On the other hand, you can check that the map $u(x,y) = \ln\sqrt{x^2+y^2}$ does satisfy the 2 dimensional Laplace equation, and this map u does represent the potential surrounding a wire.

5.18 Theorem: *If $f(z) = u(z) + i\,v(z)$ is holomorphic (or conjugate-holomorphic) in U then u and v are both harmonic functions. When $f = u + i\,v$ is holomorphic, we say that v is the* **harmonic conjugate** *of u.*

Proof: The Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ imply that

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}$$

and likewise $v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$.

5.19 Example: Let $f(z) = e^z$. Verify that u is harmonic, where $u = \mathrm{Re}\,(f)$.

Solution: Since $e^{x+i\,y} = e^x(\cos y + i\,siny)$, we have $u(x+i\,y) = e^x \cos y$. So $u_x = e^x \cos y$ and $u_{xx} = e^x \cos y$, while $u_y = -e^x \sin y$ and $u_{yy} = -e^x \cos y = -u_{xx}$.

5.20 Example: Let $f(z) = z^3$. Verify that u is harmonic, where $u = \mathrm{Re}\,(f)$.

Solution: We have $f(x+i\,y) = (x+i\,y)^3 = (x^3-3xy^2)+i\,(3x^2y-y^3)$ so $u = x^3-3xy^2$. We have $u_x = 3x^2-3y^2$ so $u_{xx} = 6x$ and $u_y = 3x^2-6xy$ and so $v_{yy} = -6x = -u_{xx}$.

5.21 Note: There is a partial converse to the above note (which we may prove in a later chapter) which says that for certain sets U, for example when U is convex, if u is harmonic in U then there exists a harmonic function v such that the map $f = u+i\,v$ is holomorphic in U. The following example shows how to find v.

5.22 Example: Let $u = 2x^2+3xy-2y^2$. Check that u is harmonic in $\mathbf{C}$, and find a harmonic conjugate v.

Solution: We have $u_x = 4x+3y$, $u_{xx} = 4$, $u_y = 3x-4y$ and $u_{yy} = -4 = -u_{xx}$, so u is harmonic. To find v such that $u+i\,v$ is holomorphic, we need to find v such that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. To get $v_y = u_x = 4x+3y$ we must take $v = \int 4x+3y\;dy = 4xy+\frac{3}{2}y^2+c(x)$. Then we have $v_x = 4y+c'(x)$. To get $v_x = -u_y = 4y-3x$ we need to have $c'(x) = -3x$, so we choose $c(x) = -\frac{3}{2}x^2$. In this way we obtain $v = 4xy+\frac{3}{2}(y^2-x^2)$. The function $f = u+i\,v$ should be holomorphic, and indeed you can check that $f(z) = (2-i\,\frac{3}{2})z^2$.

5.23 Example: A long strip of heat conducting material is modelled by the set $S = \{x+i\,y|0<y<1\}$. Find the steady state temperature $u(x+i\,y)$ at each point in the strip given that the bottom edge is held at a constant temperature of a° and the top edge is held at b°. Describe the **isotherms**, that is the curves of constant temperature.

Solution: We must find a map $u:\overline{S}\to\mathbf{R}$ which is continuous on $\overline{S}$ and harmonic in S such that $u(x,0) = a$ and $u(x,1) = b$ for all x. We can take

$$u(x+i\,y) = a+(b-a)y\,.$$

It is easy to see that u is harmonic, indeed $u_{xx} = u_{yy} = 0$. Also notice that u is the imaginary part of the holomorphic map $f(z) = a\,i+(b-a)z$.

The isotherm $u = c$ is the horizontal line $c = a+(b-a)y$, or $y = \frac{c-a}{b-a}$.

5.24 Theorem: *If $u:U\subset\mathbf{C}\to\mathbf{R}$ is harmonic and if $f:V\subset\mathbf{C}\to\mathbf{C}$ is holomorphic then $u\circ f$ is harmonic.*

Proof: Write $x+i\,y = f(s+i\,t)$, $u = u(x+i\,y)$, and $v = u\circ f$. The chain rule gives

$$v_s = u_xx_s+u_yy_s \qquad v_t = u_xx_t+u_yy_t\,.$$

Using the chain rule and the product rule, we obtain

$$\begin{aligned}v_{ss} &= (u_{xx}x_s+u_{xy}y_s)x_s+u_xx_{ss}+(u_{yx}x_s+u_{yy}y_s)y_s+u_yy_{ss}\\ v_{tt} &= (u_{xx}x_t+u_{xy}y_t)x_t+u_xx_{tt}+(u_{yx}x_t+u_{yy}y_t)y_t+u_yy_{tt}\end{aligned}$$

Adding these, using the fact that $u_{xy} = u_{yx}$ we obtain

$$v_{ss}+v_{tt} = u_{xx}({x_s}^2+{x_y}^2)+u_{yy}({y_s}^2+{y_t}^2)+u_{xy}(2x_sy_s+2x_ty_t)+u_x(x_{ss}+y_{tt})+u_y(y_{ss}+y_{tt})\,.$$

Since f is holomorphic, the Cauchy-Riemann equations $x_s = y_t$ and $x_t = -y_s$ imply that $({y_s}^2+{y_t}^2) = ({x_s}^2+{x_t}^2)$ and that $(2x_sy_s+2x_ty_t) = 0$ and that x and y are each harmonic so that $(x_{ss}+x_{tt}) = 0$ and $(y_{ss}+y_{tt}) = 0$. So we are left with

$$v_{ss}+v_{tt} = (u_{xx}+u_{yy})({x_s}^2+{x_y}^2)\,.$$

Finally, since u is harmonic, we have $(u_{xx}+u_{tt}) = 0$ and hence $v_{ss}+v_{tt} = 0$.

5.25 Note: We shall now consider problems of the following kind: given an open set $U \subset \mathbf{C}$, find a harmonic function $u : U \to \mathbf{R}$ which satisfies some given condition on the boundary ∂U; this kind of problem is called a **boundary value problem**. We solved an easy boundary value problem in example 5.23, in which the open set was the strip $S = \{x + i\,y | 0 < y < 1\}$. The above theorem allows us to use a solution to one boundary value problem on a set U to obtain a solution to another problem on a set V by mapping the set U to the set V using a holomorphic map.

5.26 Example: The upper half-plane $H = \{x + i\,y | y > 0\}$ is a model for a large heat conducting plate. Find the steady state temperature $v(z)$ at each point in the plate if the temperature along the bottom edge is held at a° for $x > 0$ and b° for $x < 0$. Also, describe the isotherms.

Solution: Notice that we can map the strip $S = \{x + i\,y | 0 < y < 1\}$ (which appeared in the example 5.23) to the half-plane $H = \{x + i\,y | y > 0\}$ using the map $f(z) = e^{\pi z}$. The bottom edge of S is mapped to the positive x-axis, and the top edge of S is mapped to the negative x-axis. To map H back to S we use the inverse map $g(z) = \frac{1}{\pi} \log z$, where $\log z$ is the branch of the logarithm given by $\log z = \ln |z| + i\theta(z)$ where $0 \leq \theta(z) \leq \pi$.

From example 5.23, the map $u(z) = \mathrm{Im}\,(a\,i + (b-a)z)$ is harmonic in the strip S with $u = a$ when $y = 0$ and $u = b$ when $y = 1$. To solve our problem in H, we take $v = u \circ g$. To be explicit, we take

$$v(z) = \mathrm{Im}\left(a\,i + \frac{b-a}{\pi}\log z\right) = a + \frac{b-a}{\pi}\theta(z)\,,$$

where $0 \leq \theta(z) \leq \pi$. The isotherm $u = c$ is the ray $c = a + \frac{b-a}{\pi}\,\theta(z)$ or $\theta(z) = \frac{c-a}{b-a}\,\pi$.

5.27 Example: Find the steady state temperature $u(z)$ inside a circular plate modelled by the disc $U = D(0,1)$, given that the top half of the boundary is held at $a^\circ = 1^\circ$ and the bottom half is held at $b^\circ = 5^\circ$. In particular, find the temperature at the point $\frac{1}{2}i$. Also describe the isotherm $u = 2$.

Solution: The map $f_1(z) = \dfrac{z+1}{2}$ maps the disc $D(0,1)$ to the disc $D(\frac{1}{2}, \frac{1}{2})$, and it sends the top half of the boundary of the first disc to the top half of the boundary of the second. The map $f_2(z) = \dfrac{1}{z}$ maps the disc $D(\frac{1}{2}, \frac{1}{2})$ to the half-plane $H_1 = \{x + i\,y | x > 1\}$, and it maps the top half of the boundary of the disc to the bottom half $\{1 + i\,y | y < 0\}$ of the boundary of H_1. The map $f_3(z) = z - 1$ translates the half-plane H_1 to $H_0 = \{x + i\,y | x > 0\}$. Finally the map $f_4(z) = i\,z$ rotates H_0 to the half-plane $H = \{x + i\,y | y > 0\}$ sending the bottom half of the boundary of H_0 to the right half $\{x > 0\}$ of the boundary of H. So we can use our solution $v(z)$ from the previous example to obtain the solution $u = v \circ f_4 \circ f_3 \circ f_2 \circ f_1$. To be explicit, we have $f_2(f_1(z)) = \dfrac{2}{1+z}$ and $f_3(f_2(f_1(z))) = \left(\dfrac{2}{1+z} - 1\right) = \left(\dfrac{1-z}{1+z}\right)$ and $f_4(f_3(f_2(f_1(z)))) = i\left(\dfrac{1-z}{1+z}\right)$, so our solution is

$$u(z) = a + \frac{b-a}{\pi}\,\theta\left(i\,\frac{1-z}{1+z}\right) = 1 + \frac{4}{\pi}\,\theta\left(i\,\frac{1-z}{1+z}\right)\,,$$

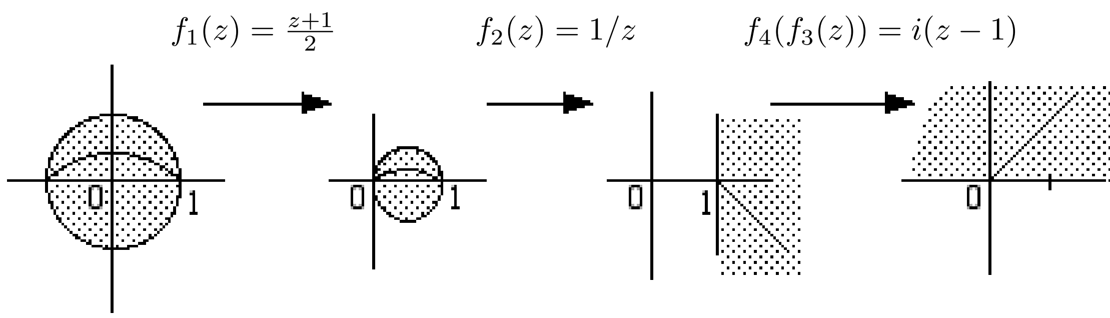
where $0 \leq \theta\big(i\,\frac{1-z}{1+z}\big) \leq \pi$. Since $\theta(i\,\frac{1-z}{1+z}) = \theta(\frac{1-z}{1+z}) + \frac{\pi}{2}$, we have

$$u(z) = 3 + \frac{4}{\pi}\,\theta\left(\frac{1-z}{1+z}\right)\,,$$

where $-\frac{\pi}{2} \le \theta\big(\frac{1-z}{1+z}\big) \le \frac{\pi}{2}$. In particular, the temperature at $\frac{1}{2}\,i$ is $u(\frac{1}{2}\,i) = 3 + \frac{4}{\pi}\,\theta\big(\frac{1-\frac{1}{2}i}{1+\frac{1}{2}i}\big) = 3 + \frac{4}{\pi}\,\theta\big(\frac{3-4i}{5}\big) = 3 + \frac{4}{\pi}\tan^{-1}\big(-\frac{4}{3}\big) \cong 1.82°$.

To find the isotherm $u = 2$, we recall that the corresponding isotherm $v = c = 2$ in example 5.26 was the ray $\theta(z) = \frac{c-a}{b-a}\,\pi = \frac{2-1}{5-1}\,\pi = \frac{\pi}{4}$. This ray is rotated by ${f_4}^{-1}(z) = -iz$ to the ray $\theta(z) = -\frac{\pi}{4}$, then translated by ${f_3}^{-1} = z+1$ to the ray $\theta(z-1) = -\frac{\pi}{4}$, this ray is the portion below the x-axis of the line whose nearest point to the origin is $\frac{1}{2}(1+i)$ and so it is mapped by ${f_2}^{-1}(z) = 1/z$ to the portion above the x-axis of the circle with diameter 0, $\frac{2}{1+i} = 1-i$, that is the circle $|z - \frac{1-i}{2}| = \frac{\sqrt{2}}{2}$, and finally this arc is translated and scaled by the map ${f_1}^{-1}(z) = 2z - 1$ to the portion above the x-axis of the circle $|z+i| = \sqrt{2}$. Thus the isotherm $u = 2$ is the arc $|z+i| = \sqrt{2}$, $z \in D(0,1)$.

We also remark that $\theta(\frac{1-z}{1+z}) = \mathrm{Im}\,\Big(\log(\frac{1-z}{1+z})\Big) = -2\,\mathrm{Im}\,(\tanh^{-1} z)$.



5.28 Example: Find the steady state temperature $u(z)$ in the plate shaped like the semi-infinite strip $U = \{x + i\,y|\,-1 < x < 1, y > 0\}$ given that the temperature along the bottom edge and the right edge is held at $a° = 10°$ and the temperature along the left edge of the boundary is held at $b° = 40°$. Also, find the temperature at $z = i$.

Solution: The map $f_1(z) = \frac{\pi}{2}z$ widens the strip U by a factor of $\frac{\pi}{2}$, and then the map $f_2(z) = \sin z$ sends the strip to the half plane $H = \{x + i\,y|y > 0\}$. The left edge of the boundary of U is mapped to the portion of the real axis with $x < -1$. Lastly, the map $f_3(z) = z + 1$ sends H to itself and it sends the portion of the real axis with $x < -1$ to the portion with $x < 0$. We can again use our solution $v(z)$ from example 5.26 to obtain the solution to this problem. We take $u = u \circ f_3 \circ f_2 \circ f_1$. To be explicit, we have $f_3(f_2(f_1(z))) = 1 + \sin\frac{\pi}{2}z$ and so

$$u(z) = a + \frac{b-a}{\pi}\,\theta\big(1 + \sin(\tfrac{\pi}{2}z)\big) = 10 + \frac{30}{\pi}\,\theta\big(1 + \sin(\tfrac{\pi}{2}z)\big)\,,$$

where $0 \le \theta\big(1 + \sin(\frac{\pi}{2}\,z)\big) \le \pi$. In particular, we have $u(i) = 10 + \frac{30}{\pi}\,\theta\big(1 + \sin(i\,\frac{\pi}{2})\big) = 10 + \frac{30}{\pi}\,\theta\big(1 + i\sinh\frac{\pi}{2}\big) = 10 + \frac{30}{\pi}\tan^{-1}\big(\sinh\frac{\pi}{2}\big) \cong 21.1°$

5.29 Example: Find the steady state temperature $v(z)$ at each point on a plate modelled by the half-plane $H = \{x + i\,y|y > 0\}$ given that the temperature along the boundary is held constant at $a°$ for $x > 1$, $b°$ for $-1 < x < 1$ and at $c°$ for $x < -1$.

Solution: We can use the fact that the sum of two harmonic maps will also be harmonic. We use the solution from example 5.26 to get one harmonic map v_1 in H with $v_1 = a$ along the portion of the x-axis with $x > 1$ and $v_1 = b$ along the portion with $x < 1$, and we get another harmonic map v_2 in H with $v_2 = 0$ along the portion of the x-axis with $x > 1$ and $v_2 = c - b$ along the portion with $x < -1$. Then we add them to get $v = v_1 + v_2$. To be explicit, $v_1(z) = a + \dfrac{b-a}{\pi}\,\theta(z-1)$ and $v_2(z) = \dfrac{c-b}{\pi}\,\theta(z+1)$ and so

$$v(z) = a + \frac{b-a}{\pi}\,\theta(z-1) + \frac{c-b}{\pi}\,\theta(z+1)\,,$$

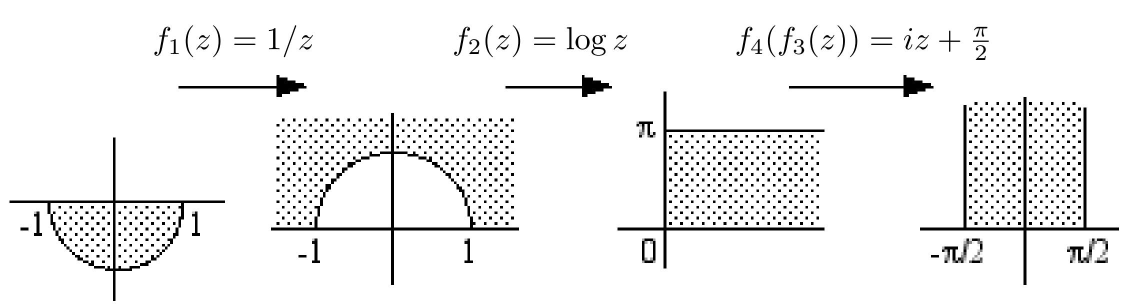
where $0 \le \theta(z-1), \theta(z+1) \le \pi$.

