Chapter 6. Completeness and Compactness

Completeness

6.1 Definition: A sequence $(x_k)_{k\geq p}$ in a metric space X is called **Cauchy** when

$$\forall\,\epsilon>0\ \exists\, m\in\mathbb{Z}_{\geq p}\ \forall\, k,l\in\mathbb{Z}_{\geq p}\ \big(k,\ell\geq m \Longrightarrow d(x_k,x_\ell)<\epsilon\big).$$

A metric space X is called **complete** when every Cauchy sequence in X converges in X. We remark that a complete inner product space is called a **Hilbert space**, and a complete normed linear space is called a **Banach space**.

6.2 Example: $\mathbb{R}$ is complete by the Cauchy Criterion for Convergence (Theorem 1.26).

6.3 Theorem: *Let X be a metric space.*
(1) Every Cauchy sequence in X is bounded.
(2) Every convergent sequence in X is Cauchy.
(3) If some subsequence of a Cauchy sequence (x_n) converges, then (x_n) converges.

Proof: To prove Part 1, let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Choose $m\in\mathbb{Z}^+$ such that $k,\ell\geq m\Longrightarrow d(x_k,x_\ell)\leq 1$ and note that, in particular, we have $d(x_k,x_m)\leq 1$ for all $k\geq m$. Let $a=x_m$ and choose $r>\max\big\{d(x_1,a),d(x_2,a),\cdots,d(x_{m-1},a),1\big\}$. Then for all $n\in\mathbb{Z}^+$ we have $d(x_n,a)<r$ so the sequence (x_n) is bounded, as required.

To Prove Part 2, let $(x_n)_{n\geq 1}$ be a convergent sequence in X and let $a=\lim\limits_{n\to\infty}x_n$. Let $\epsilon>0$. Choose $m\in\mathbb{Z}^+$ such that $n\geq m\Longrightarrow d(x_n,a)<\frac{\epsilon}{2}$. Then for all $k,\ell\geq m$ we have

$$d(x_k,x_\ell)\leq d(x_k,a)+d(a,x_\ell)<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon,$$

so the sequence (x_n) is Cauchy, as required.

To prove Part 3, let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X, let $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$, suppose tha $(x_{n_k})_{k\geq 1}$ converges, and let $a=\lim\limits_{k\to\infty}x_{n_k}$. Let $\epsilon>0$. Since (x_n) is Cauchy we can choose $m\in\mathbb{Z}^+$ so that $k,\ell\geq m\Longrightarrow d(x_k,x_\ell)<\frac{\epsilon}{2}$. Since $\lim\limits_{k\to\infty}n_k=\infty$ and $\lim\limits_{k\to\infty}x_{n_k}=a$, we can choose an index ℓ such that $n_\ell\geq m$ and $d(x_{n_\ell},a)<\frac{\epsilon}{2}$. Then for all $k\geq m$ we have

$$d(x_k,a)\leq d(x_k,x_{n_\ell})+d(x_{n_\ell},a)<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.$$

6.4 Theorem: *Let X be a complete metric space and let $A\subseteq X$. Then A is complete if and only if A is closed in X*

Proof: Suppose that A is closed in X. Let (x_n) be a Cauchy sequence in A. Since X is complete, (x_n) converges in X. Since A is closed in X and (x_n) is a sequence in A which converges in X, we have $\lim\limits_{n\to\infty}x_n\in A$ by Theorem 3.5 (The Sequential Characterization of Closed Sets). Thus every Cauchy sequence in A converges in A, so A is complete.

Suppose, conversely, that A is complete. Let $a\in A'$, that is let $a\in X$ be a limit point of A. Since $a\in A'$, by Theorem 5.16 (The Sequential Characterization of Limit Points) we can choose a sequence (x_n) in A (indeed in $A\setminus\{a\}$) with $\lim\limits_{n\to\infty}x_n=a$. Since (x_n) converges in X, it is Cauchy. Since (x_n) is Cauchy and A is complete, (x_n) converges in A, that is $a=\lim\limits_{n\to\infty}x_n\in A$.

The Completeness of $\mathbb{R}^m$

6.5 Theorem: *(Bolzano-Weierstrass Theorem) Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence (using the standard metric in $\mathbb{R}^m$).*

Proof: For this proof, we shall label the components of an element in $\mathbb{R}^m$ using superscripts rather than subscripts, se we shall write an element $x \in \mathbb{R}^m$ as $(x^1, x^2, \cdots, x^m)$. Let $(x_n)_{n\geq 1}$ be a bounded sequence in $\mathbb{R}^m$. Then the first component sequence $(x_n^1)_{n\geq 1}$ is a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass Theorem in $\mathbb{R}$ (Theorem 1.23), we can choose a convergent subsequence $(x_{n_\ell}^1)_{\ell\geq 1}$. Since the second component sequence $(x_n^2)_{n\geq 1}$ is bounded, the subsequence $(x_{n_\ell}^2)_{\ell\geq 1}$ is also bounded so (by Theorem 1.23 again) we can choose a convergent subsequence $(x_{n_{\ell_k}}^2)_{k\geq 1}$. Since $(x_{n_\ell}^1)_{\ell\geq 1}$ converges, so does the subsequence $(x_{n_{\ell_k}}^1)_{k\geq 1}$. Since the third component sequence $(x_n^3)_{n\geq 1}$ is bounded, the subsequence $(x_{n_{\ell_k}}^3)_{k\geq 1}$ is also bounded so (by Theorem 1.23) we can choose a convergent subsequence $(x_{n_{\ell_{k_j}}}^3)_{j\geq 1}$. Since the component sequences $(x_{n_\ell}^1)$ and $(x_{n_\ell}^2)$ both converge, so do the subsequences $(x_{n_{\ell_k}}^1)$ and $(x_{n_{\ell_k}}^2)$. Thus the subsequence $(x_{n_{\ell_k}})_{k\geq 1}$ of (x_n) has the property that the first 3 component sequences $(x_{n_{\ell_k}}^1)$, $(x_{n_{\ell_k}}^2)$ and $(x_{n_{\ell_k}}^3)$ all converge. We repeat the procedure until we obtain a subsequence of (x_n) whose m component sequences all converge. This subsequence converges in $\mathbb{R}^m$ by Theorem 5.4 (Component Sequences in $\mathbb{R}^m$).

