AMATH/PMATH 331 Real Analysis, Problems for Chapter 8

1: Let $f:\mathbf{R}\to\mathbf{R}$ be the 2π-periodic function with $f(x)=x^3-\pi^2x$ for $-\pi\le x\le\pi$.

(a) Find the coefficients of the (real) Fourier series for f.

(b) Show that $s_m(f)\to f$ uniformly on $\mathbf{R}$.

(c) By evaluating at $x=\frac{\pi}{2}$, evaluate $\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$.

2: Let $f:\mathbf{R}\to\mathbf{R}$ be a 2π-periodic function whose restriction to $[-\pi,\pi]$ is continuous.

(a) Use Integration by Parts to show that if f is $\mathcal{C}^1$ (meaning that the derivative f' exists and is continuous) then $\big|c_n(f)\big|\le\frac{M}{|n|}$ for all $n\in\mathbf{Z}$ where $M=\|f'\|_\infty=\max\limits_{-\pi\le t\le\pi}\big|f'(t)\big|$.

(b) Use induction to show that if f in $\mathcal{C}^k$ (meaning that the k^{th} derivative of f exists and is continuous) then $\big|c_n(f)\big|\le\frac{M}{(2\pi)^{k-1}|n|^k}$ for all $n\in\mathbf{Z}$ where $M=\big\|f^{(k)}(x)\big\|_\infty=\max\limits_{-\pi\le x\le\pi}\big|f^{(k)}(x)\big|$.

(c) Show that if $f\in\mathcal{C}^2$ then $s_m(f)\to f$ uniformly on $\mathbf{R}$.

3: Let $f\in\mathcal{R}(T)$, let $(c_n)_{n\ge0}$ and $(d_n)_{n\ge1}$ be sequences in $\mathbf{R}$ and let $p_m(x)=c_0+\sum\limits_{n=1}^{m}c_n\cos nx+\sum\limits_{n=1}^{m}d_n\sin nx$.

(a) Show that if $p_m\to f$ in $\big(\mathcal{R}(T),\|\ \|_1\big)$ then $c_0=a_0(f)$ and $c_n=a_n(f)$ and $d_n=b_n(f)$ for all $n\in\mathbf{Z}^+$. for all $n\in\mathbf{Z}$.

(b) Show that if $p_m\to f$ in $\big(\mathcal{R}(T),\|\ \|_\infty\big)$ then $c_0=a_0(f)$ and $c_n=a_n(f)$ and $d_n=b_n(f)$ for all $n\in\mathbf{Z}^+$.

(c) Show that if $p_m\to f$ in $\big(\mathcal{R}(T),\|\ \|_2\big)$ then $c_0=a_0(f)$ and $c_n=a_n(f)$ and $d_n=b_n(f)$ for all $n\in\mathbf{Z}^+$.

4: Let $f:\mathbf{R}\to\mathbf{R}$ be the 2π-periodic function with $f(x)=\begin{cases}1 \text{ if } 0<x<\pi,\\ -1 \text{ if } -\pi<x<0,\\ 0 \text{ if } x=0,\pm\pi.\end{cases}$

(a) Find the coefficients of the (real) Fourier series for f.

(b) By recognizing $s_{2m}(f)\big(\frac{\pi}{2m}\big)$ as a Riemann sum, show that $\lim\limits_{m\to\infty}s_{2m}(f)\big(\frac{\pi}{2m}\big)=\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx$.

(c) Using a computer to approximate the value of $\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx$, show that there exists $\ell\in\mathbf{Z}^+$ such that for all $m\ge\ell$ we have $\|s_m(f)-f\|_\infty>0.17$.

(d) (Optional Challenge) Show that $\big\{s_m(f)(x)\big\}$ converges for all x.