5.30 Example: Find the steady-state temperature $u(z)$ in the semi-circular plate modelled by $U = \{x + i\,y | x^2 + y^2 < 1, y < 0\}$ given that the temperature along the boundary is held constant at $a^\circ = 5^\circ$ when $y = 0$ and $x > 0$, and at $b^\circ = 10^\circ$ when $y = -\sqrt{1-x^2}$ and at $c^\circ = 20^\circ$ when $y = 0$ and $x < 0$. In particular, find the temperature at $z = -\frac{1}{2}\,i$.

Solution: The map $f_1 = \dfrac{1}{z}$ sends U to the region V above the x-axis and outside the unit circle $V = \{x + i\,y | x^2 + y^2 > 1, y > 0\}$. Then $f_2(z) = \log z$, the branch of the logarithm with $0 \le \theta \le \pi$ maps V to the semi-infinite strip $W = \{x + i\,y | x > 0, 0 < y < \pi\}$. We rotate the strip by 90° using $f_3(z) = i\,z$ then shift it to the right by $\frac{\pi}{2}$ using the map $f_4(z) = z + \frac{\pi}{2}$ (so that its base is centred at the origin), and then we use the map $f_5(z) = \sin z$ to map the strip to the half-plane $H = \{x + i\,y | y > 0\}$. The portions of the boundary which are to be held constant at a°, b° and c° are mapped to the portions of the x-axis with $x > 1$, $-1 < x < 1$ and $x < -1$ respectively, so we can use our solution $v(z)$ from the previous example. Our solution is $u = v \circ f_5 \circ \ldots \circ f_1$. To be explicit, we have $f_5(f_4(f_3(z))) = \sin(i\,z + \frac{\pi}{2}) = \cos(i\,z) = \cosh z$, and $f_5(f_4(f_3(f_2(z)))) = \cosh(\log z) = \dfrac{e^{\log z} + e^{-\log z}}{2} = \dfrac{z + \frac{1}{z}}{2}$, and so $(f_5 \circ \ldots \circ f_1)(z) = \dfrac{\frac{1}{z} + z}{2} = \dfrac{1+z^2}{2z}$. Our solution is

$$\begin{aligned} u(z) &= a + \frac{b-a}{\pi}\,\theta\left(\frac{1+z^2}{2z} - 1\right) + \frac{c-b}{\pi}\,\theta\left(\frac{1+z^2}{2z} + 1\right) \\ &= 5 + \frac{5}{\pi}\,\theta\left(\frac{1+z^2}{2z} - 1\right) + \frac{10}{\pi}\,\theta\left(\frac{1+z^2}{2z} + 1\right)\,. \end{aligned}$$

In particular, $u(-i/2) = 5 + \frac{5}{\pi}\theta\big(\frac{3/4}{-i} - 1\big) + \frac{10}{\pi}\,\theta\big(\frac{3/4}{-i} + 1\big) = 5 + \frac{5}{\pi}\,\theta\big(-1 + i\,\frac{3}{4}\big) + \frac{10}{\pi}\,\theta\big(1 + i\,\frac{3}{4}\big) = 5 + \frac{5}{\pi}\big(\pi - \tan^{-1}\frac{3}{4}\big) + \frac{10}{\pi}\tan^{-1}\frac{3}{4} = 10 + \frac{5}{\pi}\tan^{-1}\frac{3}{4} \cong 11.0^\circ$.

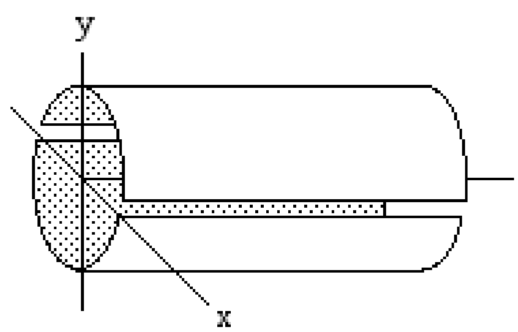


5.31 Note: All of the above examples can be re-worded so that they are asking us to find the electrostatic potential in a certain region given that the voltage along the boundary is held constant. If u is the electrostatic potential in a region, then the **electric field** E is defined by

$$E = -\nabla u\,.$$

If f is holomorphic and $u = \mathrm{Re}\,(f)$ and $v = \mathrm{Im}\,(f)$, then we have $\nabla u = u_x + i\,u_y = \overline{f_z}$ and $\nabla v = v_x + i\,v_y = -u_y + i\,u_x = i(u_x + i\,u_y) = i\,\nabla u = i\,\overline{f_z}$.

5.32 Example: Find the electostatic potential and the electric field at each point inside a long hollow metal cylinder, with unit radius, made up of two semi-cylindrical pieces separated by thin strips of insulating material, with one piece held at a potential of 1 Volt, and the other at 5 Volts. In particular, find the electrostatic potential and the electric field at points along the centre of the cylinder.



Solution: The cross-section of the cylinder is modelled by the unit disc $U = D(0,1)$. As in example 5.27, the electric potential is $u(z) = 3 + \frac{4}{\pi}\,\theta\left(\frac{1-z}{1+z}\right)$. Note that $u = \mathrm{Re}\,(f)$, where $f(z) = 3 - \frac{4}{\pi}\,i\,\log\left(\frac{1-z}{1+z}\right)$. The electric field is $E = -\nabla u = -\overline{f_z} = \overline{\frac{4}{\pi}\,i\,\frac{1+z}{1-z}\,\frac{-2}{(1+z)^2}} = \frac{8\,i}{\pi(1-\overline{z}^2)}$. In particular, we have $u(0) = 3$ and $E(0) = \frac{8}{\pi}\,i$.

5.33 Example: Find all solutions $v(z)$ to Laplace's equation in $\mathbf{C}^*$ such that $v(r\,e^{i\,\theta}) = f(r)$ for some function f (the solution will model the electrostatic potential at each point around a long charged rod).

Solution: The exponential function maps $\mathbf{C}$ onto $\mathbf{C}^*$. If $v(z)$ is harmonic in $\mathbf{C}^*$ then $u(z) = v(e^z)$ will be harmonic in $\mathbf{C}$. If v is of the form $v(r\,e^{i\,\theta}) = f(r)$ then we have $u(x + i\,y) = v(e^x e^{iy}) = f(e^x)$. Since u is independent of y, Laplace's equation becomes $u_{xx} = 0$, and the only solutions are of the form $u(x + i\,y) = ax + b = \mathrm{Re}\,(az + b)$ for some $a, b \in \mathbf{R}$. Thus we have $v(z) = u(\log z) = \mathrm{Re}\,(a \log z + b) = a \ln|z| + b$.

5.34 Example: Find the electrostatic potential $v(z)$ and the electric field $E(z)$ at each point inside a long grounded cylinder, of unit radius, which encloses a charged wire centred inside the cylinder.

Solution: We look for a harmonic map $v(z)$ defined on the punctured disc $U = D^*(0,1)$ with $v(z) = 0$ when $|z| = 1$. From the previous example, we can take $v(z) = a \ln|z|$. The constant a depends on the charge per unit length and on the choice of units. In fact

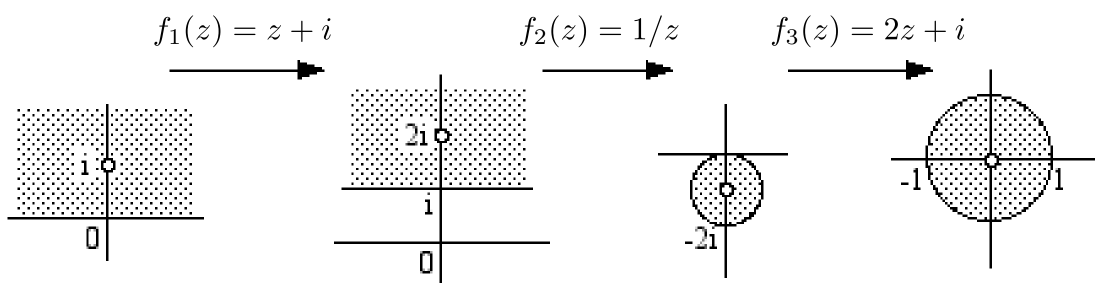
$$v(z) = -2kq \ln|z|\,,$$

where q is the charge on the rod in coulombs per meter and $k \cong 9 \times 10^9\ \frac{N\,m^2}{C^2}$. Since $v = \mathrm{Re}\,(f)$, where $f(z) = -2kq \log(z)$, we have $E(z) = -\overline{f_z} = 2kq/\overline{z}$.

5.35 Example: A charged wire at $x = 0, y = 1$ lies inside the region in space given by $y > 0$, and the boundary of the region is grounded. Find the potential $u(z)$ at each point in the region and around the wire.

Solution: Let U be the punctured half-plane $U = \{z|\text{Im}\, z > 0, z \neq i\}$. The map $f_1(z) = z + i$ maps U to the set $V = \{z|\text{Im}\, z > 1, z \neq 2i\}$, the map $f_2(z) = 1/z$ maps V to the punctured disc $W = D^*(-\frac{1}{2}i, \frac{1}{2})$, and the map $f_3(z) = 2z + i$ maps W to the punctured disc $D^*(0, 1)$. So our solution is $u = v \circ f_3 \circ f_2 \circ f_1$ where $v(z) = -2kq \ln |z|$ is the solution from the previous example. Check that

$$u(z) = -2kq \ln \left| \frac{z - i}{z + i} \right| .$$



5.36 Note: The velocity field F of a flow (of perfect fluid) and the velocity potential v are related (like the electric field and electric potential) by

$$F = -\nabla v .$$

5.37 Example: Find the velocity potential $v(z)$ of the constant flow with velocity field $F(x + i\,y) = c$ in the upper half plane $H = \{x + i\,y|y > 0\}$.

Solution: We must have $F = -\nabla v$ so we need $c = -(v_x + i\, v_y)$, that is $v_x = -c$ and $v_y = 0$. Since $v_y = 0$, v is independent of y, and since $v_x = -c$ we have

$$v = -c\,x = \text{Re}\,(-c\,z) .$$

We could add a constant to this solution.

5.38 Example: Use the previous example to find the velocity potential for the region $U = \{x + i\,y|x^2 + y^2 > 1, y > 0\}$ given that as $z \to \infty$ the flow tends to the constant flow $F = k$. Also, determine the speed of the flow near $z = i$, that is, at the top of the bump.

Solution: As in example 5.32, the map $f(z) = \cosh(\log z) = \frac{1}{2}(z + 1/z)$ sends the region U to the upper half-plane $H = \{x + i\,y|y > 0\}$. We use the potential v from the previous example, and we take $u(z) = v(f(z)) = \text{Re}\, g(z)$, where $g(z) = -\frac{c}{2}(z + 1/z)$. The velocity field is $F = -\overline{g_z} = \frac{c}{2}(1 - 1/\overline{z}^2)$. As $z \to \infty$ we have $F(z) \to c/2$ so we must take $c = 2k$. Thus our solution is

$$v(z) = \text{Re}\,\big(- k(z + z^{-1})\big) \quad , \quad F(z) = k(1 - 1/\overline{z}^2) .$$

We have $F(i) = 2k$, so the velocity at the top of the bump is twice the velocity at ∞.

Chapter 6. Integration

6.1 Definition: For a map $\alpha : [t_1, t_2] \subset \mathbf{R} \to \mathbf{C}$ we define the **integral** of α to be

$$\int_{t_1}^{t_2} \alpha(t)\, dt = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha(c_i)(s_i - t_{s-1}),$$

where the limit denotes a limit of Riemann sums; it is taken over all choices of s_i and c_i with $t_1 = s_0 < s_1 < \ldots < s_n = t_2$ and $s_{i-1} \le c_i \le s_i$ and $(s_i - s_{i-1}) \to 0$ as $n \to \infty$. We also define $\int_{t_2}^{t_1} \alpha(t)\, dt = -\int_{t_1}^{t_2} \alpha(t)\, dt$. If α is piecewise continuous and bounded, then the limit will exist and be finite, and if we write $\alpha(t) = x(t) + i\, y(t)$, then

$$\int_{t_1}^{t_2} \alpha(t)\, dt = \int_{t_1}^{t_2} x(t)\, dt \; + i \int_{t_1}^{t_2} y(t)\, dt\,.$$

6.2 Example: Let $\alpha(t) = e^{i\,t}$ for $t \in \mathbf{R}$. For $\theta \in \mathbf{R}$, find $\int_0^\theta \alpha(t)\, dt$.

Solution: $\int_0^\theta \alpha(t)\, dt = \int_0^\theta \cos t + i \sin t\, dt = \int_0^\theta \cos t\, dt + i \int_0^\theta \sin t\, dt = \Big[\sin t\Big]_0^\theta + i\Big[-\cos t\Big]_0^\theta$ $= \sin\theta + i\,(1 - \cos\theta) = i - i\, e^{i\,\theta}$. Note that as θ varies, this traces out a circle.

6.3 Theorem:

a) i) $\int_{t_1}^{t_2} c\,\alpha(t)\, dt = c \int_{t_1}^{t_2} \alpha(t)\, dt$ *ii)* $\int_{t_1}^{t_2} \alpha(t) + \beta(t)\; dt = \int_{t_1}^{t_2} \alpha(t)\, dt + \int_{t_1}^{t_2} \beta(t)\, dt$

b) $\int_{t_1}^{t_2} \alpha(t)\, dt + \int_{t_2}^{t_3} \alpha(t)\, dt = \int_{t_1}^{t_3} \alpha(t)\, dt$

c) (The Fundamental Theorem of Calculus) $\int_{t_1}^{t_2} \alpha'(t)\, dt = \alpha(t_2) - \alpha(t_1)$

d) (The Estimation Theorem) $\left|\int_{t_1}^{t_2} \alpha(t)\, dt\right| \le \int_{t_1}^{t_2} |\alpha(t)|\, dt$

e) (Change of Parameter) Let $t = t(s)$ be $\mathcal{C}^1$. Then $\int_{s_1}^{s_2} \alpha(t(s))t'(s)\, ds = \int_{t(s_1)}^{t(s_2)} \alpha(t)\, dt$

Proof: All the parts except part d) follow immediately from the corresponding results for real valued functions. For example, to prove the Fundamental Theorem of Calculus, write $\alpha(t) = x(t) = i\, y(t)$ and we have $\int_{t_1}^{t_2} \alpha'(t)\, dt = \int_{t_1}^{t_2} x'(t) + i\, y'(t)\; dt = \int_{t_1}^{t_2} x'(t)\, dt \; + i \int_{t_1}^{t_2} y'(t)\, dt = \big(x(t_2) - x(t_1)\big) + i\,\big(y(t_2) - y(t_1)\big) = \alpha(t_2) - \alpha(t_1)$.

Part d) is easier to prove using Riemann sums. For any $\epsilon > 0$ we can choose a partition $t_1 = s_0 < \ldots < s_n = t_2$ and the points c_i so that

$$\Big|\int_{t_1}^{t_2} \alpha(t)\, dt\Big| + \epsilon \le \Big|\sum_{i=1}^{n} \alpha(c_i)(s_i - s_{i-1})\Big| \le \sum_{i=n}^{n} \big|\alpha(c_i)\big|(s_i - s_{i-1}) \le \int_{t_1}^{t_2} |\alpha(t)|\, dt - \epsilon\,.$$

Since ϵ is arbitrary, we must have $\Big|\displaystyle\int_{t_1}^{t_2}\alpha(t)\,dt\Big| \leq \int_{t_1}^{t_2}\int_{t_1}^{t_2}|\alpha(t)|\,dt$.