6.6 Theorem: *(The Completeness of $\mathbb{R}^m$) For every sequence in $\mathbb{R}^m$, the sequence converges if and only if it is Cauchy (where we are using the standard metric in $\mathbb{R}^m$).*

Proof: Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^m$. If $(x_n)_{n\geq 1}$ converges, then it is Cauchy by Part 2 of Theorem 6.3. Suppose, conversely, that $(x_n)_{n\geq 1}$ is Cauchy. Choose $N \in \mathbb{Z}^+$ so that when $k, \ell \geq N$ we have $|x_k - x_\ell| < 1$. Then for all $k \in \mathbb{Z}^+$ we have $|x_k - x_N| < 1$ and hence $|x_k| \leq |x_k - x_N| + |x_N| < 1 + |x_N|$, and so the sequence $(x_n)_{n\geq 1}$ is bounded by $\max\big\{|x_1|, |x_2|, \cdots, |x_{N-1}|, 1+|x_N|\big\}$. By the Bolzano-Weierstrass Theorem, we can choose a convergent subsequence $(x_{n_k})_{k\geq 1}$. Since (x_n) is Cauchy and has a convergent subsequence, it follows that (x_n) converges by Part 3 of Theorem 6.3.

6.7 Theorem: *Every finite-dimensional normed linear space is complete.*

Proof: Let U be an m-dimensional normed linear space. Let $\{u_1, \cdots, u_m\}$ be a basis for the vector space U and let $F : \mathbb{R}^m \to U$ be the associated vector space isomorphism given by $F(t) = \sum_{k=1}^m t_k u_k$. Recall, from Theorem 5.38, that both F and F^{-1} are Lipschitz continuous. Let L be a Lipschitz constant for F and let M be a Lipschitz constant for F^{-1}. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in U. For each $n \in \mathbb{Z}^+$, let $t_n = F^{-1}(x_n) \in \mathbb{R}^m$. Note that $(t_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}^m$ because

$$\|t_k - t_\ell\| = \big\|F^{-1}(x_k) - F^{-1}(x_\ell)\big\| \leq M\|x_k - x_\ell\|.$$

Since (t_n) is a Cauchy sequence in $\mathbb{R}^m$ and $\mathbb{R}^m$ is complete, (t_n) converges in $\mathbb{R}^m$. Let $s = \lim_{n\to\infty} t_n \in \mathbb{R}^m$ and let $a = F(s) \in U$. Then we have $\lim_{n\to\infty} x_n = a$ because

$$\|x_n - a\| = \big\|F(t_n) - F(s)\big\| \leq L\|t_n - s\|.$$

6.8 Corollary: *The metric spaces $(\mathbb{R}^m, d_1)$, $(\mathbb{R}^m, d_2)$ and $(\mathbb{R}^m, d_\infty)$ are all complete.*

6.9 Corollary: *Let U be a finite dimensional normed linear space and let $A \subseteq U$. Then A is complete if and only if A is closed in U.*

The Completeness of Spaces of Sequences and Spaces of Functions

6.10 Theorem: *The metric spaces* (ℓ_1, d_1), (ℓ_2, d_2) *and* (ℓ_∞, d_∞) *are all complete.*

Proof: We prove that (ℓ_1, d_1) is complete and we leave the proof that (ℓ_2, d_2) and (ℓ_∞, d_∞) are complete as an exercise. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in ℓ_1. For each $n \in \mathbb{Z}^+$, write $a_n = (a_{n,k})_{k\geq 1} = (a_{n,1}, a_{n,2}, a_{n,3}, \cdots)$. Since $a_n \in \ell_1$ we have $\sum_{k=1}^{\infty} |a_{n,k}| < \infty$. Since $(a_n)_{n\geq 1}$ is Cauchy, for every $\epsilon > 0$ we can choose $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ we have $\|a_n - a_m\|_1 < \epsilon$, that is $\sum_{k=1}^{\infty} |a_{n,k} - a_{m,k}| < \epsilon$. For each fixed $k \in \mathbb{Z}^+$, note that for $n, m \geq N$ we have $|a_{n,k} - a_{m,k}| \leq \sum_{j=1}^{\infty} |a_{n,j} - a_{m,j}| < \epsilon$, and so the sequence $(a_{n,k})_{n\geq 1}$ is Cauchy in $\mathbb{R}$, so it converges. For each $k \in \mathbb{Z}^+$, let $b_k = \lim_{n\to\infty} a_{n,k} \in \mathbb{R}$ and let $b = (b_k)_{k\geq 1}$.

We claim that $b \in \ell_1$. Since $(a_n)_{n\geq 1}$ is Cauchy, for every $\epsilon > 0$ we can choose $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ we have $\|a_n - a_m\|_1 < \epsilon$, that is $\sum_{k=1}^{\infty} |a_{n,k} - a_{m,k}| < \epsilon$. By the Triangle Inequality, for $n, m \geq N$ we have $\big| \|a_n\|_1 - \|a_m\|_1 \big| \leq \|a_n - a_m\|_1 < \epsilon$ It follows that the sequence $\big(\|a_n\|\big)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Let $M = \lim_{n\to\infty} \|a_n\|_1 \in \mathbb{R}$. For each fixed $K \in \mathbb{Z}^+$ we have

$$\sum_{k=1}^{K} |b_k| = \sum_{k=1}^{K} \big| \lim_{n\to\infty} a_{n,k} \big| = \lim_{n\to\infty} \sum_{k=1}^{K} |a_{n,k}| \leq \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k}| = \lim_{n\to\infty} \|a_n\|_1 = M.$$

Since $\sum_{k=1}^{K} |b_k| \leq M$ for all $K \in \mathbb{Z}^+$ it follows that $\sum_{k=1}^{\infty} |b_k| \leq M$, so $b \in \ell_1$, as claimed.

