AMATH/PMATH 331 Real Analysis, Problems for Chapter 7

1: (a) Find an example of a function $f:\mathbf{R}\to\mathbf{R}$ such that $\big|f(y)-f(x)\big|<|y-x|$ for all $x,y\in\mathbf{R}$ with $x\neq y$, but f has no fixed point in $\mathbf{R}$.

(b) Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=\displaystyle\int_0^x f(t)\,dt$. Show that F is not a contraction map but that $F^2=F\circ F$ is.

(c) Use the Banach Fixed Point Theorem to show that there exists a unique function $f\in\mathcal{C}[0,1]$ such that $f(x)=x+\displaystyle\int_0^x t\,f(t)\,dt$ for all $x\in[0,1]$.

2: (a) Let $A=\left\{\sum_{k=1}^{n} f_k(x)g_k(y)\,\big|\,n\in\mathbf{Z}^+,\,f_k,g_k\in\mathcal{C}[0,1]\right\}$. Show that A is dense in $\big(\mathcal{C}\big([0,1]\times[0,1]\big),d_\infty\big)$.

(b) Let $A=\left\{\sum_{k=0}^{n}\big(a_k\sin(kx)+b_k\cos(kx)\big)\,\Big|\,0\leq n\in\mathbf{Z},\,a_k,b_k\in\mathbf{R}\right\}$. Show that A is dense in $\big(\mathcal{C}[0,r],d_\infty\big)$ for every $0<r<2\pi$, but A is not dense in $\big(\mathcal{C}[0,2\pi],d_\infty\big)$.

(c) Show that there does exist $0\neq f\in\mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^{2}x^{2n}f(x)\,dx=0$ for all $0\leq n\in\mathbf{Z}$ but there does not exist $0\neq f\in\mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^{2}x^{3n}f(x)\,dx=0$ for all $0\leq n\in\mathbf{Z}$.