Alternatively, we can use the corresponding result for real valued functions as follows. Write $\int_{t_1}^{t_2} f(t)\,dt$ in polar coordinates as $\int_{t_1}^{t_2} f(t)\,dt = \big|\int_{t_1}^{t_2} f(t)\,dt\big| e^{i\,\theta}$. Then

$$\left|\int_{t_1}^{t_2} f(t)\,dt\right| = e^{-i\,\theta}\int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} e^{-i\,\theta} f(t)\,dt = \left|\mathrm{Re}\int_{t_1}^{t_2}\big(e^{-i\theta}f(t)\big)\,dt\right|\,,$$

where the last equality holds since for $r \geq 0$ we have $r = |\mathrm{Re}\,(r)|$, and

$$\left|\mathrm{Re}\int_{t_1}^{t_2}\big(e^{-i\theta}f(t)\big)\,dt\right| = \left|\int_{t_1}^{t_2}\mathrm{Re}\,\big(e^{-i\theta}f(t)\big)\,dt\right| \leq \int_{t_1}^{t_2}\big|\mathrm{Re}\,\big(e^{i\,\theta}f(t)\big)\big|\;dt \leq \int_{t_1}^{t_2}\big|f(t)\big|\,dt\,,$$

since $|\mathrm{Re}\,(e^{i\,\theta}f(t))| \leq |e^{i\theta}f(t)| = |f(t)|$.

6.4 Definition: Let $\alpha : [t_1, t_2] \subset \mathbf{R} \to \mathbf{C}$. If α is continuous and α' is piecewise continuous and bounded, then we call α a **path**. If, in addition, we have $\alpha(t_1) = \alpha(t_2)$ then α is called a **loop**. The **arclength** of a path $\alpha : [t_1, t_2] \to \mathbf{C}$ is

$$L(\alpha) = \int_{t_1}^{t_2}|\alpha'(t)|\,dt\,.$$

Since α' is piecewise continuous and bounded, the arclength exists and is finite.

6.5 Example: Find the arclength of the path $\alpha(t) = t^2 + i\,t^3$, $0 \leq t \leq 1$.

Solution: $L(\alpha) = \displaystyle\int_0^1|\alpha'(t)|\,dt = \int_0^1|2t + i\,3t^2|\,dt = \int_0^1\sqrt{4t^2+9t^4}\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt$ $= \displaystyle\int_4^{13}\frac{1}{18}\sqrt{u}\,du = \left[\frac{1}{27}u\sqrt{u}\right]_4^{13} = \frac{1}{27}\big(13\sqrt{13}-8\big)$.

6.6 Definition: Given a path $\alpha : [t_1, t_2] \subset \mathbf{R} \to \mathbf{C}$ and a piecewise continuous bounded map $f : \mathrm{Image}\,(\alpha) \subset \mathbf{C} \to \mathbf{C}$ we define

$$\int_\alpha f = \int_\alpha f(z)\,dz := \int_{t_1}^{t_2} f(\alpha(t))\alpha'(t)\,dt\,.$$

This kind of integral is called a **path integral**.

6.7 Example: Let $c \in \mathbf{C}$ and let α be the line segment $\alpha(t) = a+(b-a)\,t$, $0 \leq t \leq 1$. Then $\displaystyle\int_\alpha c\,dz = \int_0^1 c\,\alpha'(t)\,dt = \int_0^1 c\,(b-a)\,dt = \int_0^1\mathrm{Re}\,\big(c\,(b-a)\big)\,dt + i\int_0^1\mathrm{Im}\,\big(c\,(b-a)\big)\,dt =$ $\mathrm{Re}\,\big(c\,(b-a)\big) + i\,\mathrm{Im}\,\big(c\,(b-a)\big) = c\,(b-a)$.

6.8 Note: The complex path integral is related to real path integrals in the following way. Write $z = \alpha(t) = x(t) + i\,y(t)$ and $f(z) = u(z) + i\,v(z)$ with $x, y, u, v \in \mathbf{R}$. Then $\displaystyle\int_\alpha f(z)\,dz = \int_{t_1}^{t_2} f(\alpha(t))\alpha'(t)\,dt = \int_{t_1}^{t_2}\big(u(\alpha(t)) + i\,v(\alpha(t))\big)\big(x'(t) + i\,y'(t)\big)\,dt =$ $\displaystyle\int_{t_1}^{t_2} u(\alpha(t))x'(t) - v(\alpha(t))y'(t)\;dt + i\int_{t_1}^{t_2} v(\alpha(t))x'(t) + u(\alpha(t))y'(t)\;dt = \int_\alpha(u\,dx - v\,dy) +$ $\displaystyle i\int_\alpha(v\,dx + u\,dy)$. This can easily be remembered by defining $dz = dx + i\,dy$ and then writing $\displaystyle\int_\alpha f(z)\,dz = \int_\alpha(u + i\,v)(dx + i\,dy) = \int_\alpha(u\,dx - v\,dy) + i\int_\alpha(v\,dx + u\,dy)$. In a similar way we could define the path integral $\displaystyle\int_\alpha f(z)\,d\overline{z}$, where $d\overline{z} = dx - i\,dy$

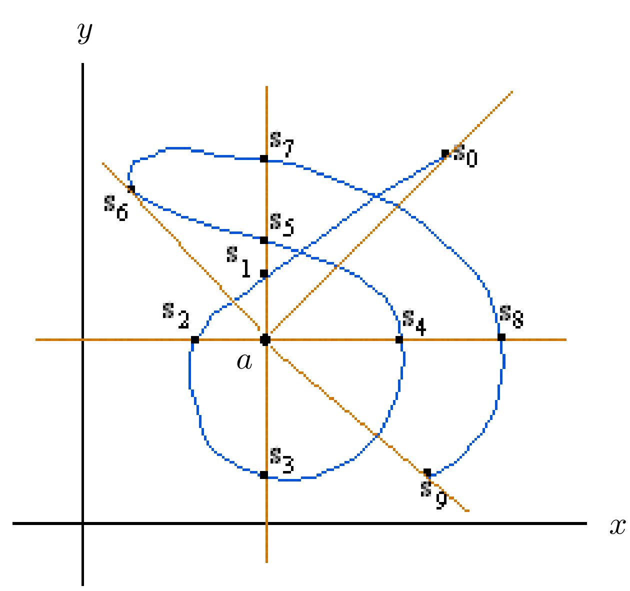
6.9 Definition: For a path $\alpha : [t_1, t_2] \to \mathbf{C}$ and a point $a \in \mathbf{C} \setminus \text{Image}(\alpha)$, we define the **winding number** $\eta(\alpha, a)$ of α about a as follows. We write α as $\alpha(t) = a + r(t)e^{i\,\theta(t)}$ where $r(t) = |\alpha(t) - a|$ and $\theta(t)$ is chosen continuously with $0 \le \theta(t_1) < 2\pi$ (it can be shown that the map $\theta(t)$ is uniquely determined), and then we set

$$\eta(\alpha, a) = \frac{\theta(t_2) - \theta(t_1)}{2\pi}\,.$$

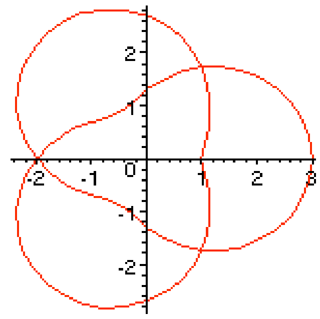
If α is a loop then we have $\alpha(t_1) = \alpha(t_2)$ and so $e^{i\,\theta(t_1)} = e^{i\,\theta(t_2)}$ and hence $\theta(t_2) - \theta(t_1)$ will be a multiple of 2π. Thus $\eta(\alpha, a) \in \mathbf{Z}$.

6.10 Example: The loop which goes k times around the circle $|z - a| = r$ can be given parametrically by $\alpha(t) = a + r(t)\,e^{i\,\theta(t)}$ with $r(t) = r$, $\theta(t) = t$ and $0 \le t \le 2\pi k$. We have $\eta(\alpha, a) = \dfrac{\theta(2\pi k) - \theta(0)}{2\pi} = \dfrac{2\pi k - 0}{2\pi} = k$.

6.11 Example: It is not hard to find the winding number $\eta(\alpha, a)$ from a picture of the path α. For example, for α and a as shown below, we can choose values $t = s_i$ (as shown) for which the points $\alpha(s_i)$ lie on the horizontal and vertical lines through a. From the picture, we can see that $\theta(s_0) \cong \frac{\pi}{4}$, and then $\theta(t)$ increases (since we move counterclockwise around a) with $\theta(s_1) = \frac{\pi}{4}$, $\theta(s_2) = \pi$, $\theta(s_3) = \frac{3\pi}{2}$, $\theta(s_4) = 2\pi$ and $\theta(s_5) = \frac{5\pi}{2}$, and then $\theta(t)$ reaches its maximum at $\theta(s_6) \cong \frac{11\pi}{4}$ and begins to decrease (since we now begin moving clockwise around a) with $\theta(s_7) = \frac{5\pi}{2}$, $\theta(s_8) = 2\pi$) and finally $\theta(s_9) \cong \frac{7\pi}{4}$. Thus we have $\eta(\alpha, a) = \dfrac{\theta(s_9) - \theta(s_0)}{2\pi} \cong \dfrac{\frac{7\pi}{4} - \frac{\pi}{4}}{2\pi} = \dfrac{3}{4}$.



6.12 Example: If α is the pretzel curve $\alpha(t) = r(t)e^{i\,\theta(t)}$, where $r(t) = (2 + \cos 3t)$ and $\theta(t) = 2t$ with $0 \le t \le 2\pi$ (as shown below), then the winding number of α about 0 is $\eta(\alpha, 0) = \frac{\theta(2\pi) - \theta(0)}{2\pi} = \frac{4\pi - 0}{2\pi} = 2$. The winding number about other points is hard to compute from the given equation of α, but is easy to find using a sketch of the curve. For example we have $\eta(\alpha, 2) = \eta(\alpha, 2e^{i\,2\pi/3}) = \eta(\alpha, 2e^{i\,4\pi/3}) = 1$ and $\eta(\alpha, 4) = 0$.



6.13 Theorem: *For the path $\alpha(t)=a+r(t)e^{i\,\theta(t)}$, with $r(t)>0$ and $t_1\le t\le t_2$, we have*

$$\int_\alpha \frac{dz}{z-a}=\Big[\ln r(t)\Big]_{t_1}^{t_2}+i\,\Big[\theta(t)\Big]_{t_1}^{t_2}=\ln r(t_2)-\ln r(t_1)+i\,\big(\theta(t_2)-\theta(t_1)\big)\,.$$

In particular, when α is a loop we have $r(t_1)=r(t_2)$ so

$$\eta(\alpha,a)=\frac{1}{2\pi i}\int_\alpha \frac{dz}{z-a}$$

Proof: $\displaystyle\int_\alpha \frac{dz}{z-a}=\int_{t_1}^{t_2}\frac{r'e^{i\,\theta}+i\,r\,\theta' e^{i\,\theta}}{r\,e^{i\,\theta}}\,dt=\int_{t_1}^{t_2}\frac{r'}{r}\,dt+i\int_{t_1}^{t_2}\theta'\,dt=\Big[\ln r(t)\Big]_{t_1}^{t_2}+i\,\Big[\theta(t)\Big]_{t_1}^{t_2}.$

6.14 Definition: For a path $\alpha:[t_1,t_2]\to\mathbf{C}$ we define the path $\alpha^{-1}:[t_1,t_2]\to\mathbf{C}$ by

$$\alpha^{-1}(t)=\alpha(t_1+t_2-t)\,,$$

so that α and α^{-1} have the same image, but α^{-1} traces the image in the oposite direction. Also, for a path $\alpha:[t_1,t_2]\to\mathbf{C}$ and a path $\beta:[t_2,t_3]\to\mathbf{C}$ with $\alpha(t_2)=\beta(t_2)$, we define $\alpha*\beta:[t_1,t_3]\to\mathbf{C}$ by

$$\alpha*\beta(t)=\begin{cases}\alpha(t)\ \text{for}\ t_1\le t\le t_2\\ \beta(t)\ \text{for}\ t_2\le t\le t_3\end{cases}$$

This path first traces out the image of α then traces out the image of β.

6.15 Theorem:
a) i) $\displaystyle\int_\alpha c\,f=c\int_\alpha f$ *ii)* $\displaystyle\int_\alpha (f+g)=\int_\alpha f+\int_\alpha g$

b) $\displaystyle\int_{\alpha*\beta} f=\int_\alpha f+\int_\beta f$

c) (The Fundamental Theorem of Calculus) Let $\alpha:[t_1,t_2]\to U\subset\mathbf{C}$ be a path in U, and let f be holomorphic in U. Then $\displaystyle\int_\alpha f'(z)\,dz=f(\alpha(t_2))-f(\alpha(t_1))$

d) (The Estimation Theorem) Let $L=L(\alpha)$ be the length of the path $\alpha:[t_1,t_2]\to\mathbf{C}$ and let $M=\max\limits_{z=\alpha(t)}\big|f(z)\big|$. Then $\displaystyle\Big|\int_\alpha f(z)\,dz\Big|\le\int_{t_1}^{t_2}\big|f(\alpha(t))\alpha'(t)\big|\,dt\le ML.$

e) (Change of Parameter) Let $t=t(s)$ be monotonic and $\mathcal{C}^1$. Let $\beta(s)=\alpha(t(s))$. Then $\displaystyle\int_\beta f(z)\,dz=\pm\int_\alpha f(z)\,dz$. *We use $+$ when $t(s_1)<t(s_2)$ and we use $-$ when $t(s_1)>t(s_2)$. In particular* $\displaystyle\int_\alpha f=-\int_{\alpha^{-1}} f.$

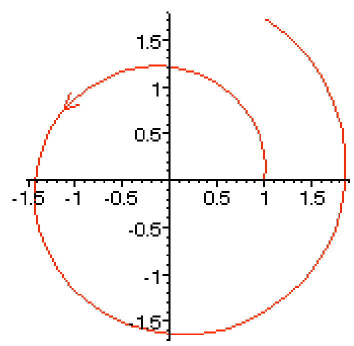
Proof: These all follow immediately from the theorem 6.3. For example to prove part e), note that since $\beta(s)=\alpha(t(s))$ we have $\beta'(s)=\alpha'(t(s))t'(s)$ and so

$$\int_\beta f = \int_{s_1}^{s_2} f(\beta(s))\beta'(s)\,ds = \int_{s_1}^{s_2} f(\alpha(t(s)))\alpha'(t(s))t'(s)\,ds = \int_{t(s_1)}^{t(s_2)} f(\alpha(t))\alpha'(t)\,dt$$

by part e) of the previous theorem. If $t(s_1) < t(s_2)$ then the integral on the right is equal to $\int_\alpha f$, but if $t(s_1) > t(s_2)$ then we will have $\int_\alpha f = \int_{t(s_2)}^{t(s_1)} f(\alpha(t))\alpha'(t)\,dt$.

6.16 Example: Let α be the path $\alpha(t) = \left(1+\frac{3}{7}t\right)e^{i\,\pi t}$ for $0\le t\le \frac{7}{3}$. Find $\int_\alpha z^n\,dz$.

Solution: We have $\alpha(0)=1$ and $\alpha\left(\frac{7}{3}\right) = 2e^{i\,\pi/3}$. If $n\neq 1$ then by the FTC we have $\int_\alpha z^n\,dz = \left[\frac{1}{n+1}\,z^{n+1}\right]_1^{2e^{i\pi/3}} = \frac{1}{n+1}\left(2^{n+1}e^{i\,(n+1)\pi/3}-1\right)$. If $n=-1$ then by theorem 6.13 we have $\int_\alpha \frac{dz}{z} = \Big[\ln(r(t))\Big]_0^{7/3} + i\,\Big[\theta(t)\Big]_0^{7/3} = \ln 2 + i\,\frac{7\pi}{3}$, where $r(t) = 1+\frac{3}{7}t$ and $\theta(t)=\pi\,t$.



6.17 Note: Let $U\subset \mathbf{C}$ be open, and let α be a path which runs counterclockwise around the boundary of a closed set $E\subset U$. Recall that Green's theorem (for real path integrals) states that if $u,v:U\subset\mathbf{C}\to\mathbf{R}$ are $\mathcal{C}^1$ maps, then

$$\int_\alpha u\,dx + v\,dy = \int\int_E (v_x-u_y)\,dx\,dy\,.$$

Let $f:U\to\mathbf{C}$ be holomorphic, and let $u=\mathrm{Re}\,(f)$ and $v=\mathrm{Im}\,(f)$. If we suppose that u and v are $\mathcal{C}^1$, then Green's Theorem and the Cauchy-Riemann equations imply that

$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_\alpha (u+i\,v)(dx+i\,dy)\\ &= \int_\alpha u\,dx - v\,dy + i\int_\alpha v\,dx + u\,dy\\ &= \int\int_E(-v_x-u_y)dx\,dy + i\int\int_E(u_x-v_y)dx\,dy\\ &= 0\,.\end{aligned}$$

We shall now prove a series of theorems which generalize this result (which is known as **Cauchy's Theorem**) and which do not require the assumption that u and v are $\mathcal{C}^1$. Indeed, we shall be able to show that every holomorphic map is $\mathcal{C}^\infty$.

6.18 Definition: Let $f, g : U \subset \mathbf{C} \to \mathbf{C}$. If $g'(z) = f(z)$ for all $z \in U$ then we write $g = \int f$ and we say that g is an **antiderivative** of f in U.