Finally, we claim that $\lim_{n\to\infty} a_n = b$ in ℓ_1. Let $\epsilon > 0$. Choose $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ we have $\|a_n - a_m\|_1 < \epsilon$. Then for each $K \in \mathbb{Z}^+$ we have

$$\begin{aligned} \sum_{k=1}^{K} |a_{n,k} - b_k| &= \sum_{k=1}^{K} \big| a_{n,k} - \lim_{m\to\infty} a_{m,k} \big| = \lim_{m\to\infty} \sum_{k=1}^{K} |a_{n,k} - a_{m,k}| \\ &\leq \lim_{m\to\infty} \sum_{k=1}^{\infty} |a_{n,k} - a_{m,k}| = \lim_{m\to\infty} \|a_n - a_m\|_1 \leq \epsilon \end{aligned}$$

Since $\sum_{k=1}^{K} |a_{n,k} - b_k| \leq \epsilon$ for all $K \in \mathbb{Z}^+$ it follows that $\|a_n - b\|_1 = \sum_{k=1}^{\infty} |a_{n,k} - b_k| \leq \epsilon$.

6.11 Exercise: After showing that (ℓ_∞, d_∞) is complete, show that (ℓ_1, d_∞) and (ℓ_2, d_∞) are not closed in (ℓ_∞, d_∞) and so they are not complete.

6.12 Definition: For a metric space X, we define

$$\begin{aligned} \mathcal{B}(X) &= \mathcal{B}(X, \mathbb{R}) = \big\{ f : X \to \mathbb{R} \big| \; f \text{ is bounded} \big\} \\ \mathcal{C}(X) &= \mathcal{C}(X, \mathbb{R}) = \big\{ f : X \to \mathbb{R} \big| \; f \text{ is continuous} \big\}, \\ \mathcal{C}_b(X) &= \mathcal{C}_b(X, \mathbb{R}) = \big\{ f : X \to \mathbb{R} \big| \; f \text{ is bounded and continuous} \big\}. \end{aligned}$$

Note that $\mathcal{B}(X)$ is a normed linear space using the **supremum norm** given by

$$\|f\|_\infty = \sup_{x\in X} \big| f(x) \big|$$

and a metric space under the **supremum metric** given by $d_\infty(f, g) = \sup_{x\in X} \big| f(x) - g(x) \big|$.

6.13 Definition: For a sequence (f_n) of functions $f_n : X \to \mathbb{R}$ and a function $g : X \to \mathbb{R}$, we say that (f_n) **converges uniformly** to g on X, and write $f_n \to g$ uniformly on X, when

$$\forall \epsilon > 0\ \exists m \in \mathbb{Z}^+\ \forall x \in X\ \forall n \in \mathbb{Z}^+ \big(n \geq m \Longrightarrow |f_n(x) - g(x)| < \epsilon\big).$$

6.14 Note: For a sequence $(f_n) \in \mathcal{B}(X)$ and for $g \in \mathcal{B}(X)$, note that $|f_n(x) - g| < \epsilon$ for every $x \in X$ if and only if $\|f_n - g\|_\infty < \epsilon$. It follows that $f_n \to g$ uniformly on X if and only if $f_n \to g$ in the metric space $\big(\mathcal{B}(X), d_\infty\big)$.

6.15 Theorem: *Let X be a metric space. Then the metric spaces $\big(\mathcal{B}(X), d_\infty\big)$ and $\big(\mathcal{C}_b(X), d_\infty\big)$ are complete.*

Proof: Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\big(\mathcal{B}(X), d_\infty\big)$. Note that for each $x \in X$, we have $\big|f_n(x) - f_m(x)\big| \leq \sup_{y\in X} \big|f_n(y) - f_m(y)\big| = \|f_n - f_m\|_\infty$, and so the sequence $\big(f_n(x)\big)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Thus we can define a function $g : X \to \mathbb{R}$ by $g(x) = \lim\limits_{n\to\infty} f_n(x)$ and then we have $f_n \to g$ pointwise in X.

We claim that $g \in \mathcal{B}(X)$, that is we claim that g is bounded. Since (f_n) is a Cauchy sequence in $\mathcal{B}(X)$, it is bounded (by Part 1 of Theorem 6.3) so we can choose $M \geq 0$ such that $\|f_n\|_\infty \leq M$ for all indices n. Then for all $x \in X$ we have $|f_n(x)| \leq \|f_n\|_\infty \leq M$ and hence $|g(x)| = \lim\limits_{n\to\infty} |f_n(x)| \leq M$. Thus g is a bounded function, that is $g \in \mathcal{B}(X)$.

We know that $f_n \to g$ pointwise on X. We must show that $f_n \to g$ uniformly on X. Let $\epsilon > 0$. Since (f_n) is Cauchy in $\big(\mathcal{B}(X), d_\infty\big)$, we can choose $m \in \mathbb{Z}^+$ such that $\|f_k - f_\ell\|_\infty < \epsilon$ for all $k, \ell \geq m$. Then for all $k \geq m$ and for all $x \in X$ we have

$$\big|f_k(x) - g(x)\big| = \lim_{\ell\to\infty} \big|f_k(x) - f_\ell(x)\big| \leq \epsilon.$$

It follows that $f_n \to g$ uniformly on X, that is $f_n \to g$ in the metric space $\big(\mathcal{B}(X), d_\infty\big)$. Thus $\big(\mathcal{B}(X), d_\infty\big)$ is complete.

To show that $\big(\mathcal{C}_b(X), d_\infty\big)$ is complete, it suffices (by Theorem 6.4) to show that $\mathcal{C}_b(X)$ is closed in $\mathcal{B}(X)$. Let (f_n) be a sequence in $\mathcal{C}_b(X)$ which converges in $\big(\mathcal{B}(X), d_\infty\big)$. Let $g = \lim\limits_{n\to\infty} f_n$ in $\mathcal{B}(X)$. We need to show that g is continuous. Let $\epsilon > 0$ and let $a \in X$. Since $f_n \to g$ in $\big(\mathcal{B}(X), d_\infty\big)$ we know that $f_n \to g$ uniformly on X, so we can choose $m \in \mathbb{Z}^+$ such that $\big|f_m(x) - g(x)\big| < \frac{\epsilon}{3}$ for all $n \geq m$ and all $x \in X$. Since f_m is continuous at a we can choose $\delta > 0$ such that for all $x \in X$ with $d(x, a) < \delta$ we have $\big|f_m(x) - f_m(a)\big| < \frac{\epsilon}{3}$. Then for all $x \in X$ with $d(x, a) < \delta$ we have

$$\big|g(x) - g(a)\big| \leq \big|g(x) - f_m(x)\big| + \big|f_m(x) - f_m(a)\big| + \big|f_m(a) - g(a)\big| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

Thus g is continuous at a. Since a was arbitrary, g is continuous on X, hence $g \in \mathcal{C}_b(X)$. By the Sequential Characterization of Closed Sets (Part 3 of Theorem 5.16) it follows that $\mathcal{C}_b(X)$ is closed in $\mathcal{B}(X)$, as required.

6.16 Corollary: *The metric space $\big(\mathcal{C}[a, b], d_\infty\big)$ is complete.*

Proof: Since every continuous function $f : [a, b] \to \mathbb{R}$ is bounded, we have $\mathcal{C}[a, b] = \mathcal{C}_b[a, b]$.

6.17 Exercise: Show that the metric spaces $\big(\mathcal{C}[a, b], d_1\big)$ and $\big(\mathcal{C}[a, b], d_2\big)$ are not complete. Hint: in the case $[a, b] = [-1, 1]$, consider $f_n : [-1, 1] \to \mathbb{R}$ given by $f_n(x) = x^{1/2n-1}$ for $n \in \mathbb{Z}^+$. Show that if (f_n) did converge, either in $\big(\mathcal{C}[-1, 1], d_1\big)$ or in $\big(\mathcal{C}[-1, 1], d_2\big)$, then it would necessarily converge to a function g with $g(x) = 1$ when $x > 0$ and $g(x) = -1$ when $x < 0$, but such a function g cannot be continuous.

Compactness

6.18 Definition: Let X be a metric space (or a topological space) and let $A \subseteq X$. An **open cover** for A (in X) is a set S of open sets in X such that $A \subseteq \bigcup S = \bigcup_{U \in S} U$. When S is an open cover for A in X, a **subcover** of S for A is a subset $T \subseteq S$ such that $A \subseteq \bigcup T = \bigcup_{U \in T} U$. We say that A is **compact** (in X) when every open cover for A has a finite subcover.

6.19 Theorem: *Let $A \subseteq X \subseteq Y$ where Y is a metric space (or a topological space). Then A is compact in X if and only if A is compact in Y.*

Proof: Suppose that A is compact in X. Let T be an open cover for A in Y. For each $V \in T$, let $U_V = V \cap X$. By Theorem 4.49 (or Remark 4.50), each set U_V is open in X. Since $A \subseteq X$ and $A \subseteq \bigcup_{V \in T} V$, we also have $A \subseteq \bigcup_{V \in T} (V \cap X) = \bigcup_{V \in T} U_V$. Thus the set $S = \left\{ U_V \middle| V \in T \right\}$ is an open cover for A in X. Since A is compact in X we can choose a finite subcover, say $\left\{ U_{V_1}, \cdots U_{V_n} \right\}$ of S, where each $V_i \in T$. Since $A \subseteq \bigcup_{i=1}^n U_{V_i} = \bigcup_{i=1}^n (V_i \cap X)$, we also have $A \subseteq \bigcup_{i=1}^n V_i$ and so $\{V_1, \cdots, V_n\}$ is a finite subcover of T.

Suppose, conversely, that A is compact in Y. Let S be an open cover for A in X. For each $U \in S$, by Theorem 4.49 (or by Remark 4.50) we can choose an open set V_U in Y such that $U = V_U \cap X$. Then $T = \left\{ V_U \middle| U \in S \right\}$ is an open cover of A in Y. Since A is compact in Y we can choose a finite subcover, say $\left\{ V_{U_1}, \cdots, V_{U_n} \right\}$ of T, where each $U_i \in S$. Then we have $A \subseteq \bigcup_{i=1}^n (V_{U_i} \cap X) = \bigcup_{i=1}^n U_i$ and so $\{U_1, \cdots, U_n\}$ is a finite subcover of S.

6.20 Remark: Let $A \subseteq X$ where X is a metric space (or a topological space). By the above theorem, note that A is compact in X if and only if A is compact in itself. For this reason, we do not usually say that A is compact in X, we simply say that A is compact.

6.21 Theorem: *Let X be a metric space and let $A \subseteq X$. If A is compact then A is closed and bounded.*

Proof: Suppose that A is compact. We claim that A is closed. Let $b \in A^c$. For each $a \in A$, let $r_a = d(a, b) > 0$, let $U_a = B\big(b, \frac{r_a}{2}\big)$, and let $V_a = B\big(b, \frac{r_a}{2}\big)$ so that U_a and V_a are disjoint. Note that the set $S = \left\{ V_a \middle| a \in A \right\}$ is an open cover for A. Since A is compact we can choose a finite subcover, say $\{V_{a_1}, \cdots, V_{a_n}\}$ where each $a_k \in A$. Let $r = \min\{r_{a_1}, \cdots, r_{a_n}\}$ so that $B\big(b, \frac{r}{2}\big) \subseteq U_{a_k}$ for all k, and hence $B\big(b, \frac{r}{2}\big)$ is disjoint from each set V_{a_k}. Since $B\big(b, \frac{r}{2}\big)$ is disjoint from each set V_{a_k} and the sets V_{a_k} cover A, it follows that $B\big(b, \frac{r}{2}\big)$ is disjoint from A, hence $B\big(b, \frac{r}{2}\big) \subseteq A^c$. Thus A^c is open, hence A is closed.

We claim that A is bounded. Let $a \in A$. For each $n \in \mathbb{Z}^+$, let $U_n = B(a, n)$. Then the set $S = \{U_1, U_2, U_3, \cdots\}$ is an open cover for A. Since A is compact, we can choose a finite subcover, say $\{U_{n_1}, U_{n_2}, \cdots, U_{n,\ell}\} \subseteq S$, with each $n_i \in \mathbb{Z}^+$. Let $m = \max\{n_1, n_2, \cdots, n_\ell\}$ so that $U_{n_i} \subseteq U_m$ for all indices i. Then we have $A \subseteq \bigcup_{i=1}^{\ell} U_{n_i} = U_m = B(a, m)$ and so A is bounded.