6.19 Example: Since complex functions have the same derivative formulas as real functions, they have the same antiderivative formulas also. For example, we can use Integration by Parts to get $\int z\, e^z\, dz = z\, e^z - \int e^z\, dz = (z-1)\, e^z + c$.

6.20 Example: Let $U_\alpha = \left\{ r\, e^{i\,\theta} | r > 0, \alpha < \theta < \alpha + 2\pi \right\}$, and let $f(z) = 1/z$. Then the antiderivatives of f in U_α are the maps of the form $g(z) = \log z + c$ where $\log z = |z| + i\,\theta(z)$ with $\alpha < \theta(z) < \alpha + 2\pi$. However, $f(z)$ does not have an antiderivative in $\mathbf{C}^*$ because none of the maps $g(z)$ can be extended continuously to $\mathbf{C}^*$.

6.21 Theorem: *If α is a loop in U and if f has an antiderivative in U then* $\displaystyle\int_\alpha f = 0$.

Proof: Say $\alpha : [t_1, t_2] \to U$ and say $g' = f$ in U. Then by the Fundamental Theorem of Calculus, we have $\displaystyle\int_\alpha f = \int_\alpha g' = g(\alpha(t_2)) - g(\alpha(t_1)) = 0$ since $\alpha(t_1) = \alpha(t_2)$.

6.22 Example: Let α be any loop in $\mathbf{C}^*$ and let $f(z) = \sum\limits_{n=-k}^{l} a_n\, z^n$ where k and l are positive integers and $a_n \in \mathbf{C}$. Show that $\displaystyle\int_\alpha f(z)\, dz = 2\pi\, i\, \eta(\alpha, 0)\, a_{-1}$.

Solution: For $n \neq -1$, the map z^n has an antiderivative in $\mathbf{C}^*$, namely $\frac{1}{n+1}\, z^{n+1}$, so for $n \neq -1$ we have $\displaystyle\int_\alpha z^n\, dz = 0$. And so

$$\int_\alpha \sum_{n=-k}^{l} a_n\, z^n\, dz = \sum_{n=-k}^{l} a_n \int_\alpha z^n\, dz = a_{-1} \int_\alpha z^{-1}\, dz = a_{-1} 2\pi\, i\, \eta(\alpha, 0)\,.$$

6.23 Theorem: *(Cauchy's Theorem in a Triangle) Suppose that $f : U \subset \mathbf{C} \to \mathbf{C}$ is holomorphic in U. Let Δ be a closed solid triangle in U and let α be a loop around the boundary of the triangle. Then* $\displaystyle\int_\alpha f = 0$.

Proof: Let $I = \left| \int_\alpha f \right|$ and set $I_0 = I$, $\Delta_0 = \Delta$, $\alpha_0 = \alpha$ and $L_0 = L(\alpha)$. Divide Δ into four congruent triangles Δ_{01}, Δ_{02}, Δ_{03} and Δ_{04}, let $\alpha_{01}, \ldots, \alpha_{04}$ be loops around these triangles, and let $I_{0j} = \left| \int_{\alpha_{0j}} f \right|$ for $j = 1, 2, 3, 4$. Choose k so that I_{0k} is the largest of these, and then set $I_1 = I_{0k}$, $\Delta_1 = \Delta_{0k}$, $\alpha_1 = \alpha_{0k}$ and $L_1 = L(\alpha_1)$. Since the triangles Δ_{0j} are half as big as Δ_0 we have $L_0 = 2L_1$. Also, since $I_1 \geq I_{0j}$ for all j, we have

$$I_0 = \left| \int_{\alpha_0} f \right| = \left| \sum_{j=1}^{4} \int_{\alpha_{0j}} f \right| \leq \sum_{j=1}^{4} \left| \int_{\alpha_{0j}} f \right| = \sum_{j=1}^{n} I_{0j} \leq 4I_1\,.$$

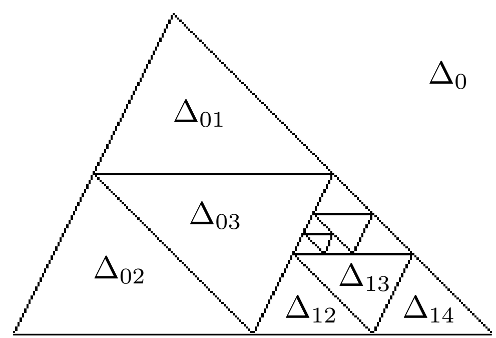
Next we subdivide Δ_1 into four congruent triangles and repeat the procedure. In this way we obtain a sequence of congruent triangles $\Delta_0 \supset \Delta_1 \supset \ldots$ with a loop α_k around each triangle, and we have $I_0 \leq 4I_1 \leq 4^2 I_2 \leq \ldots$ and $L_0 = 2L_1 = 2^2 L_2 = \ldots$, where $I_k = \left| \int_{\alpha_k} f \right|$ and $L_k = L(\alpha_k)$. Let a be the (unique) point which lies in all of the Δ_k. (The *proof* that the point a exists uses the fact that the triangles Δ_k are compact).

Now let $\epsilon > 0$. Since f is holomorphic at a, we can choose δ so that for $|z-a| < \delta$ we have $\left|f'(a) - \frac{f(z)-f(a)}{z-a}\right| < \epsilon$ so that $\left|f(z) - \big(f(a) + f'(a)(z-a)\big)\right| < \epsilon|z-a|$. Choose N so that for $n \geq N$ we have $\Delta_n \subset D(a,\delta)$, and note that for all $z \in \Delta_n$ we have $|z-a| \leq L_n$. So for $z \in \Delta_n$ we have $|z-a| < \delta$ which implies $\left|f(z) - \big(f(a) + f'(a)(z-a)\big)\right| < \epsilon|z-a| < \epsilon L_n$. Since $f(a) + f'(a)(z-a)$ has an antiderivative, namely $f(a)\,z + f'(a)(\frac{1}{2}z^2 - az)$, we know that $\int_{\alpha_n} f(a) + f'(a)(z-a)\;dz = 0$. Using the Estimation Theorem we obtain

$$I_n = \left|\int_{\alpha_n} f(z)\,dz\right| = \left|\int_{\alpha_n} f(z) - \big(f(a) + f'(a)(z-a)\big)\,dz\right| \leq M_n L_n \leq \epsilon\, {L_n}^2 = \epsilon\,\frac{{L_0}^2}{4^n}\,,$$

where $M_n = \max\limits_{z=\alpha(t)} \big(f(z) - (f(a) + f'(a)(z-a)\big)$. Thus $I_0 \leq 4^n I_n \leq \epsilon {L_0}^2$. Since ϵ was arbitrary, we must have $I_0 = 0$.

In the picture below, $\Delta_1 = \Delta_{04}$, $\Delta_2 = \Delta_{11}$ and $\Delta_3 = \Delta_{22}$.



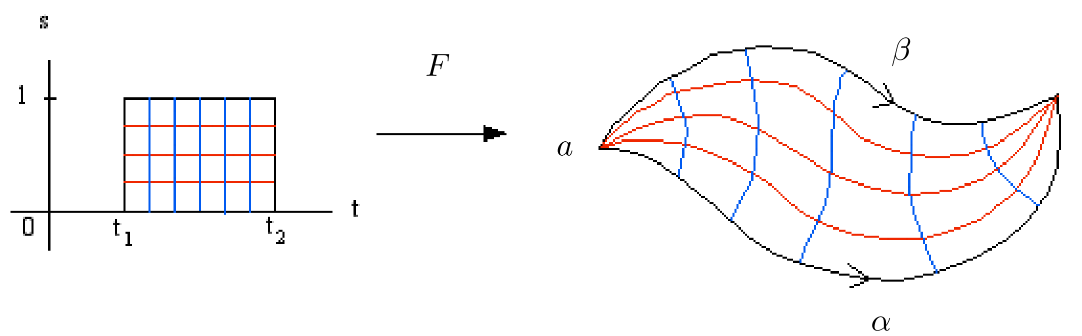
6.24 Theorem: *(Cauchy's Theorem in a Convex Region) Suppose that $f : U \subset \mathbf{C} \to \mathbf{C}$ is holomorphic in U, where U is open and convex. Then f has an antiderivative in U. Consequently, $\displaystyle\int_\alpha f = 0$ for all loops α in U.*

Proof: Choose any point $a \in U$. For each $z \in U$ set $g(z) = \displaystyle\int_\alpha f$ where α is the line segment from a to z (that is $\alpha(t) = a + (z-a)\,t$, $0 \leq t \leq 1$). We claim that $g'(z) = f(z)$ for all $z \in U$. Indeed, given $h \in \mathbf{C}$ (small enough so that $z + h \in U$) we let β be the line segment from z to $z+h$ and we let γ be the line segment from $z+h$ to a, so by the definition of g we have $g(z+h) = \int_{\gamma^{-1}} f = -\int_\gamma f$, and by Cauchy's Theorem in a Triangle we have $\int_\alpha f + \int_\beta f + \int_\gamma f = 0$, and so

$$\begin{aligned}
\left|f(z) - \frac{g(z+h)-g(z)}{h}\right| &= \left|f(z) + \frac{1}{h}\Big(\int_\alpha f(w)\,dw + \int_\gamma f(w)\,dw\Big)\right| \\
&= \left|f(z) - \frac{1}{h}\int_\beta f(w)\,dw\right| \\
&= \left|\frac{1}{h}\int_\beta f(z)\,dw - \frac{1}{h}\int_\beta f(w)\,dw\right| \\
&= \left|\frac{1}{h}\int_\beta f(z) - f(w)\;dw\right| \\
&\leq \max_{w=\beta(t)} \big|f(z) - f(w)\big|\,.
\end{aligned}$$

As $h \to 0$ we have $w = \beta(t) \to z$ and so $|f(z) - f(w)| \to 0$, since f is continuous.

6.25 Definition: Let $\alpha, \beta : [t_1, t_2] \to U \subset \mathbf{C}$ be paths with $\alpha(t_1) = \beta(t_1) = a$ and $\alpha(t_2) = \beta(t_2) = b$. A **path-homotopy** (or **deformation of paths**) from α to β in U is a continuous map $F : [t_1, t_2] \times [0, 1] \to U$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all t, and also $F(t_1, s) = a$ and $F(t_2, s) = b$ for all s. If such a homotopy exists, then we say that α is (path)-**homotopic** to β in U and we write $\alpha \cong \beta$. Note that for each fixed s, $F_s(t) := F(t, s)$ is a continuous curve from a to b.



6.26 Example: In a convex set U we can find a path-homotopy between any two paths $\alpha, \beta : [t_1, t_2] \to U$ with $\alpha(t_1) = \beta(t_1)$ and $\alpha(t_2) = \beta(t_2)$. Indeed, we can take $F(t, s) = \alpha(t) + s\big(\beta(t) - \alpha(t)\big)$.

6.27 Example: In $\mathbf{C}^*$, the maps $\alpha, \beta : [0, \pi] \to \mathbf{C}^*$ given by $\alpha(t) = e^{i\,t}$ and $\beta(t) = e^{-i\,t}$ are not homotopic. This follows from the following version of Cauchy's theorem.

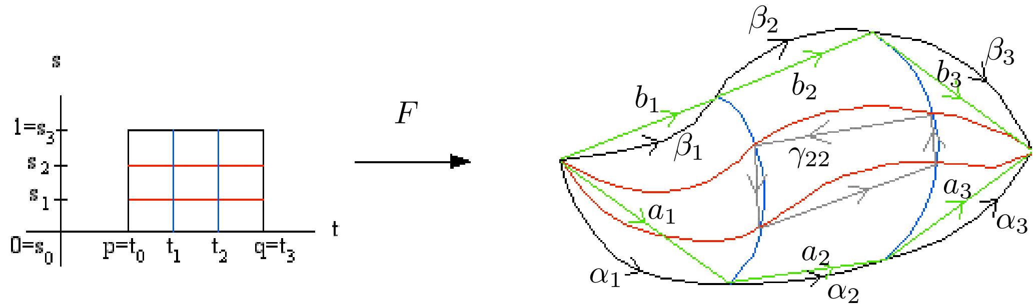
6.28 Theorem: *(Cauchy's Theorem for Paths) If f is holomorphic in U and if $\alpha \cong \beta$ in U then* $\displaystyle\int_\alpha f = \int_\beta f$.

Proof: Say $\alpha, \beta : [p, q] \to U$. Choose an path-homotopy $F : [p, q] \times [0, 1] \to U$ from α to β in U. Choose partitions $p = t_0 < t_1 < \ldots t_k = q$ and $0 = s_1 < s_2 < \ldots s_l = 1$ with the property that if $\square_{ij} = \{F(t, s) | t_{i-1} \le t \le t_i, s_{j-1} \le s \le s_j\}$ then each $\square_{ij}$ is contained in a convex set (a disc if you like) which is contained in U. (To *prove* that such partitions can be found, you must use the fact that $[p, q] \times [0, 1]$ is compact). For each i and j, let $\alpha_i : [t_{i-1}, t_i] \to U$ be a segment of the path α, let $\beta_i : [t_{i-1}, t_i] \to U$ be a segment of β, let a_i be the line segment from $\alpha(t_{i-1})$ to $\alpha(t_i)$, let b_i be the line segment from $\beta(t_{i-1})$ to $\beta(t_i)$, and let γ_{ij} be the loop around the polygon with vertices at $F(t_{i-1}, s_{j-1})$, $F(t_i, s_{j-1})$, $F(t_i, s_j)$ and $F(t_{i-1}, s_j)$. Then by Cauchy's Theorem for convex sets, we have

$$\int_{\alpha_i} f = \int_{a_i} f \quad , \quad \int_{\beta_i} f = \int_{b_i} f \quad \text{and} \quad \int_{\gamma_i} f = 0\ .$$

When we consider all of the paths ${a_i}^{-1}$, b_i and γ_{ij}, every line segment occurs twice, once in each direction, and so the path integrals all cancel with each other to give

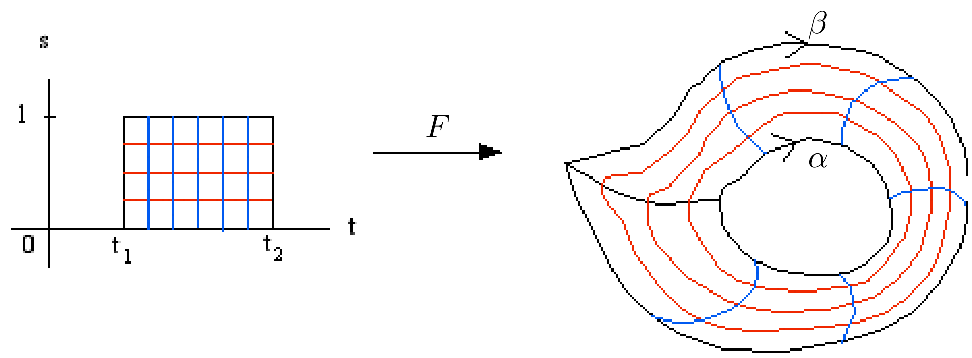
$$\begin{aligned} 0 &= \sum_i \int_{b_i} f - \sum_i \int_{a_i} f + \sum_{i,j} \int_{\gamma_{ij}} f \\ &= \sum_i \int_{\beta_i} f - \sum_i \int_{\alpha_i} f \\ &= \int_\beta f - \int_\alpha f \end{aligned}$$



6.29 Example: Let $\alpha, \beta : [0, \pi] \to \mathbf{C}^*$ be given by $\alpha(t) = e^{i\,t}$ and $\beta(t) = e^{-i\,t}$. Show that α and β are not homotopic in $\mathbf{C}^*$.

Solution: Let $f(z) = 1/z$. Then f is holomorphic in $\mathbf{C}^*$ and we have $\int_\alpha f = i\,\pi$ and $\int_\beta f = -i\,\pi$. Since $\int_\alpha f \neq \int_\beta f$ we know that α is not homotopic to β.

6.30 Definition: Let $\alpha, \beta : [t_1, t_2] \to U \subset \mathbf{C}$ be loops in U. A **loop-homotopy** (or **deformation of loops**) from α to β in U is a continuous map $F : [t_1, t_2] \times [0, 1] \to U$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all t and $F(t_1, s) = F(t_2, s)$ for all s. If such a homotopy exists then we say that α is (loop)-**homotopic** to β in U and we write $\alpha \sim \beta$.



6.31 Example: In a convex set U, any two loops are homotopic. Indeed, given loops $\alpha, \beta : [t_1, t_2] \to U$ we can take $F(t, s) = \alpha(t) + s\,\big(\beta(t) - \alpha(t)\big)$.

6.32 Theorem: *(Cauchy's Theorem for Loops) If f is holomorphic in U and if $\alpha \sim \beta$ then $\int_\alpha f = \int_\beta f$.*

Proof: The proof is the same as the proof of Cauchy's theorem for paths.