6.22 Theorem: *Let X be a metric space (or a topological space) and let $A \subseteq X$. If X is compact and A is closed in X, then A is compact.*

Proof: Suppose that X is compact and A is closed in X. Let S be an open cover for A. Then $S \cup \{A^c\}$ is an open cover for X. Since X is compact, we can choose a finite subcover T of $S \cup \{A^c\}$. Note that T may or may not contain the set A^c but, in either case, $T \setminus \{A^c\}$ is an open cover for A with $T \setminus \{A^c\} \subseteq S$, so that $T \setminus \{A^c\}$ is a finite subcover of S.

Compactness in R^n

6.23 Definition: A **closed bounded rectangle** in $\mathbb{R}^n$ is a set of the form

$$\begin{aligned} R &= [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n] \\ &= \left\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n \middle| a_j\le x_j\le b_j \text{ for all } j\right\}. \end{aligned}$$

6.24 Theorem: *(Nested Rectangles) Let $(R_k)_{k\ge1}$ be a sequence of closed bounded rectangles in $\mathbb{R}^n$ with $R_1\supseteq R_2\supseteq R_3\supseteq\cdots$. Then*

$$\bigcap_{k=1}^{\infty} R_k \neq \emptyset.$$

Proof: Let $R_k=[a_{k,1},b_{k,1}]\times[a_{k,2},b_{k,2}]\times\cdots\times[a_{k,n},b_{k,n}]$. Since $R_1\supseteq R_2\supseteq\cdots$ it follows that for each index j with $1\le j\le n$ we have $[a_{1,j},b_{1,j}]\supseteq[a_{2,j},b_{2,j}]\supseteq[a_{3,j},b_{3,j}]\supseteq\cdots$. By the Nested Interval Theorem (Theorem 1.19), for each index j with $1\le j\le n$ we can choose $u_j\in\bigcap\limits_{k=1}^{\infty}[a_{k,j},b_{k,j}]$. Then for $u=(u_1,u_2,\cdots,u_n)$ we have $u\in\bigcap\limits_{k=1}^{\infty}R_k$.

6.25 Theorem: *(Compactness of Rectangles) Every closed bounded rectangles in $\mathbb{R}^n$ is compact (using the standard topology in $\mathbb{R}^n$).*

Proof: Let $R=I_1\times I_2\times\cdots\times I_n$ where $I_j=[a_j,b_j]$ with $a_j\le b_j$. Let d be the diameter of R, that is $d=\text{diam}(R)=\Big(\sum\limits_{j=1}^{n}(b_j-a_j)^2\Big)^{1/2}$. Let S be an open cover of R. Suppose, for a contradiction, that S does not have a finite subset which covers R. Let $a_{1,j}=a_j$, $b_{1,j}=b_j$, $I_{1,j}=I_j=[a_{1,j},b_{1,j}]$ and $R_1=R=I_{1,1}\times\cdots\times I_{1,n}$. Recursively, we construct rectangles $R=R_1\supseteq R_2\supseteq R_3\supseteq\cdots$, with $R_k=I_{k,1}\times\cdots\times I_{k,n}$ where $I_{k,j}=[a_{k,j},b_{k,j}]$, and $d_k=\text{diam}(R_k)=\Big(\sum\limits_{j=1}^{n}(b_{k,j}-a_{k,j})^2\Big)^{1/2}=\frac{d}{2^{k-1}}$, such that the open cover S does not have a finite subset which covers any of the rectangles R_k. We do this recursive construction as follows. Having constructed one of the rectangles R_k, we partition each of the intervals $I_{k,j}=[a_{k,j},b_{k,j}]$ into the two equal-sized subintervals $[a_{k,j},\frac{a_{k,j}+b_{k,j}}{2}]$ and $[\frac{a_{k,j}+b_{k,j}}{2},b_{k,j}]$, and we thereby partition the rectangle R_k into 2^n equal-sized sub-rectangles. We choose R_{k+1} to be equal to one of these 2^n sub-rectangles with the property that the open cover S does not have a finite subset which covers R_{k+1} (if each of the 2^n sub-rectangles could be covered by a finite subset of S then the union of theses 2^n finite subsets would be a finite subset of S which covers R_k).

By the Nested Rectangles Theorem, we can choose an element $u\in\bigcap\limits_{k=1}^{\infty}R_k$. Since $u\in R$ and S covers R we can choose an open set $U\in S$ such that $u\in U$. Since U is open we can choose $r>0$ such that $B(u,r)\subseteq U$. Since $d_k\to0$ we can choose k so that $d_k<r$. Since $u\in R_k$ and $\text{diam}R_k=d_k<r$ we have $R_k\subseteq B(u,r)\subseteq U$. Thus S does have a finite subset, namely $\{U\}$, which covers R_k, giving the desired contradiction.

6.26 Theorem: *(The Heine-Borel Theorem) Let $A\subseteq\mathbb{R}^n$. Then A is compact if and only if A is closed and bounded (using the standard topology in $\mathbb{R}^n$).*

Proof: If A is compact then A is closed and bounded by Theorem 6.21. Suppose that A is closed and bounded. Since A is bounded we can choose $r>0$ so that $A\subseteq B(0,r)$. Let $R=\left\{x\in\mathbb{R}^n \middle| |x_k|\le r \text{ for all } k\right\}$. Note that $B(0,r)\subseteq R$ since if $x=(x_1,\cdots,x_n)\in B(0,r)$, then for each index k we have $|x_k|=\left({x_k}^2\right)^{1/2}\le\left(\sum\limits_{i=1}^{n}{x_i}^2\right)^{1/2}=\|x\|<r$. Since A is closed and $A\subseteq R$ and R is compact, it follows that A is compact, by the Theorem 6.22.