6.33 Example: Let $\alpha, \beta : [t_1, t_2] \to \mathbf{C}^*$ be loops. Show that if $\eta(\alpha, 0) \neq \eta(\beta, 0)$ then α and β are not homotopic in $\mathbf{C}^*$.

Solution: Let $f(z) = 1/z$. Then f is holomorphic and we have $\int_\alpha f = 2\pi\,i\;\eta(\alpha, 0)$ and $\int_\beta f = 2\pi\,i\;\eta(\beta, 0)$, so if $\eta(\alpha, 0) \neq \eta(\beta, 0)$ then α and β cannot be homotopic in $\mathbf{C}^*$.

6.34 Definition: A set $U \subset \mathbf{C}$ is called **simply connected** if any two loops $\alpha, \beta : [t_1, t_2] \to U$ are homotopic in U. Roughly speaking, a connected set will be simply connected if it doesn't have any holes in it.

6.35 Example: Any convex set is simply connected, but $\mathbf{C}^*$ is not.

6.36 Theorem: *(Cauchy's Theorem in a Simply Connected Region) If U is a simply connected open set and if f is holomorphic in U then $\int_\alpha f = 0$ for any loop α in U.*

Proof: Since U is simply connected, any loop $\alpha : [t_1, t_2] \to U$ will be homotopic to the constant loop e given by $e(t) = \alpha(t_1)$ for all t, so $\int_\alpha f = \int_e f = \int_{t_1}^{t_2} f(\alpha(a))e'(t)\,dt = 0$ since $e'(t) = 0$.

6.37 Theorem: *(Cauchy's Integral Formulas) Let U be a convex open set, let f be holomorphic in U and let α be a loop in U. Then for any point $a \in U \setminus \text{Image}\,(\alpha)$ we have*
a) $2\pi i\ \eta(\alpha, a)\, f(a) = \int_\alpha \frac{f(z)}{z-a}\,dz$.
b) All the derivatives $f^{(n)}(a)$ exist, and $2\pi i\ \eta(\alpha, a)\, f^{(n)}(a) = n! \int_\alpha \frac{f(z)}{(z-a)^{n+1}}\,dz$.

Proof: First we prove part a). For any $\epsilon > 0$, let α_ϵ denote the path $\alpha_\epsilon(t) = a + \epsilon(\alpha(t) - a)$. Note that $\alpha \sim \alpha_\epsilon$ in the set $U \setminus \{a\}$, indeed a homotopy is given by $F(t,s) = \alpha(t) + s(\alpha_\epsilon(t) - \alpha(t))$. Also note that the map $\frac{f(z)-f(a)}{z-a}$ is holomorphic in $U \setminus \{a\}$. So we have

$$\begin{aligned}\left|\int_\alpha \frac{f(z)}{z-a}\,dz - 2\pi i\,\eta(\alpha,a)\,f(a)\right| &= \left|\int_\alpha \frac{f(z)}{z-a}\,dz - \int_\alpha \frac{f(a)}{z-a}\,dz\right| \\ &= \left|\int_\alpha \frac{f(z)-f(a)}{z-a}\,dz\right| \\ &= \left|\int_{\alpha_\epsilon} \frac{f(z)-f(a)}{z-a}\,dz\right| \\ &\le M_\epsilon\, L(\alpha_\epsilon)\,,\end{aligned}$$

where $M_\epsilon = \max_{z=\alpha_\epsilon(t)} \left|\frac{f(z)-f(a)}{z-a}\right|$. As $\epsilon \to 0$ we have $\frac{f(z)-f(a)}{z-a} \to f'(a)$ so $M_\epsilon \to |f'(a)|$, and also $L(\alpha_\epsilon) = \epsilon\, L(\alpha) \to 0$

Next we prove part b) using induction. Suppose that

$$2\pi i\ \eta(\alpha,a)\, f^{(n)}(a) = n! \int_\alpha \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Then we have

$$\begin{aligned}2\pi i\ \eta(\alpha,a)\left(\frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}\right) &= \frac{n!}{h}\int_\alpha \frac{f(z)}{(z-(a+h))^{n+1}} - \frac{f(z)}{(z-a)^{n+1}}\,dz \\ &= \frac{n!}{h}\int_\alpha f(z)\left(\frac{1}{(z-(a+h))^{n+1}} - \frac{1}{(z-a)^{n+1}}\right)dz \\ &= \frac{(n+1)!}{h}\int_\alpha f(z)\int_\lambda \frac{1}{(z-w)^{n+2}}\,dw\,dz\,,\end{aligned}$$

where λ is the line segment from a to $a+h$. So we have

$$\begin{aligned} L &:= \left| 2\pi i\, \eta(\alpha,a)\, \frac{f^{(n)}(a+h)-f^{(n)}(a)}{h} - (n+1)! \int_\alpha \frac{f(z)}{(z-a)^{n+2}}\, dz \right| \\ &= \left| \frac{(n+1)!}{h} \left(\int_\alpha f(z) \int_\lambda \frac{1}{(z-w)^{n+2}}\, dw\; dz - h \int_\alpha \frac{f(z)}{(z-a)^{n+2}} \right) dz \right| \\ &= \left| \frac{(n+1)!}{h} \int_\alpha f(z) \left(\int_\lambda \frac{1}{(z-w)^{n+2}}\, dw - h\, \frac{1}{(z-a)^{n+2}} \right) dz \right| \\ &= \left| \frac{(n+1)!}{h} \int_\alpha f(z) \int_\lambda \frac{1}{(z-w)^{n+2}} - \frac{1}{(z-a)^{n+2}}\, dw\; dz \right| \\ &= \left| \frac{(n+2)!}{h} \int_\alpha f(z) \int_\lambda \int_\tau \frac{1}{(z-u)^{n+3}}\, du\; dw\; dz \right|, \end{aligned}$$

where τ is the line segment from a to w. Choose $r>0$ so that $D(a,2r)\subset U\setminus \text{Image}\,(\alpha)$, and let $|h|<r$. For $w\in \text{Image}\,(\lambda)$ and $u\in\text{Image}\,(\tau)$ we have w between a and $a+h$, and u between a and w, so $u\in D(a,r)$, and so we have $|z-u|\geq r$ hence $\dfrac{1}{|z-u|}\leq \dfrac{1}{r}$. By the estimation theorem, $L\leq \dfrac{(n+2)!}{|h|}\, L(\alpha) \max\limits_{z=\alpha(t)} |f(z)|\,|h|\,|h|\, \dfrac{1}{r^{n+3}} \to 0$ as $|h|\to 0$.

6.38 Example: Let $\alpha(t)=2\,e^{i\,t}$ for $0\leq t\leq 2\pi$ and let $f(z)=\dfrac{z+1}{z^2+1}$. Find $\displaystyle\int_\alpha f(z)\,dz$.

Solution: We shall find the integral in several ways. First, we shall use partial fractions. To write $\dfrac{z+1}{z^2+1}=\dfrac{z+1}{(z+i)(z-i)}$ in the form $\dfrac{A}{z+i}+\dfrac{B}{z-i}$ we need $A(z-i)+B(z+i)=z+1$ for all z. Setting $z=i$ gives $B(2i)=i+1$ so $B=\dfrac{i+1}{2i}=\dfrac{i-1}{2}$. Setting $z=-i$ gives $A(-2i)=-i+1$ so $A=\dfrac{-i+1}{-2i}=\dfrac{1+i}{2}$. And so we have $\displaystyle\int_\alpha \frac{z+1}{z^2+1}\,dz=\frac{1+i}{2}\int_\alpha\frac{dz}{z+i}+\frac{1-i}{2}\int_\alpha\frac{dz}{z-i}=\frac{1+i}{2}\,2\pi i\;\eta(\alpha,-i)+\frac{1-i}{2}\,2\pi i\;\eta(\alpha,i)=\frac{1+i}{2}\,2\pi\, i+\frac{1-i}{2}\,2\pi\, i=\pi\,(i-1)+\pi\,(i+1)=2\pi\, i$.

Now we shall find the integral again by immitating the proof of Cauchy's integral formula. Notice that f is holomorphic except at $z=\pm i$. Let α_1 be the loop around the top half of the circle, and let α_2 be the loop around the bottom half, to be explicit, we take $\alpha_1(t)=\begin{cases} 2\,e^{i\,t} & \text{for } 0\leq t\leq\pi \\ \frac{2}{\pi}t-3 & \text{for } \pi\leq t\leq 2\pi\end{cases}$ and $\alpha_2(t)=\begin{cases} 1-\frac{2}{\pi}\,t & \text{for } 0\leq t\leq\pi \\ 2\,e^{i\,\pi} & \text{for } \pi\leq t\leq 2\pi\end{cases}$ and then we will have $\int_\alpha f=\int_{\alpha_1} f+\int_{\alpha_2} f$. Next we deform the paths α_1 and α_2 into the circular paths σ_1 and σ_2, where $\sigma_1(t)=i+r\,e^{i\,t}$ and $\sigma_2(t)=-i+r\,e^{i\,t}$ for $0\leq t\leq 2\pi$, where $0<r<1$. We have $\displaystyle\int_{\alpha_1} f=\int_{\sigma_1} f=\int_0^{2\pi} f(\sigma_1(t))\,\sigma_1{}'(t)\,dt=\int_0^{2\pi}\frac{1+i+r\,e^{i\,t}}{-1+2i\,re^{it}+r^2e^{i\,2t}+1}\,i\,re^{it}\,dt \to \int_0^{2\pi}\frac{1+i}{2}\,dt=\pi(1-i)$ as $r\to 0$, and we have $\displaystyle\int_{\alpha_2} f=\int_{\sigma_2} f=\int_0^{2\pi} f(\sigma_2(t))\,\sigma_2{}'(t)\,dt=\int_0^{2\pi}\frac{1-i+r\,e^{it}}{-2i\,re^{it}+r^2e^{i\,2t}}\,i\,re^{it}\,dt\to\int_0^{2\pi}\frac{1-i}{-2}\,dt=\pi\,(i-1)$ as $r\to 0$. So $\displaystyle\int_\alpha f=2\pi\, i$.

Finally, we shall compute the integral a third time using Cauchy's formula. Taking α_1 and α_2 as above, we have $\displaystyle\int_{\alpha_1} f = \int_{\alpha_1} \frac{(z+1)/(z+i)}{z-i}\,dz = \int_{\alpha_1} \frac{F(z)}{z-i}\,dz = 2\pi i\,F(i) = 2\pi i\,\frac{i+1}{2i} = \pi(i+1)$, where $F(z) = (z+1)/(z+i)$, and $\displaystyle\int_{\alpha_2} f = \int_{\alpha_2} \frac{(z+1)/(z-i)}{z+i}\,dz = \int_{\alpha_2} \frac{G(z)}{z+i}\,dz = 2\pi i\,G(-i) = 2\pi i\,\frac{1-i}{-2i} = \pi\,(i-1)$, where $G(z) = (z+1)/(z-i)$. Again we obtain $\displaystyle\int_\alpha f = 2\pi i$.

6.39 Example: Let $\alpha(t) = 2\,e^{i\,t}$ for $0 \le t \le 2\pi$, and let $f(z) = \dfrac{e^z}{z^2-1}$. Find $\int_\alpha f$.

Solution: Of the three methods we used above, only the third works here. Notice that f is holomorphic except at $z = \pm 1$. Let α_1 be the loop around the right half of the circle, and let α_2 be the loop around the left half, so we have $\int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_2} f$. Deform α_1 and α_2 to the circles σ_1 and σ_2 with $\sigma_1(t) = 1 + re^{it}$ and $\sigma_2(t) = -1 + re^{it}$. Then we have $\displaystyle\int_{\alpha_1} f = \int_{\alpha_1} \frac{e^z/(z+1)}{z-1}\,dz = \int_{\alpha_1} \frac{F(z)}{z-1}\,dz = 2\pi i\,F(1) = 2\pi i\,\frac{e}{2} = i\,\pi\,e$, and $\displaystyle\int_{\alpha_2} f = \int_{\alpha_2} \frac{e^z/(z-1)}{z+1}\,dz = \int_{\alpha_2} \frac{G(z)}{z+1}\,dz = 2\pi i\,G(-1) = 2\pi i\frac{e^{-1}}{-2} = -i\pi e^{-1}$. So the integral of f over α is equal to $i\,\pi\left(e - \dfrac{i}{e}\right)$.

6.40 Example: Let $\alpha(t) = 2\,e^{it}$ for $0 \le t \le 2\pi$ and let $f(z) = \dfrac{z+1}{z^3(z-1)^2}$. Find $\displaystyle\int_\alpha f$.

Solution: We shall solve this integral using two methods. First we use partial fractions. To write f in the form $\dfrac{z+1}{z^3(z-1)^2} = \dfrac{A}{z} + \dfrac{B}{z^2} + \dfrac{C}{z^3} + \dfrac{D}{z-1} + \dfrac{E}{(z-1)^2}$ we need to have $Az^2(z-1)^2 + Bz(z-1)^2 + C(z-1)^2 + Dz^3(z-1) + Ez^3 = z+1$ for all z. Equating coefficients gives fives 5 equations: $A + D = 0$, $-2A + B - D + E = 0$, $A - 2B + C = 0$, $B - 2C = 1$ and $C = 1$. Solving these gives $A = 5$, $B = 3$, $C = 1$, $D = -5$ and $E = 2$. So $\displaystyle\int_\alpha f = \int_\alpha \frac{5}{z} + \frac{3}{z^2} + \frac{1}{z^3} - \frac{5}{z-1} + \frac{2}{(z-1)^2}\,dz = 2\pi i\big(5\eta(\alpha,0) - 5\eta(\alpha,1)\big) = 2\pi i(5-5) = 0.$

Now we compute the integral again using Cauchy's formulas. Notice that f is holomorphic except at $z = 0, 1$. Let α_1 be the loop around the portion of the circle which lies to the right of the line $y - \frac{1}{2}$ and let α_0 be the loop around the portion to the left of $y = \frac{1}{2}$, so that $\int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_0} f$. We have $\displaystyle\int_{\alpha_0} f = \int_{\alpha_0} \frac{(z+1)/z^3}{(z-1)^2}\,dz = \int_{\alpha_0} \frac{F(z)}{z^3}\,dz = \frac{2\pi i}{2!}\,F''(0)$. From $F(z) = \dfrac{z+1}{(z-1)^3}$, we calculate $F'(z) = \dfrac{-z-3}{(z-1)^3}$ and $F''(z) = \dfrac{2z+10}{(z-1)4}$ to get $F''(0) = 10$, so we have $\displaystyle\int_{\alpha_1} f = 10\pi i$. Also, $\displaystyle\int_{\alpha_1} f = \int_{\alpha_1} \frac{(z+1)/z^3}{(z-1)^2}\,dz = \int_{\alpha_1} \frac{G(z)}{(z-1)^2}\,dz = \frac{2\pi i}{1!}G'(1)$. From $G(z) = \dfrac{z+1}{z^3}$ we find $G'(z) = \dfrac{-2z-3}{z^4}$ to get $G'(1) = -5$, so we have $\displaystyle\int_{\alpha_2} f = -10\pi i$. Again we obtain $\displaystyle\int_\alpha f = 0$.

6.41 Theorem: *(Liouville's Theorem) If* $f : \mathbf{C} \to \mathbf{C}$ *is holomorphic and bounded, then* f *is constant.*

Proof: Suppose that f is holomorphic in $\mathbf{C}$ with $|f(z)| \leq M$ for all z. Let a and b be any two distinct points in $\mathbf{C}$. Let $\alpha(t) = a + r|b-a|e^{it}$ for $0 \leq t \leq 2\pi$, where $r > 1$. Then

$$\begin{aligned} \big|f(a) - f(b)\big| &= \left| \frac{1}{2\pi\, i} \int_\alpha \frac{f(z)}{z-a} - \frac{f(z)}{z-b}\; dz \right| \\ &= \frac{1}{2\pi} \left| \int_\alpha f(z) \frac{a-b}{(z-a)(z-b)}\; dz \right| \\ &\leq \frac{1}{2\pi} 2\pi\, r|b-a|\, M\, |b-a| \frac{1}{r|b-a|} \frac{1}{(r-1)|b-a|} \\ &= \frac{M}{r-1} \to 0 \;\text{ as } r \to \infty\, . \end{aligned}$$

6.42 Theorem: *(The Fundamental Theorem of Algebra) Every non-constant polynomial has a root in* $\mathbf{C}$.

Proof: Suppose that p is a non-constant polynomial with no roots. Since p is a non-constant polynomial, we have $p(z) \to \infty$ as $z \to \infty$, and so we can choose R large enough that when $|z| \geq R$ we have $|p(z)| \geq 1$ and so $1/|p(z)| \leq 1$. Note that since p has no roots, $1/p$ is holomorphic in $\mathbf{C}$. In particular, $1/p$ is continuous in $\overline{D}(0,R)$ and so it attains its maximum value. Since $1/p$ is bounded in $\overline{D}(0,R)$ and $|1/p| \leq 1$ outside $D(0,R)$, we know that $1/p$ is bounded in $\mathbf{C}$. By Liouville's Theorem, $1/p$ must be a constant. But this would imply that p is constant.