Compact Sets and Continuous Maps

6.27 Theorem: *Let X and Y be metric spaces (or topological spaces) and let $f : X \to Y$. If X is compact and f is continuous then $f(X)$ is compact.*

Proof: Suppose that X is compact and f is continuous. Let T be an open cover for $f(X)$ in Y. Since f is continuous, so that $f^{-1}(V)$ is open in X for each $V \in T$, the set $S = \big\{f^{-1}(V)\big|V \in T\big\}$ is an open cover for X. Since X is compact, we can choose a finite subcover, say $\{f^{-1}(V_1), f^{-1}(V_2), \cdots, f^{-1}(V_n)\}$ of S, with each $V_i \in T$. Then the set $\{V_1, V_2, \cdots, V_n\}$ is a finite subcover of T for $f(X)$.

6.28 Example: Note that continuous maps do not necessarily send closed sets to closed sets. For example, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{2}{\pi} \tan^{-1}(x)$ sends the closed set $\mathbb{R}$ homeomorphically to the open interval $(-1, 1)$.

6.29 Theorem: *(The Extreme Value Theorem) Let X be a nonempty compact metric space (or topological space) and let $f : X \to \mathbb{R}$ be continuous. Then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*

Proof: Since X is compact and f is continuous, $f(X)$ is compact, hence $f(X)$ is closed and bounded. By the Supremum and Infimum Properties of $\mathbb{R}$, since $f(X)$ is nonempty and bounded, $M = \sup f(X)$ and $m = \inf f(X)$ are finite real numbers. By the Approximation Property of the Supremum and Infimum, M and m are both limits of sequences in $f(X)$, so they both lie in the closure of $f(X)$. Since $f(X)$ is closed in $\mathbb{R}$, we have $m, M \in f(X)$.

6.30 Theorem: *Let X and Y be metric spaces (or topological spaces) with X compact. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.*

Proof: Let $g = f^{-1} : Y \to X$. We need to prove that g is continuous. Let $A \subseteq X$ be closed in X. Since X is compact and $A \subseteq X$ is closed, it follows (from Theorem 6.22) that A is compact. Since the map $f : A \to Y$ is continuous and A is compact, it follows (from Theorem 6.27) that $f(A)$ is compact. Since $f(A)$ is compact it follows (from Theorem 6.21) that $f(A)$ is closed. Since $g = f^{-1}$ we have $g^{-1}(A) = f(A)$, which is closed. Since $g^{-1}(A)$ is closed in Y for every closed set A in X, it follows (by taking complements) that $g^{-1}(U)$ is open in Y for every open set U in X. Thus g is continuous, by the Topological Characterization of Continuity.

6.31 Example: In the above theorem, the requirement that X is compact is necessary. For example, if X is the interval $X = [0, 2\pi)$ and Y is the unit circle $Y = \big\{z \in \mathbb{C}\big|\|z\| = 1\big\}$, then the map $f : X \to Y$ given by $f(t) = e^{i\,t}$ is continuous and bijective, but the inverse map is not continuous at 1.

6.32 Theorem: *Let X and Y be metric spaces with X compact and let $f : X \to Y$ be continuous. Then f is uniformly continuous.*

Proof: Let $\epsilon > 0$. For each $a \in X$, since f is continuous at a we can choose $\delta_a > 0$ such that for all $x \in X$ with $d(x, a) < \delta_a$ we have $d\big(f(x), f(a)\big) < \frac{\epsilon}{2}$. The set of open balls $B\big(a, \frac{1}{2}\delta_a\big)$ with $a \in X$ is an open cover for X, and X is compact, so we can choose $a_1, a_2, \cdots, a_n \in X$ such that $X = B\big(a_1, \frac{1}{2}\delta_{a_1}\big) \cup \cdots \cup B\big(a_n, \frac{1}{2}\delta_{a_n}\big)$. Let $\delta = \min\big\{\frac{1}{2}\delta_{a_1}, \cdots, \frac{1}{2}\delta_{a_n}\big\}$. We claim that for all $x, y \in X$ with $d(x, y) < \delta$, we have $d\big(f(x), f(y)\big) < \epsilon$. Let $x, y \in X$ with $d(x, y) < \delta$. Since $X = B\big(a_1, \frac{1}{2}\delta_{a_1}\big) \cup \cdots \cup B\big(a_n, \frac{1}{2}\delta_{a_n}\big)$, we can choose an index k so that $x \in B\big(a_k, \frac{1}{2}\delta_{a_k}\big)$. Since $d(x, a_k) < \frac{1}{2}\delta_{a_k}$ and $d(x, y) < \delta \leq \frac{1}{2}\delta_{a_k}$, we have $d(y, a_k) < \delta_{a_k}$. Since $d(x, a_k) < \delta_{a_k}$ we have $d\big(f(x), f(a_k)\big) < \frac{\epsilon}{2}$ and since $d(y, a_k) < \delta_{a_k}$ we have $d\big(f(y), f(a_k)\big) < \frac{\epsilon}{2}$. Thus $d\big(f(x), f(y)\big) \leq d\big(f(x), f(a_k)\big) + d\big(f(a_k), f(y)\big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

6.33 Theorem: *Let X and Y be metric spaces (or topological spaces), and let $f : X \to Y$ be a homeomorphism Which means that f is bijective and both f and f^{-1} are continuous). Then for every set $A \subseteq X$, A is compact (in X) if and only if $f(A)$ is compact (in Y).*

Proof: This follows immediately from Theorem 6.27. Indeed, if A is compact (in X) then since $f : A \subseteq X \to Y$ is continuous (on A), it follows that $f(A)$ is compact (in Y) and, conversely, if $B = f(A)$ is compact (in Y) then since $f^{-1} : B \subseteq Y \to X$ is continuous it follows that $A = f^{-1}(B)$ is compact (in X).

6.34 Remark: When X and Y are metric spaces and $f : X \to Y$ is a homeomorphsm and $A \subseteq X$, it is *not* always the case that for every $A \subseteq X$, A is complete if and only if $f(A)$ is complete. For example, the map $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ given by $f(x) = \tan x$ is a homeomorphism, but $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is not complete and $\mathbb{R}$ is complete.