Chapter 7. Power Series

7.1 Definition: A **sequence** of complex numbers is a function $\alpha:\{k,k+1,k+2\ldots\}\to\mathbf{C}$. We usually write $a_n=\alpha(n)$ and $\{a_n|n\geq k\}=\alpha$ or simply $\{a_n\}=\alpha$. We say that the sequence $\{a_n\}$ **converges** to $a\in\mathbf{C}$ and write

$$\lim_{n\to\infty}a_n=a\quad\text{or}\quad a_n\to a$$

if for all $\epsilon>0$ there exists $N\in\mathbf{Z}$ such that $n\geq N\Rightarrow a_n\in D(a,\epsilon)$. If the sequence converges to some $a\in\mathbf{C}$, then we say it **converges**, otherwise we say it **diverges**. We say that the sequence $\{a_n\}$ **diverges** to ∞, and write

$$\lim_{n\to\infty}a_n=\infty\quad\text{or}\quad a_n\to\infty$$

if for all $R>0$ there exists $N\in\mathbf{Z}$ such that $n\geq N\Rightarrow a_n\notin\overline{D}(0,R)$.

7.2 Example: If $a_n=1/n$ then $a_n\to 0$. If $b_n=2+\left(\frac{1}{2}(1+i)\right)^n$ then $b_n\to 2$. If $c_n=(1+i)^n$ then $c_n\to\infty$. If $d_n=i^n$ then $\{d_n\}$ diverges.

7.3 Theorem: *Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers, and let $c\in\mathbf{C}$.*
a) Write $a_n=x_n+i\,y_n$ and $a=x+i\,y$. Then $a_n\to a$ if and only if ($x_n\to x$ and $y_n\to y$).
b) If $a_n\to a$ and $b_n\to b$ then
- *i) $(c\,a_n)\to c\,a$*
- *ii) $(a_n\pm b_n)\to a\pm b$*
- *iii) $(a_nb_n)\to ab$*
- *iv) $(a_n/b_n)\to a/b$, provided that $b\neq 0$ (and hence $b_n\neq 0$ for large n)*
- *v) $|a_n|\to|a|$*

All parts (suitably modified) hold for sequences in $\mathbf{R}^n$ except parts a)iii) and iv).

Proof: We shall only show how to prove parts a) and b)iii) (the proofs of the other parts are similar).

To prove part a), suppose first that $a_n\to a$. Note that $(x_n-x)=\mathrm{Re}\,(a_n-a)$ so $|x_n-x|\leq|a_n-a|$. So given $\epsilon>0$ we choose $N\in\mathbf{Z}$ so that $n\geq N\Rightarrow|a_n-a|<\epsilon$, and then for $n\geq N$ we have $|x_n-x|\leq|a_n-a|<\epsilon$. This shows that $x_n\to x$. Similarly, we can show that $y_n\to y$. Next we suppose that $x_n\to x$ and that $y_n\to y$. By the triangle inequality we have $|a_n-a|\leq|x_n-x|+|y_n-y|$. So given $\epsilon>0$ we choose $N\in\mathbf{Z}$ so that $n\geq N\Rightarrow\left(|x_n-x|<\frac{1}{2}\epsilon\text{ and }|y_n-y|<\frac{1}{2}\epsilon\right)$. Then for $n\geq N$ we will have $|a_n-a|\leq|x_n-x|+|y_n-y|<\epsilon$. This shows that $a_n\to a$.

We shall now use part a), together with known results about sequences of *real* numbers, to prove part b)iii). We write $a_n=x_n+i\,y_n$, $a=x+i\,y$, $b_n=u_n+i\,v_n$ and $b=u+i\,v$. We suppose that $a_n\to a$ and $b_n\to b$ so that from part a) we have $x_n\to x$, $y_n\to y$, $u_n\to u$ and $v_n\to v$. We have $a_nb_n=(x_n+i\,y_n)(u_n+i\,v_n)=(x_nu_n-y_nv_n)+i\,(x_nv_n+y_nu_n)$. From our knowlege of sequences of *real* numbers, we know that $(x_nu_n-y_nv_n)\to xu-yv$ and that $(x_nv_n+y_nu_n)\to xv-yu$, and so using part a) again, we see that $a_nb_n=(x_nu_n-y_nv_n)+i\,(x_nv_n+y_nu_n)\to(xu-yv)+i\,(xv+yu)=ab$.

7.4 Definition: We write $\sum\limits_{n=0}^{\infty} a_n$ to denote the sequence $\{s_n\}$ where $s_n = \sum\limits_{i=0}^{n} a_i$. This kind of sequence is called a **series**, and the finite sums s_n are called the **partial sums**. We say the series $\sum a_n$ **converges** or **diverges** according to whether the sequence $\{s_n\}$ converges or diverges. We also write $\sum\limits_{n=0}^{\infty} a_n$ to denote the limit of $\{s_n\}$, if it exists, and we call the limit the **sum** of the series. If $s_n \to s$ then we write $\sum\limits_{n=0}^{\infty} a_n = s$. The series $\sum a_n$ is said to **converge absolutely** if the series $\sum |a_n|$ converges.

7.5 Theorem: *a) i)* $\sum c\, a_n = c \sum a_n$ *ii)* $\sum (a_n + b_n) = \sum a_n + \sum b_n$
b) If $\sum a_n$ *converges then* $|a_n| \to 0$.
c) If $\sum |a_n|$ *converges then* $\sum a_n$ *converges and* $|\sum a_n| \leq \sum |a_n|$.
d) (The Ratio Test)

i) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum |a_n|$ *converges.*

ii)If $\exists N \in \mathbf{Z}$ *s.t.* $n \geq N \Rightarrow \left|\dfrac{a_{n+1}}{a_n}\right| \geq 1$ *then* $|a_n| \not\to 0$ *and so* $\sum a_n$ *diverges.*

e) (The Root Test)

i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$ *then* $\sum |a_n|$ *converges and so* $\sum a_n$ *converges, too.*

ii) If $\exists N \in \mathbf{Z}$ *s.t.* $n \geq N \Rightarrow \sqrt[n]{|a_n|} \geq 1$ *then* $|a_n| \not\to 0$ *and so* $\sum a_n$ *diverges.*

Proof: We shall only prove the ratio test here. Suppose first that $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = p < 1$. Choose r with $p < r < 1$. Choose N such that for $n \geq N$ we have $\left|\dfrac{a_{n+1}}{a_n}\right| \leq r$. Then we have $|a_{N+1}| \leq r|a_N|$, $a_{N+2} \leq r|a_{N+1}| \leq r^2|a_N|$, $|a_{N+3}| \leq r|a_{N+2}| \leq r^3|a_N|$ and so on. Hence $\sum\limits_{n=0}^{\infty} |a_n| \leq |a_0| + \cdots + |a_{N-1}| + |a_N|\,(1 + r + r^2 + r^3 + \cdots)$ so it converges (by the comparison test for series of positive real numbers).

On the other hand, if we suppose that there exists $N \in \mathbf{Z}$ such that for $n \geq N$ we have $\left|\dfrac{a_{n+1}}{a_n}\right| \geq 1$ the we have $|a_N| \leq |a_{N+1}| \leq |a_{N+2}| \leq \cdots$ and so $|a_n| \not\to 0$.

7.6 Example: The sum $\sum\limits_{n=0}^{\infty} \dfrac{1}{(n+i)^2}$ converges by part c) since for $n \geq 2$ we have $|n+i| \geq n-1$ so $\left|\dfrac{1}{(n+i)^2}\right| \leq \dfrac{1}{(n-1)^2}$, and we know that $\sum \dfrac{1}{(n-1)^2}$ converges.

7.7 Definition: A **power series centred at** $a \in \mathbf{C}$ is a series of the form $\sum\limits_{n=0}^{\infty} c_n(z-a)^n$, where $c_n \in \mathbf{C}$. A power series is a series for each value of $z \in \mathbf{C}$. It will converge for certain values of z and diverge for others.

7.8 Example: The geometric series $\sum\limits_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$ is a power series centred at a=0. Its partial sums are given by $s_n = \sum\limits_{i=0}^{n} z^n = \dfrac{1 - z^{n+1}}{1-z}$. For $|z| < 1$ we have $z^n \to 0$ as $n \to \infty$ and so $s_n \to \dfrac{1}{1-z}$ hence $\sum\limits_{n=0}^{n} z^n = \dfrac{1}{1-z}$. On the other hand, for $|z| \geq 1$ we have $|z^n| \geq 1$ for all n so $|z^n| \not\to 0$ and hence $\sum\limits_{n=0}^{\infty} z^n$ diverges.

7.9 Theorem: *Let* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *be a power series.*

a) There exists a number R *with* $0 \le R \le \infty$*, called the* **radius of convergence** *of the power series, such that*

i) if $|z-a| < R$ *then* $\sum c_n(z-a)^n$ *converges absolutely.*

ii) if $|z-a| > R$ *then* $|c_n(z-a)^n| \not\to 0$ *and so* $\sum c_n(z-a)^n$ *diverges.*

b) The power series $\sum n\, c_n(z-a)^{n-1}$ *has the same radius of convergence.*

c) If $R > 0$ *then the function* f *defined by* $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ *for* $z \in D(a,R)$ *is holomorphic with* $f'(z) = \sum_{n=1}^{\infty} n\, c_n(z-a)^{n-1}$ *and* $\int f = c + \sum_{n=0}^{\infty} \frac{1}{n+1} c_n(z-a)^{n+1}$.

d) The function $f(z)$ *as above has derivatives of all orders and* $c_n = \dfrac{f^{(n)}(a)}{n!}$*, so we have* $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$.

e) If $\sum b_n(z-a)^n = \sum c_n(z-a)^n$ *for all* $z \in D(a,R)$ *then we have* $b_n = c_n$ *for all* n.

Proof: We shall give the proof in the case that $a = 0$.

To prove part a), we shall show that if $\sum_{n=0}^{\infty} c_n w^n$ converges, where $w \in \mathbf{C}$ then $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely for all z with $|z| < |w|$. So we suppose that $\sum c_n w^n$ converges and that $|z| < |w|$. Since $\sum c_n w^n$ converges, we know that $|c_n w^n| \to 0$ as $n \to \infty$ and so we can choose $M > 0$ so that $M \ge |c_n w^n|$ for all n. Then we have $|c_n z^n| = \left|c_n w^n \dfrac{z^n}{w^n}\right| = |c_n w^n| \left|\dfrac{z^n}{w^n}\right| \le M \left|\dfrac{z}{w}\right|^n$. Since $\left|\dfrac{z}{w}\right| < 1$, the series $\sum_{n=0}^{\infty} M \left|\dfrac{z}{w}\right|$ converges and hence the series $\sum_{n=0}^{\infty} |c_n z^n|$ converges too (by the comparison test for series of positive real terms). The radius of convergence is $R = \max\big\{|w| \big| w \in \mathbf{C}, \sum c_n w^n \text{ converges}\big\}$. If $R = \infty$ then the series converges for all z.

Next we prove part b). Let R be the radius of convergence of the series $\sum c_n z^n$ and let S be the radius of convergence of the series $\sum n\, c_n z^{n-1}$. First we show that $R \ge S$. If $S \ne 0$ then let z be any point with $|z| < S$. Then by part a), the series $\sum |n\, c_n z^{n-1}|$ converges, and so $\sum |c_n z^{n-1}| = \sum \frac{1}{n} |n\, c_n z^{n-1}|$ also converges by comparison, and hence $\sum |c_n z^n| = |z| \sum |c_n z^{n-1}|$ also converges. This implies that $R \ge |z|$. Since z was arbitrary, we have $R \ge S$.

It is a bit harder to show that $R \le S$. If $R \ne 0$ then let z be any point with $0 < |z| < R$. Choose $\rho > 0$ with $|z| < \rho < R$. We have $|n\, c_n z^{n-1}| = \dfrac{n}{|z|}\big(|z|/\rho\big)^n |c_n \rho^n|$. The series (of positive real terms) $\sum n\big(|z|/\rho\big)^n$ converges by the Ratio Test, so we know that $n\big(|z|/\rho\big)^n \to 0$ and hence we can choose $M > 0$ so that $M \ge n\big(|z|/\rho\big)^n$ for all n. Then we have $|n\, c_n z^{n-1}| \le \dfrac{M}{|z|} |c_n \rho^n|$. Since $\rho < R$ we know that the series $\sum |c_n \rho^n|$ converges, so the series $\sum \frac{M}{|z|} |c_n \rho^n| = \frac{M}{|z|} \sum |c_n \rho^n|$ also converges, and hence the series $\sum |n\, c_n z^{n-1}|$ also converges by comparison. Thus $S \ge |z|$, and since z was arbitrary, $S \ge R$.

Now we prove part c). Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and let $g(z) = \sum_{n=1}^{\infty} n\, c_n z^{n-1}$ for all $z \in D(0,R)$. We claim that $f'(z) = g(z)$. Given $z \in D(0,R)$ choose $\rho > 0$ with $|z| < \rho < R$.

Then for $|w| < \rho$ we have

$$\begin{aligned}
\left|\frac{f(w)-f(z)}{w-z} - g(z)\right| &= \left|\sum_{n=2}^{\infty} c_n\Big(\frac{w^n - z^n}{w-z} - n\,z^{n-1}\Big)\right| \\
&= \left|\sum_{n=2}^{\infty} c_n\big(w^{n-1} + w^{n-2}z + \cdots + w\,z^{n-2} + z^{n-1} - n\,z^{n-1}\big)\right| \\
&= \left|\sum_{n=2}^{\infty} c_n(w-z)\big(w^{n-2} + 2\,w^{n-3}z + +3\,w^{n-4}z^2 + \cdots + (n-1)z^{n-2}\big)\right| \\
&\leq \sum_{n=2}^{\infty} |c_n||w-z|\big(|w|^{n-2} + 2\,|w|^{n-3}z + \cdots + (n-1)|z|^{n-2}\big) \\
&\leq \sum_{n=2}^{\infty} |c_n||w-z|\rho^{n-2}\big(1 + 2 + \cdots + (n-1)\big) \\
&= \frac{|w-z|}{2}\sum_{n=0}^{\infty} n(n-1)\,|c_n|\,\rho^{n-2}\,.
\end{aligned}$$

But notice that by part b), the series $\sum c_n z^n$, $\sum n\,c_n z^{n-1}$ and $\sum n(n-1)\,c_n z^{n-2}$ all have the same radius of convergence R and so since $\rho < R$ we know that $\sum n(n-1)|c_n|\,\rho^{n-2}$ converges. Thus $|w-z|\,\frac{1}{2}\sum n(n-1)|c_n|\rho^{n-2} \to 0$ as $w \to z$. This proves part c).

Part d) follows from part c). If $f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ then we have $f'(z) = c_1 + 2\,c_2 z + 3\,c_3 z^2 + 4\,c_4 z^3 + \cdots$, $f''(z) = 2\cdot 1\,c_2 + 3\cdot 2\,c_3 z + 4\cdot 3\,c_4 z^2 + 5\cdot 4\,c_5 z^3 + \cdots$ and $f'''(z) = 3\cdot 2\cdot 1\,c_3 + 4\cdot 3\cdot 2\,c_4 z + 5\cdot 4\cdot 3\,c_5 z^2 + \cdots$ and so on, and we have $f(0) = c_0$, $f'(0) = 1\,c_1$, $f''(0) = 2!\,c_2$, $f'''(0) = 3!\,c_3$ and so on. Using induction you can show that $f^{(n)}(0) = n!\,c_n$.

Finally, part e) follows immediately from part d).

7.10 Theorem: *(Taylor's Theorem) If $f(z)$ is holomorphic in $D(a,R)$ and $0 < r < R \leq \infty$ then*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_\sigma \frac{f(z)}{(z-a)^{n+1}}\,dz\,,$$

where σ is the circle $\sigma(t) = a + r\,e^{i\,t}$ with $0 \leq t \leq 2\pi$.

Proof: We give the proof in the case that $a = 0$. Fix $z \in D(0,R)$ and choose r with $|z| < r < R$. Then by Cauchy's integral formula,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\,i}\int_\sigma \frac{f(w)}{w-z}\,dw \\
&= \frac{1}{2\pi\,i}\int_\sigma f(w)\,\frac{1}{w}\,\frac{1}{1-(z/w)}\,dw \\
&= \frac{1}{2\pi\,i}\int_\sigma f(w)\,\frac{1}{w}\left(1 + \frac{z}{w} + \Big(\frac{z}{w}\Big)^2 + \cdots + \Big(\frac{z}{w}\Big)^{N-1} + \frac{(z/w)^N}{1-(z/w)}\right)dw \\
&= \sum_{n=0}^{N-1} \frac{1}{2\pi\,i}\int_\sigma \frac{f(w)}{w^{n+1}}\,z^n\,dw \; + \frac{1}{2\pi\,i}\int_\sigma \frac{f(w)(z/w)^N}{w-z}\,dw \\
&= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!}\,z^n \; + R_N\,,
\end{aligned}$$

where $R_N = \frac{1}{2\pi\, i}\int_\sigma \frac{f(w)(z/w)^N}{w-z}\,dw$. Setting $M = \max_{w=\sigma(t)} |f(w)|$, the estimation theorem gives $|R_N| \leq \frac{1}{2\pi}\frac{M(|z|/r)^N}{(r-|z|)}\,2\pi r$. Since $|z|<r$, we have $R_N \to 0$ as $N\to\infty$

7.11 Example: The elementary complex functions have the same derivative formulas as their real counterparts, and so they have the same Taylor series centred at the origin (or centred at any real number). For all $z\in\mathbf{C}$ we have

$$e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$$

$$\sin z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_{n=0}^\infty \frac{z^{2n+1}}{(2n+1)!} \qquad \cosh z = \sum_{n=0}^\infty \frac{z^{2n}}{(2n)!}$$

For $|z|<1$ we have

$$\frac{1}{1-z} = \sum_{n=0}^\infty z^n \qquad \frac{1}{1+z} = \sum_{n=0}^\infty (-1)^n\, z^n \qquad \frac{1}{1+z^2} = \sum_{n=0}^\infty (-1)^n\, z^{2n}$$

When $|z|<1$, the principal branch of logarithm and inverse tangent are given by

$$\log(1-z) = -\sum_{n=1}^\infty \frac{z^n}{n} \qquad \log(1+z) = \sum_{n=1}^\infty (-1)^{n+1}\frac{z^n}{n}$$

$$\tan^{-1}(z) = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{2n+1}$$

For $|z|<1$ and for $a\in\mathbf{R}$, the principal branch of $(1+z)^a$ is given by

$$(1+z)^a = \sum_{n=0}^\infty \binom{a}{n}\, z^n = 1 + a\,z + \frac{a(a-1)}{2!}\,z^2 + \frac{a(a-1)(a-2)}{3!}\,z^3 + \cdots$$

This last power series is called the **Binomial series**.

7.12 Note: We should point out two important differences between Taylor series of complex functions and Taylor series of real functions. The first difference is that holomorphic functions are always equal to their Taylor series. This is not the case for real $\mathcal{C}^\infty$ functions. The standard example is the real function $f(x) = \begin{cases} e^{-1/x^2}, & x\neq 0 \\ 0\,, & x=0 \end{cases}$. This function is $\mathcal{C}^\infty$ at $x=0$, but all its derivatives vanish so its Taylor series is equal to 0. The second difference we would like to mention is that a real function might be $\mathcal{C}^\infty$ in a large interval while its Taylor series might converge only in a small interval, but notice that if a function is holomorphic in a disc, then its Taylor series will converge in that disc. An example which illustrates this difference is the real function $f(x) = 1/(1+x^2)$. This function is $\mathcal{C}^\infty$ for all x, but its Taylor series only converges for $|x|<1$. The reason for this is that when we extend f to the complex numbers, so $f(z) = 1/(1+z^2)$, then we find that $f(z) = \frac{1}{(z-i)(z+i)}$ so that f is holomorphic in $\mathbf{C}\setminus\{\pm i\}$. The radius of convergence is equal to 1 because the disc $D(0,1)$ is the largest disc (centred at 0) which can be contained in the domain of $f(z)$.

7.13 Note: If f and g are both holomorphic at a then The product fg will also be holomorphic at a. The coefficients of the Taylor series of fg at a are given by $(fg)^{(n)}(a)/n!$, and so they can be computed, using the product rule, from the coefficients of the Taylor series for f and for g. One can show that the Taylor series at a for fg is obtained from the Taylor series at a of f and of g by multiplying the power series together as if they were polynomials. We have

$$\left(\sum_{n=0}^{\infty} a_n(z-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-a)^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right) z^n$$

Also, if f and g are holomorphic at a and $g(a) \neq 0$, then we can solve the equation $hg = f$ for h to obtain the Taylor series of $h = f/g$ centred at a from the Taylor series of f and of g. This is equivalent to calculating f/g using long division as if the power series were polynomials.

Also, if f is holomorphic at a and g is holomorphic at $b = f(a)$ then the composite $g \circ f$ is holomorphic at a and hence has a Taylor series centred at a. Using the chain rule, one can show that the Taylor series for $g \circ f$ at a can be computed by composing the Taylor series of g at b with that of f at a as if the power series were polynomials.

7.14 Example: Find the Taylor series at 0 for $f(z) = \dfrac{1}{(1-z)^2}$.

Solution: We give several solutions. But first we note that since $f(z)$ is holomorphic in $\mathbf{C} \setminus \{1\}$, we know that the Taylor series at 0 converges in $D(0,1)$.

For our first solution, we calculate the derivatives: $f(z) = (1-z)^{-2}$, $f'(z) = 2(1-z)^{-3}$, $f''(z) = 3!(1-z)^{-4}$, and so on. So $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$ and so on. Thus

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Our second solution uses the Binomial series:

$$\begin{aligned} f(z) = (1-z)^{-2} &= 1 + \frac{-2}{1!}(-z)^1 + \frac{(-2)(-3)}{2!}(-z)^2 + \frac{(-2)(-3)(-4)}{3!}(-z)^3 + \cdots \\ &= 1 + 2z + 3z^2 + \cdots \end{aligned}$$

Our third solution is to differentiate both sides of $\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ to obtain

$$f(z) = 0 + 1 + 2z + 3z^2 + \cdots$$

Our fourth solution is to mutiply the Taylor series for $\dfrac{1}{1-z}$ by itself as if it was a polynomial to obtain

$$\begin{aligned} f(z) &= (1 + z + z^2 + z^3 + \cdots)(1 + z + z^2 + z^3 + \cdots) \\ &= 1 + (1+1)z + (1+1+1)z^2 + (1+1+1+1)z^3 + \cdots \\ &= 1 + 2z + 3z^2 + 4z^3 + \cdots \end{aligned}$$

7.15 Example: Find the Taylor series for $f(z) = e^z/(1-z)$.

Solution: We have $f(z) = e^z\,\frac{1}{1-z} = \left(\sum\limits_{n=0}^{\infty} \frac{1}{n!}z^n\right)\left(\sum\limits_{n=0}^{\infty} z^n\right) = \sum\limits_{n=0}^{\infty}\left(\sum\limits_{i=0}^{n}\frac{1}{n!}\right)z^n$. We can write out the first few terms: $f(z) = 1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + \frac{65}{24}z^4 + \cdots$.

7.16 Example: Find the first few terms of the Taylor series about 0 for $f(z) = \tan z$.

Solution: We have $\tan z = \dfrac{\sin z}{\cos z}$. We can use long division:

$$\begin{array}{r|l} & z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots \\ \hline 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots & z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots \\ & z - \frac{1}{2}z^3 + \frac{1}{24}z^5 - \cdots \\ \hline & \frac{1}{3}z^3 - \frac{1}{30}z^5 + \cdots \\ & \frac{1}{3}z^3 - \frac{1}{6}z^5 + \cdots \\ \hline & \frac{2}{15}z^5 + \cdots \end{array}$$

We find that $f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$. We can also easily find the radius of convergence. Since $\cos z = 0 \iff z = \frac{\pi}{2} + \pi k$ for some $k \in \mathbf{Z}$, we know that $f(z)$ is holomorphic for $z \neq \frac{\pi}{2} + \pi k$, so the radius of convergence is $R = \frac{\pi}{2}$.

7.17 Example: Find the Taylor series centred at $2i$ for $f(z) = \dfrac{1}{z}$.

Solution: $f(z) = \dfrac{1}{z} = \dfrac{1}{z - 2i + 2i} = \dfrac{1}{2i}\,\dfrac{1}{1 + \frac{z-2i}{2i}} = -\dfrac{i}{2}\,\dfrac{1}{1 - \frac{i(z-2i)}{2}} = -\dfrac{i}{2}\sum_{n=0}^{\infty}\Big(\dfrac{i(z-2i)}{2}\Big)^n$
$= \sum_{n=0}^{\infty} -\Big(\dfrac{i}{2}\Big)^{n+1}(z - 2i)^n$. The disc of convergence is $D(2i, 2)$.

7.18 Theorem: *(The Identity Theorem) Let f and g be holomorphic in the disc $D(a, r)$ where $0 < r \leq \infty$. Let $\{a_n\}$ be a sequence with $a_n \to a$. If $f(a_n) = g(a_n)$ for all n then $f(z) = g(z)$ for all $z \in D(a, r)$.*

Proof: Suppose that $f(a_n) = g(a_n)$ for all n. Let $h = f - g$. Then $h(a_n) = 0$ for all n. Since h is continuous, $h(a) = 0$. Since its holomorphic it is equal to its Taylor series $h(z) = \sum_{n=0}^{\infty} c_n(z - a)^n$. We want to show that all the coefficients c_n are zero. Suppose not, and say m is the smallest integer such that $c_m \neq 0$. Let $k(z) = h(z)(z - a)^{-m}$ Then we have $k(z) = c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^{m+2} + \cdots$, so $k(z)$ is holomorphic in $D(a, r)$ and $k(a) = c_m \neq 0$. Since $k(z)$ is continuous with $k(a) \neq 0$, we can find $s > 0$ such that $k(z) \neq 0$ for all $z \in D(a, s)$. But since $(z - a)^m \neq 0$ in $D^*(a, s)$ and since $h(z) = k(z)(z - a)^m$, this would imply that $h(z) \neq 0$ in $D^*(a, s)$. This gives us a contradiction since we assumed that $h(a_n) = 0$ for all n.

7.19 Note: We have studied power series. We are also interested in series of the form

$$\sum_{n=-\infty}^{\infty} c_n(z - a)^n = \sum_{n=-\infty}^{-1} c_n(z - a)^n + \sum_{n=0}^{\infty} c_n(z - a)^n c = \sum_{n=1}^{\infty} c_{-n}w^n + \sum_{n=0}^{\infty} c_n(z - a)^n \,.$$

where we have written $w = 1/(z - a)$. If the first series has radius of convergence $1/R$ and the second has radius of convergence S, then the first converges when $|w| < 1/R$, that is when $|z - a| > R$, and the second converges for $|z - a| < S$. They both converge in the annulus $A = \big\{z \in \mathbf{C}\big|R < |z - a| < S\big\}$. The next theorem shows that every function which is holomorphic in an annulus can be expressed as a series of this form.

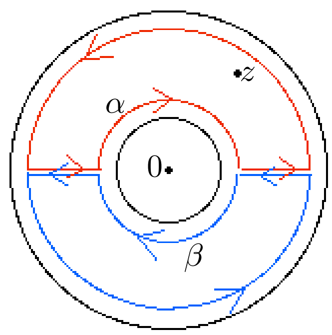
7.20 Theorem: *(Laurent's Theorem) Let $0 \le R < \rho < S \le \infty$ and let $a \in \mathbf{C}$. Suppose that f is holomorphic in the annulus $A = \{z \in \mathbf{C} \big| R < |z-a| < S\}$. Then for all $z \in A$,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \text{where} \quad c_n = \frac{1}{2\pi\, i} \int_\sigma \frac{f(z)}{(z-a)^{n+1}}\, dz\,,$$

where σ is the circle $\sigma(t) = a + \rho\, e^{i\,t}$ with $0 \le t \le 2\pi$. In particular, we have

$$c_{-1} = \frac{1}{2\pi i} \int_\sigma f(z)\, dz\,.$$

Proof: To simplify notation, we take $a = 0$, so $A = \{z | R < |z| < S\}$. For $z \in A$ pick r and s so that $R < r < |z| < s < S$. Again to simplify notation, suppose that $\text{Im}\,(z) > 0$. Let α be the loop in A which follows the semicircle counterclockwise from s to $-s$, then the line segment from $-s$ to $-r$, then the semicircle clockwise from $-r$ to r, and then the line segment from r to s. Let β be the loop which follows the line segment from s to r, then the semicircle clockwise from r to $-r$, then the line segment from $-r$ to $-s$, and then the semicircle counterclockwise from $-s$ to s.



Since $\eta(\alpha, z) = 1$ and $\eta(\beta, z) = 0$, Cauchy's theorem tells us that $\displaystyle\int_\alpha \frac{f(w)}{w-z}\, dw = 2\pi\, i\; f(z)$ and $\displaystyle\int_\beta \frac{f(w)}{w-z}\, dw = 0$. Also, since the integrals along the line segments cancel, we have $\displaystyle\int_\alpha \frac{f(w)}{w-z}\, dw + \int_\beta \frac{f(w)}{w-z}\, dw = \int_{\sigma_s} \frac{f(w)}{w-z}\, dw - \int_{\sigma_r} \frac{f(w)}{w-z}\, dw$, where σ_r and σ_s are the circles $\sigma_r(t) = r\, e^{i\,t}$ and $\sigma_s(t) = s\, e^{i\,t}$ for $0 \le t \le 2\pi$. So we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\left(\int_{\sigma_s} f(w)\, \frac{1}{w-z}\, dw - \int_{\sigma_r} f(w)\, \frac{1}{w-z}\, dw \right) \\
&= \frac{1}{2\pi i}\left(\int_{\sigma_s} f(w)\, \frac{1}{w}\, \frac{1}{1-\frac{z}{w}}\, dw - \int_{\sigma_r} f(w)\, \frac{-1}{z}\, \frac{1}{1-\frac{w}{z}} \right) \\
&= \frac{1}{2\pi i}\left(\int_{\sigma_s} f(w) \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}\, dw + \int_{\sigma_r} f(w) \sum_{m=0}^{\infty} \frac{w^m}{z^{m+1}}\, dw \right) \\
&= \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty} \int_{\sigma_s} \frac{f(w)\, z^n}{w^{n+1}}\, dw + \sum_{m=0}^{\infty} \int_{\sigma_r} \frac{f(w) w^m}{z^{m+1}}\, dw \right) \\
&= \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty} \Big(\int_\sigma \frac{f(w)}{w_{n+1}}\, dw \Big)\, z^n + \sum_{n=-\infty}^{-1} \Big(\int_\sigma \frac{f(w)}{w^{n+1}}\, dw \Big)\, z^n \right) \\
&= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \Big(\int_\sigma \frac{f(w)}{w^{n+1}}\, dw \Big)\, z^n
\end{aligned}$$

In the second last equlity, we replaced m by $-n-1$, and we used the fact that each of the loops σ_s and σ_r is homotopic to σ in A. The interchange of summation and integration in the third equality should be justified. We can justify it as follows. For any positive integer N we have

$$\begin{aligned}\int_{\sigma_s} f(w)\,\frac{1}{w}\sum_{n=0}^{\infty}\big(\frac{z}{w}\big)^n\,dw &= \int_{\sigma_s} f(w)\,\frac{1}{w}\bigg(\sum_{n=0}^{N-1}\big(\frac{z}{w}\big)^n\;+\frac{(z/w)^N}{1-(z/w)}\bigg)\,dw\\ &= \sum_{n=0}^{N-1}\int_{\sigma_s} f(w)\,\frac{1}{w}\,\frac{z^n}{w^n}\,dw\;+\int_{\sigma_s} f(w)\,\frac{1}{w}\,\frac{(z/w)^N}{1-(z/w)}\,dw\end{aligned}$$

As $N\to\infty$ the first term tends to the infinite sum $\displaystyle\sum_{n=0}^{\infty}\int_{\sigma_s} f(w)\frac{z^n}{w^{n+1}}\,dw$ and the second term may be estimated using the Estimation Theorem:

$$\left|\int_{\sigma_s} f(w)\,\frac{(z/w)^N}{w-z}\,dw\right| \le \max_{|w|=s}\big|f(w)\big|\,\frac{(|z|/s)^N}{(s-|z|)}\,2\pi s\to 0$$

as $N\to\infty$ since $(|z|/s)<1$.

7.21 Example: Let $f(z)=\dfrac{1}{z(z^2+4)}$. Note that f is holomorphic except at $z=0$ and $z=\pm 2i$. In particular, f is holomorphic in the annulus $A=\{z|0<|z|<2\}$ and in the annulus $B=\{z|2<|z|<\infty\}$ and also in the annulus $C=\{z|0<|z-2i|<2\}$. Find the Laurent series of $f(z)$ in A and in B and in C. Also, use the Laurent series to find the path integrals $\int_\alpha f$, $\int_\beta f$ and $\int_\gamma f$, where α, β and γ are the circles $\alpha(t)=e^{i\,t}$, $\beta(t)=3\,e^{i\,t}$ and $\gamma=2i+e^{i\,t}$ for $0\le t\le 2\pi$.