6.35 Theorem: *Let A be a subset of a finite-dimensional normed linear space U. Then A is compact if and only if A is closed and bounded.*

Proof: If A is compact (in X), then A is closed and bounded by Theorem 6.21. Suppose that A is closed and bounded. Let $\{u_1, u_2, \cdots, u_n\}$ be a basis for U and let $F : \mathbb{R}^n \to U$ be the bijective linear map given by $F(t) = \sum_{k=1}^{n} t_k u_k$. Recall (from Theorem 5.35) that F and F^{-1} are Lipschitz continuous. Let L be a Lipschitz constant for F. Since A is closed in U and F^{-1} is continuous, it follows (from Theorem 6.27) that $F(A) = \left(F^{-1}\right)^{-1}(A)$ is closed in $\mathbb{R}^n$. Since A is bounded (in U) and F is Lipschitz continuous, it follows that $F(A)$ is bounded in $\mathbb{R}^n$, indeed if $A \subseteq B(0, R)$ then for all $x \in A$ we have

$$\|Fx\| = \|Fx - F0\| \le L\|x - 0\| < LR$$

so that $F(A) \subseteq B(0, LR)$. Since $F(A)$ is closed and bounded in $\mathbb{R}^n$, it follows (from the Heine-Borel Theorem) that $F(A)$ is compact (in $\mathbb{R}^n$). Since $F(A)$ is compact (in $\mathbb{R}^n$) and F^{-1} is continuous, it follows (from Theorem 6.27) that $A = F^{-1}\big(F(A)\big)$ is compact (in U).

6.36 Exercise: Recall from linear algebra (or verify) that the space $M_{n\times m}(\mathbb{R})$ of $n \times m$ matrices with entries in $\mathbb{R}$ is an inner-product space with inner product given by

$$\langle A, B\rangle = \text{trace}(B^T A) = \sum_{k=1}^{n}\sum_{\ell=1}^{m} A_{k,\ell}B_{k,\ell},$$

and with standard orthonormal basis $\left\{E_{k,\ell}\,\middle|\, 1 \le k \le n, 1 \le \ell \le m\right\}$ where $E_{k,\ell}$ is the $n \times m$ matrix whose (k, ℓ) entry is equal to 1 and all other entries are zero. The linear map $L = L_{n\times m} : M_{n\times m}(\mathbb{R}) \to \mathbb{R}^{nm}$ given by $L(E_{k,\ell}) = e_{(k-1)n+\ell}$ or, equivalently, by

$$L(u_1, \cdots, u_n) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

(where each $u_k \in \mathbb{R}^n$) is an inner product space isomorphism.

Show that the set $S = \left\{A \in M_{n\times m}(\mathbb{R}) \,\middle|\, A^T A = I\right\}$ is compact by showing that it is closed and bounded. To show that S is bounded, first show that $A \in S$ if and only if the columns of A are orthonormal. To show that S is closed, first use the isomorphisms $L_{n\times m}$ and $L_{p\times q}$ to show that a function $F : M_{n\times m}(\mathbb{R}) \to M_{p\times q}(\mathbb{R})$ is continuous if and only if each component function $F : M_{k\times\ell}(\mathbb{R}) \to \mathbb{R}$ $\big($given by $F_{k,\ell}(X) = F(X)_{k,\ell}\big)$ is continuous as a function of the entries $X_{i,j}$, of the matrix $X \in M_{n\times m}(\mathbb{R})$, hence show that the function $F : M_{n\times m}(\mathbb{R}) \to M_{m\times m}(\mathbb{R})$ given by $F(X) = X^T X$ is continuous, then show that S is closed by noting that $S = F^{-1}\big(\{I\}\big)$.

Some Characterizations of Compactness

6.37 Definition: Let X be a metric space. We say that X is **totally bounded** when for every $\epsilon > 0$ there exists a finite subset $\{a_1, a_2, \cdots, a_n\} \subseteq X$ such that $X = \bigcup_{i=1}^{n} B(a_i, \epsilon)$. We say that X has the **finite intersection property on closed sets** when for every set T of closed sets in X, if every finite subset of T has non-empty intersection, then T has non-empty intersection.

6.38 Theorem: *Let X be a metric space. Then the following are equivalent.*
(1) X is compact.
(2) X has the finite intersection property on closed sets.
(3) Every sequence (x_n) in X has a convergent subsequence.
(4) Every infinite subset $A \subseteq X$ has a limit point.
(5) X is complete and totally bounded.

Proof: First we prove that (1) implies (2). Suppose that X is compact. Let T be a set of closed sets in X. Suppose that T has empty intersection, that is suppose $\bigcap_{A \in T} A = \emptyset$. Then $\bigcup_{A \in T} A^c = X$ so the set $S = \{A^c \big| A \in T\}$ is an open cover for X. Since X is compact, we can choose a finite subcover, say $\{{A_1}^c, \cdots, {A_n}^c\}$ of S for X. Then we have $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$, showing that some finite subset of T has empty intersection.

Next we prove that (2) implies (3). Suppose X has the finite intersection property on closed sets. Let $(x_n)_{n \geq 1}$ be a sequence in X. For each $m \in \mathbb{Z}^+$, let $A_m = \overline{\{x_n | n > m\}}$ and note that each A_m is closed with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Let $T = \{A_m | m \in \mathbb{Z}^+\}$. Note that every finite subset of T has non-empty intersection because given $A_{m_1}, \cdots, A_{m_\ell} \in T$ we can let $m = \max\{m_1, \cdots, m_\ell\}$ and then we have $\bigcap_{i=1}^{\ell} A_{m_i} = A_m$ and we have $x_n \in A_m$. Since X has the finite intersection property on closed sets, it follows that T has non-empty intersection. Choose a point $a \in \bigcap_{m=1}^{\infty} A_m$. We construct a subsequence $(x_{n_k})_{k \geq 1}$ of $(x_n)_{n \geq 1}$ with $\lim\limits_{k \to \infty} x_{n_k} = a$ as follows. Since $a \in A_1 = \overline{\{x_n | n > 1\}}$ we can choose $n_1 > 1$ such that $d(x_{n_1}, a) < 1$. Since $a \in A_{n_1} = \overline{\{x_n | n > n_1\}}$ we can choose $n_2 > n_1$ such that $d(x_{n_2}, a) < \frac{1}{2}$. Since $a \in A_{n_2} = \overline{\{x_n | n > n_2\}}$ we can choose $n_3 > n_2$ such that $d(x_{n_3}, a) < \frac{1}{3}$. Repeating this procedure, we can choose $1 < n_1 < n_2 < n_3 < \cdots$ such that $d(x_{n_k}, a) < \frac{1}{k}$ for all indices k, and then we have constructed a subsequence (x_{n_k}) such that $\lim\limits_{k \to \infty} x_{n_k} = a$.