Solution: We have $f(z)=\dfrac{1}{4z}\,\dfrac{1}{1+(z/2)^2}=\dfrac{1}{4z}\displaystyle\sum_{n=0}^{\infty}(-1)^n\big(\frac{z}{2}\big)^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,z^{2n-1}$. This is the Laurent series for f in A. Since the coefficient of z^{-1} in this series is $c_{-1}=\frac{1}{4}$, we have $\displaystyle\int_\alpha f=2\pi i\,c_{-1}=\frac{1}{2}\pi\,i$.

Also, we have $f(z)=\dfrac{1}{z^3}\,\dfrac{1}{1+(2/z)^2}=\dfrac{1}{z^3}\displaystyle\sum_{n=0}^{\infty}(-1)^n\big(\frac{2}{z}\big)^{2n}=\sum_{n=0}^{\infty}(-1)^n4^n\,z^{-2n-3}$. This is the Laurent series for f in B. Since the coefficient of z^{-1} is $c_{-1}=0$, we have $\displaystyle\int_\beta f=0$.

In the third annulus we write

$$
\begin{aligned}
f(z) &= \frac{1}{z-2i}\,\frac{1}{z+2i}\,\frac{1}{z} = \frac{1}{z-2i}\,\frac{1}{(z-2i)+4i}\,\frac{1}{(z-2i)+2i} \\
&= \frac{1}{z-2i}\,\frac{1}{4i}\,\frac{1}{1+\frac{z-2i}{4i}}\,\frac{1}{2i}\,\frac{1}{1+\frac{z-2i}{2i}} \\
&= -\frac{1}{8}\,\frac{1}{z-2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2i}{4i}\right)^n\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2i}{2i}\right)^n \\
&= -\frac{1}{8}\,\frac{1}{z-2i}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^j}{(4i)^j}\,\frac{(-1)^{n-j}}{(2i)^{n-j}}\right)(z-2i)^n \\
&= -\frac{1}{8}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^n}{i^n 2^{n+j}}\right)(z-2i)^{n-1} \\
&= -\frac{1}{8}\sum_{n=0}^{\infty}\frac{i^n}{4^n}\left(\sum_{j=0}^{n}\frac{1}{2^j}\right)(z-2i)^{n-1} \\
&= -\frac{1}{8}\sum_{n=0}^{\infty}\frac{i^n(2^{n+1}-1)}{2^{2n}}(z-2i)^{n-1}
\end{aligned}
$$

This is the Laurent series in C. The coefficient of $(z-2i)^{-1}$ is $c_{-1} = -\frac{1}{8}$ so $\displaystyle\int_\gamma f = -\frac{1}{4}\pi i$.

7.22 Note: It should be remarked that all three of the path integrals in the above example are easy to compute using Cauchy's integral formula. In the following example, however, its easier to use the Laurent series to find the path integral.

7.23 Example: Let $f(z) = \dfrac{1}{z^4 \sinh z}$. Since $\sinh z = 0$ when $z = k\pi i, k \in \mathbf{Z}$ we see that f is holomorphic except at $z = k\pi i$. Find the first few terms of the Laurent series for f in the annulus $A = \{z | 0 < |z| < \pi\}$, and hence find $\int_\sigma f$ where σ is the circle $\sigma(t) = e^{i\,t}$ with $0 \le t \le 2\pi$.

Solution: We have $f(z) = \dfrac{1}{z^4}\,\dfrac{1}{\sinh z} = \dfrac{1}{z^4}\,\dfrac{1}{z(1+\frac{1}{6}z^2+\frac{1}{120}z^4+\cdots)}$. We use long division:

$$
\begin{array}{r|l}
 & 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots \\
\hline
1 + \frac{1}{6}z^2 + \frac{1}{120}z^4 + \cdots & 1 + 0\,z^2 + 0\,z^4 + \cdots \\
 & 1 + \frac{1}{6}z^2 + \frac{1}{120}z^4 + \cdots \\
\cline{2-2}
 & -\frac{1}{6}z^2 - \frac{1}{120}z^4 + \cdots \\
 & -\frac{1}{6}z^2 - \frac{1}{36}z^4 + \cdots \\
\cline{2-2}
 & \frac{7}{360}z^5 + \cdots
\end{array}
$$

We find that $f(z) = z^{-5} - \frac{1}{6}z^{-3} + \frac{7}{360}z^{-1} + \cdots$. Since $c_{-1} = \frac{7}{360}$, we have $\displaystyle\int_\sigma f = \frac{7\pi}{180}\,i$.

7.24 Definition: Suppose that f is holomorphic in an open set U which contains the punctured disc $D^*(a,R)$ and say the Laurent series of f in $D^*(a,R)$ is given by

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-a)^n\,.$$

If for all $N\in\mathbf{Z}$ there exists $n<N$ with $a_n\neq 0$ then we say that f has an **essential singularity** at a. Otherwise, let N be the smallest integer such that $a_N\neq 0$. If $N<0$ then we say that f has a **pole** at a of **order** $|N|$. If $N\geq 0$ we say that f has a **removable** singularity at a, and in this case we shall extend f so that it is holomorphic in the disc $D(a,r)$ by setting $f(a)=a_0$. If $N>0$ then we say that f has a **zero** at a of **order** N. In any case, we define the **residue** of f at a to be $\text{Res}\,(f,a)=a_{-1}$. If σ is the circle $\sigma(t)=a+r\,e^{i\,t}$ for $0\leq t\leq 2\pi$ where $0<r<R$ then we have

$$\text{Res}\,(f,a)=a_{-1}=\frac{1}{2\pi i}\int_\sigma f(z)\,dz$$

7.25 Note: If f has a removable singularity at a, then of course we have $\lim\limits_{z\to a}f(z)=a_0$. If f has a pole at a then its not hard to show that $\lim\limits_{z\to a}f(z)=\infty$. If f has an essential singularity at a, then the limit $\lim\limits_{z\to a}f(z)$ does not exist, and in fact there is a (dificult) theorem called *Picard's Theorem* which states that for all $\epsilon>0$ the image $f\big(D^*(a,\epsilon)\big)$ is either equal to $\mathbf{C}$ or to $\mathbf{C}\setminus\{p\}$ for some point p.

7.26 Note: Let U be an open set and let $p_1,p_2,\cdots,p_k$ be points in U. If f is holomorphic in $U\setminus\{p_1,p_2,\cdots p_n\}$ and if f has poles at each of the points p_i, then we say that f is **meromorphic** in U. In this case we can extend f to a holomorphic map $f:U\to\hat{\mathbf{C}}$ by setting $f(p_i)=\infty$ for each i.

7.27 Theorem: *(The Residue Theorem) Let U be an open set and let $z_1,z_2,\cdots,z_n$ be points in U. Let f be holomorphic in $U\setminus\{z_1,z_2,\cdots,z_n\}$. Let α be a loop in $U\setminus\{z_1,z_2,\cdots,z_n\}$ which is homotopic in U to a constant loop. Then*

$$\int_\alpha f(z)\,dz=2\pi\,i\sum_{i=1}^{k}\eta(\alpha,z_i)\,\text{Res}\,(f,z_i)\,.$$

Proof: Choose $R>0$ so that each puctured disc $D^*(z_k,R)$ lies inside $U\setminus\{z_1,z_2,\cdots,z_n\}$. Inside each of these puctured discs, f will be equal to its Laurent series, and we write

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_k)^n=p_k(z)+h_k(z)\ ,\ \text{where}$$
$$p_k=\sum_{n=-\infty}^{-1}a_n(z-z_k)^n\quad\text{and}\quad h_k=\sum_{n=0}^{\infty}a_n(z-z_k)^n$$

(p_k is called the **principal part** of f, and h_k is called the **holomorphic part** of f at the point z_k). We have $\text{Res}\,(f,z_k)=a_{-1}=\text{Res}\,(p_k,z_k)$. Notice that h_k is holomorhic in the disc $D(z_k,R)$ (not just the punctured disc) and notice that p_k is holomorphic in all of $\mathbf{C}\setminus\{z_k\}$ and $\int_\alpha p_k(z)\,dz = 2\pi\,i\,\eta(\alpha,z_k)\,a_{-1}$. Now we let

$$g(z) = f(z) - \sum_{k=1}^{n} p_k(z)\,.$$

Although f was only holomorphic in $U\setminus\{z_1\ldots,z_n\}$, the map g is holomorphic in all of U, indeed in $D^*(z_k,R)$ we have $g(z)=f(z)-p_k(z)-\sum_{i\neq k}p_i(z) = h_k(z)-\sum_{i\neq k}p_i(z)$. Since α is homotopic to a constant loop in U, we have

$$\begin{aligned}0 = \int_\alpha g(z)\,dz &= \int_\alpha f(z) - \sum_{k=1}^{n} p_k(z)\,dz = \int_\alpha f(z)\,dz - \sum_{k=1}^{n}\int_\alpha p_k(z)\,dz \\ &= \int_\alpha f(z)\,dz - \sum_{k=1}^{n} 2\pi\,i\,\eta(\alpha,a_k)\,\mathrm{Res}\,(f,z_k))\,.\end{aligned}$$

7.28 Example: Let α be a loop in $D(0,3)$ with $\eta(\alpha,0)=3$, $\eta(\alpha,\frac{\pi}{2})=-1$ and $\eta(\alpha,-\frac{\pi}{2})=1$, and let $f(z)=\dfrac{(z+1)\,e^z}{z\cos z}$. Find $\int_\alpha f(z)\,dz$.

Solution: Notice that f is holomorphic in $\mathbf{C}$ except at $z=0$ and $z=\frac{\pi}{2}+k\pi$, $k\in\mathbf{Z}$. In particular, f is holomorphic in $D(0,3)$ except at $z=0$ and $z=\pm\frac{\pi}{2}$. So by the Residue Theorem, $\int_\alpha f(z)\,dz = 2\pi\,i\big(3\mathrm{Res}\,(f,0)-\mathrm{Res}\,(f,\dfrac{\pi}{2})+\mathrm{Res}\,(f,-\dfrac{\pi}{2})\big)$.

By Cauchy's Integral Formula, $\mathrm{Res}\,(f,0)=\dfrac{1}{2\pi\,i}\displaystyle\int_\sigma \frac{F(z)}{z}\,dz = F(0)=1$, where $F(z)=(z+1)\,e^z/\cos z$ and where σ is a small circle centred at 0. Alternatively, we could have found $\mathrm{Res}\,(f,0)$ by finding the coefficient of z^{-1} in the Laurent series for f in $D^*(0,\frac{\pi}{2})$.

To find $\mathrm{Res}\,(f,\frac{\pi}{2})$ we use a Laurent series. Near $\frac{\pi}{2}$ we have $(z+1)=(z-\frac{\pi}{2})+(1+\frac{\pi}{2})$, and $e^z = e^{z-\pi/2+\pi/2} = e^{\pi/2}e^{z-\pi/2} = e^{\pi/2}\sum_{n=0}^{\infty}\frac{1}{n!}(z-\frac{\pi}{2})^n = e^{\pi/2}\big(1+(z-\frac{\pi}{2})+\ldots\big)$, and $\dfrac{1}{z}=\dfrac{1}{(z-\frac{\pi}{2})+\frac{\pi}{2}}=\dfrac{\frac{2}{\pi}}{1+\frac{2}{\pi}(z-\frac{\pi}{2})}=\dfrac{2}{\pi}\displaystyle\sum_{n=0}^{\infty}(-1)^n(\tfrac{2}{\pi})^n(z-\tfrac{\pi}{2})^n=\dfrac{2}{\pi}\big(1-\tfrac{2}{\pi}(z-\tfrac{\pi}{2})+\ldots\big)$, and $\cos z = -\sin(z-\frac{\pi}{2}) = -\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}(z-\frac{\pi}{2})^n = \frac{1}{(z-\frac{\pi}{2})}\big(-1+\tfrac{1}{6}(z-\tfrac{\pi}{2})^2+\ldots\big)$ so that by long division $\dfrac{1}{\cos z}=\dfrac{1}{(z-\frac{\pi}{2})}\big(-1-\tfrac{1}{6}(z-\tfrac{\pi}{2})^2+\ldots\big)$. Multiplying these series together gives $f(z)=(z+1)\,e^z\,\dfrac{1}{z}\,\dfrac{1}{\cos z}=\big((1+\frac{\pi}{2})+(z-\frac{\pi}{2})\big)e^{\pi/2}\big(1+(z-\frac{\pi}{2})+\ldots\big)\frac{2}{\pi}\big(1-\frac{2}{\pi}(z-\frac{\pi}{2})+\ldots\big)(z-\frac{\pi}{2})^{-1}\big(-1+\frac{1}{6}(z-\frac{\pi}{2})^2+\ldots\big)=(1+\frac{\pi}{2})e^{\pi/2}\frac{2}{\pi}(z-\frac{\pi}{2})^{-1}+\ldots$. Thus we have $\mathrm{Res}\,(f,\frac{\pi}{2})=\frac{2}{\pi}(1+\frac{\pi}{2})\,e^{\pi/2}=(1+\frac{2}{\pi})e^{\pi/2}$.

To find $\mathrm{Res}\,(f,-\frac{\pi}{2})$ we use a Laurent series. Near $-\frac{\pi}{2}$ we have $(z+1)=(1-\frac{\pi}{2})+(z+\frac{\pi}{2})$, $e^z=e^{(z+\pi/2)-\pi/2}=e^{-\pi/2}(1+\ldots)$, $\dfrac{1}{z}=\dfrac{1}{(z+\frac{\pi}{2})-\frac{\pi}{2}}=\dfrac{-\frac{2}{\pi}}{1-\frac{2}{\pi}(z+\frac{\pi}{2})}=-\frac{2}{\pi}(1+\ldots)$, and

$\cos z = \sin(z+\frac{\pi}{2}) = (z+\frac{\pi}{2})+\ldots$ so that $\dfrac{1}{\cos z} = (z+\frac{\pi}{2})^{-1}+\ldots$. Multiplying these together gives $f(z) = (1-\frac{\pi}{2})e^{-\pi/2}(-\frac{2}{\pi})(z+\frac{\pi}{2})^{-1}+\ldots$ so we have $\text{Res}\,(f,-\frac{\pi}{2}) = (1-\frac{2}{\pi})e^{-\pi/2}$.

Finally, we obtain $\displaystyle\int_\alpha f(z)\,dz = 2\pi\,i\big(3-(1+\tfrac{2}{\pi})e^{\pi/2}+(1-\tfrac{2}{\pi})e^{-\pi/2}\big)$

7.29 Example: Evaluate the *real* integral $\displaystyle\int_{-\infty}^{\infty}\frac{\cos t}{t^2+1}\,dt$.

Solution: Let $f(z) = \dfrac{e^{iz}}{z^2+1}$ and let σ be the loop which follows the line segment $\alpha(t) = t$ for $-R \le t \le R$ and then the semicircle $\beta(t) = R\,e^{i\,t}$ for $0 \le t \le \pi$. Notice that f has poles at $z = \pm i$ and that only the pole at $z = i$ lies inside σ. By Cauchy's Integral Formula we have $\displaystyle\int_\sigma f(z)\,dz = \int_\sigma \frac{e^{iz}/z+i}{z-i}\,dz = 2\pi\,i\,\frac{e^{-1}}{2\,i} = \frac{\pi}{e}$. On the other hand we have $\displaystyle\int_\sigma f = \int_\alpha f + \int_\beta f$. We have $\displaystyle\int_\alpha f = \int_{-R}^{R}\frac{\cos t+i\,\sin t}{t^2+1}\,dt \to \int_{-\infty}^{\infty}\frac{\cos t}{t^2+1}\,dt + i\int_{-\infty}^{\infty}\frac{\sin t}{t^2+1}\,dt$ as $R\to\infty$, and $\displaystyle\int_\beta f = \int_0^\pi \frac{e^{iRe^{it}}}{R^2e^{i2t}+1}\,dt = \int_0^\pi \frac{e^{iR(\cos t+i\sin t)}iRe^{it}}{R^2e^{i2t}+1}\,dt$ so by the Estimation Theorem $\displaystyle\left|\int_\beta f(z)\,dz\right| \le \frac{\big(\max_{0\le t\le\pi} e^{-R\sin t}\big)\,R}{R^2-1}\,\pi \le \frac{\pi\,R}{R^2-1} \to 0$ as $R\to\infty$. Comparing our two values for $\displaystyle\int_\sigma f$ we obtain $\displaystyle\int_{-\infty}^{\infty}\frac{\cos t}{t^2+1}\,dt = \frac{\pi}{e}$.