Next we prove that (3) implies (4). Suppose that every sequence (x_n) in X has a convergent subsequence. Let $A \subseteq X$ be an infinite subset. Choose a sequence (x_n) in A with the terms x_n all distinct. Choose a convergent subsequence (x_{n_k}) of (x_n) and let $a = \lim\limits_{k \to \infty} x_{n_k}$. Then a is a limit point of the set A.

Now let us prove that (4) implies (5). Suppose that every infinite subset $A \subseteq X$ has a limit point. We claim that X is complete. Let (x_n) be a Cauchy sequence in X. We claim that (x_n) has a convergent subsequence. If the set $\{x_n | n \in \mathbb{Z}^+\}$ is finite, then some term in the sequence occurs infinitely often, so we can choose indices $n_1 < n_2 < n_3 < \cdots$ such that $x_1 = x_2 = x_3 = \cdots$, and so in this case (x_n) has a constant subsequence. Suppose the set $\{x_n | n \in \mathbb{Z}^+\}$ is infinite. Let a be a limit point of the infinite set $A = \{x_n | n \in \mathbb{Z}^+\}$. Since a is a limit point of the set $\{x_n\}$ we can choose indices n_k with $n_1 < n_2 < n_3 < \cdots$ such that $0 < d(x_{n_k}, a) < \frac{1}{k}$ for each index k. Then (x_{n_k}) is a subsequence of (x_n) with $\lim\limits_{k \to \infty} x_{n_k} = a$. Since the sequence (x_n) Cauchy and has a convergent subsequence, it follows, from Part 3 of Theorem 6.3, that the sequence (x_n) converges. Thus X is complete, as claimed.

Continuing our proof that (4) implies (5), suppose that X is not totally bounded. Choose $\epsilon > 0$ such that there do not exist finitely many points $a_1, \cdots, a_n \in X$ for which $X = \bigcup_{i=1}^n B(a_i, \epsilon)$. Let $a_1 \in X$. Since $X \neq B(a_1, \epsilon)$ we can choose $a_2 \in X$ with $a_1 \notin B(a_1, \epsilon)$. Since $X \neq B(a_1, \epsilon) \cup B(a_2, \epsilon)$ we can choose $a_3 \in X$ with $a_3 \notin B(a_1, \epsilon) \cup B(a_2, \epsilon)$. Repeat this procedure to choose points $a_1, a_2, a_3, \cdots$ with $a_{n+1} \notin \bigcup_{k=1}^n B(a_k, \epsilon)$. Then the set $A = \{a_n | n \in \mathbb{Z}^+$ is an infinite subset of X which has no limit point.

Finally we prove that prove that (5) implies (1). Suppose that X is complete and totally bounded. Suppose, for a contradiction, that X is not compact, and choose an open cover S for X which has no finite subcover for X. Since X is totally bounded, we can cover X by finitely many balls of radius 1. Choose one of the balls, say $U_1 = B(a_1, 1)$ such that there is no finite subcover of S for U_1 (if there was a finite subcover for each ball, then the union of all these subcovers would be a finite subcover for X). Since X is totally bounded, we can cover X (hence also U_1) by finitely many balls of radius $\frac{1}{2}$. Choose one of these balls, say $U_2 = B(a_2, \frac{1}{2})$ such that there is no finite subcover of S for $U_1 \cap U_2$. Repeat the procedure to obtain balls $U_n = B\big(a_n, \frac{1}{n}\big)$ such that, for each n, there is no finite subcover of S for $\bigcap_{k=1}^n U_k$. In particular, each intersection $\bigcap_{k=1}^n U_k$ is nonempty so we can choose an element $x_n \in \bigcap_{k=1}^n U_k$. Since for all $k, \ell \geq m$ we have $x_k, x_\ell \in U_m = B\big(a_m, \frac{1}{m}\big)$ it follows that (x_n) is Cauchy. Since X is complete, it follows that (x_n) converges in X. Let $a = \lim\limits_{n \to \infty} x_n$. Since S covers X we can choose $U \in S$ with $a \in U$. Since U is open we can choose $r > 0$ such that $B(a, r) \subseteq U$. Since $x_n \to a$ we can choose $m > \frac{3}{r}$ such that $d(x_m, a) < \frac{r}{3}$. Then for all $x \in U_m = B\big(a_m, \frac{1}{m}\big)$ we have $d(x, a) \leq d(x, a_m) + d(a_m, x_m) + d(x_m, a) < \frac{1}{m} + \frac{1}{m} + \frac{r}{3} < r$, and so $U_m \subseteq B(a, r) \subseteq U$. But then S has a finite subcover for U_m, namely the singleton $\{U\}$, which contradicts the fact that S has no finite subcover for $\bigcap_{k=1}^m U_k$.

6.39 Example: Show that in the metric space $\big(\mathcal{C}[0,1], d_\infty\big)$, the closed unit ball $\overline{B}(0,1)$ is not compact.

Solution: Let $f_n(x) = x^n$ for $n \in \mathbb{Z}^+$. Note that $\|f_n\|_\infty = 1$ so that each $f_n \in \overline{B}(0,1)$. Note that the pointwise limit of the sequence (f_n) is the function $g : [0,1] \to \mathbb{R}$ given by $g(x) = 0$ when $x < 1$ and $g(1) = 1$, which is not continuous. If some subsequence (f_{n_k}) of (f_n) were to converge in $\big(\mathcal{C}[0,1], d_\infty\big)$ then it would need to converge uniformly on $[0,1]$ to the function g. But this is not possible since the uniform limit of a sequence of continuous functions is always continuous. Thus (f_n) has no convergent subsequence and so $\overline{B}(0,1)$ is not compact.