PMATH 321 Non-Euclidean Geometry, Solutions to the Exercises for Chapter 4

1: (a) Let C be the Euclidean circle in $\mathbb{R}^2$ with diameter from $a = (1,4)$ to $b = (3,-2)$, and let D be the Euclidean circle in $\mathbb{R}^2$ with diameter from $c = (4,2)$ to $d = (3,3)$. Find the (Euclidean) area of the image of D under the reflection F_C.

Solution: The centre of circle C is $p = (2,1)$ and the radius is $r = \sqrt{10}$. Circle D also has diameter from $u = (3,2)$ to $v = (4,3)$. Since u and v lie along the same ray from p, we see (from Theorem 4.6) that the image of D under F_C is the circle E with diameter from $F_C(u)$ to $F_C(v)$. We have

$$F_C(u) = p + \tfrac{r^2}{|u-p|^2}(u-p) = (2,1) + \tfrac{10}{2}(1,1) = (7,6) \text{ and}$$
$$F_C(v) = p + \tfrac{r^2}{|v-p|^2}(v-p) = (2,1) + \tfrac{10}{8}(2,2) = \left(\tfrac{9}{2}, \tfrac{7}{2}\right).$$

The radius of E is $s = \frac{1}{2}\left|(7,6) - \left(\frac{9}{2},\frac{7}{2}\right)\right| = \frac{1}{2}\left|\left(\frac{5}{2},\frac{5}{2}\right)\right| = \frac{5\sqrt{2}}{4}$ so the area is $A = \pi s^2 = \frac{25\pi}{8}$.

(b) Let C be the Euclidean circle in $\mathbb{R}^2$ of radius $r = 5$ centred at $p = (-1,-1)$ and let T be the Euclidean triangle in $\mathbb{R}^2$ with vertices at $a = (1,0)$, $b = (3,1)$ and $c = (0,2)$. Find the (Euclidean) area of the image of T under the reflection F_C.

Solution: Let

$$a' = F_C(a) = p + \tfrac{r^2}{|a-p|^2}(a-p) = (-1,-1) + \tfrac{25}{5}(2,1) = (9,4),$$
$$b' = F_C(b) = p + \tfrac{r^2}{|b-p|^2}(b-p) = (-1,-1) + \tfrac{25}{20}(4,2) = \left(4,\tfrac{3}{2}\right), \text{ and}$$
$$c' = F_C(c) = p + \tfrac{r^2}{|c-p|^2}(c-p) = (-1,-1) + \tfrac{25}{10}(1,3) = \left(\tfrac{3}{2},\tfrac{13}{2}\right).$$

Again, we make use of Theorem 4.6. Since a and b lie on the same ray from p, the line segment from a to b is mapped by F_C to the line segment from a' to b'. Since a is the point on line ac nearest to p, the line ac is mapped by F_C to the circle D with diameter p, a', so the line segment from a to c is mapped to the arc from a' counterclockwise to c' along D. Since c is the point on line bc nearest to p, the line bc is mapped by F_C to the circle E with diameter p, c', so the line segment from b to c is mapped to the arc from b' counterclockwise to c' along E. The circle D with diameter p, a' has centre at $\frac{1}{2}(p+a') = \left(4,\frac{3}{2}\right) = b'$ and radius $s = \frac{1}{2}|a'-p| = \frac{5\sqrt{5}}{2}$. The circle E with diameter p, c' has centre at $u' = \frac{1}{2}(p+c') = \left(\frac{1}{4},\frac{11}{4}\right)$ and radius $t = \frac{1}{2}|c'-p| = \frac{5\sqrt{10}}{4}$. The area A of the image $F_C(T)$ is equal to $\frac{1}{4}$ of the area of D (since the arc from a' to c' subtends the angle $\frac{\pi}{2}$ at b') plus the area of the triangle b', c', u' minus $\frac{1}{4}$ of the area of E (since the arc from b' to c' subtends the angle $\frac{\pi}{2}$ at u'), so we have

$$A = \tfrac{1}{4}\pi s^2 + \tfrac{1}{2}t^2 - \tfrac{1}{4}\pi t^2 = \tfrac{\pi}{4}\cdot\tfrac{125}{4} + \tfrac{1}{2}\cdot\tfrac{125}{8} - \tfrac{\pi}{4}\cdot\tfrac{125}{8} = \tfrac{125}{32}(\pi+2).$$

2: (a) Find the centre $p \in \mathbb{R}^2$ and radius $r > 0$ of the Euclidean circle C in $\mathbb{R}^2$ such that the reflection F_C sends the line L with equation $2x + y = 8$ to the circle D with equation $(x+1)^2 + y^2 = 5$ (with one point removed).

Solution: The centre of D is the point $d = (-1, 0)$. Note that if a is the point on L nearest to p and $a' = F_C(a)$, so that the circle $D = F_C(L)$ has diameter p, a', then the line p, a' passes through d and meets the line L orthogonally at a. This shows that the points p and a' both lie on the line M through d which meets L orthogonally at a. This line M has equation $x - 2y = -1$, and it meets L at the point $a = (3, 2)$ and it meets D at the two points $(-3, -1)$ and $(1, 1)$. Since a and a' must lie on the same ray from p (so p does not lie between a and a') it follows that $p = (-3, -1)$ and $a' = (1, 1)$. In order that $F_C(a) = a'$ we need $r^2 = |a - p||a' - p| = 3\sqrt{5} \cdot 2\sqrt{5} = 30$ and so $r = \sqrt{30}$. Thus $p = (-3, -1)$ and $r = \sqrt{30}$.

(b) Find the centre $p \in \mathbb{R}^2$ and the radius $r > 0$ of the Euclidean circle C in $\mathbb{R}^2$ such that the reflection F_C sends the Euclidean circle D with diameter from $a = (2, 1)$ to $b = (1, 2)$ to the Euclidean circle E with diameter from $c = (-1, 3)$ to $d = (6, 2)$.

Solution: Note that when F_C sends the circle with diameter through u and v to the circle with diameter through $u' = F_C(u)$ and $v' = F_C(v)$ where $0 \neq u \in \mathbb{R}^2$ and $v = u + t(v - u)$ with $0 \neq t \in \mathbb{R}$, all of the points u, v, u', v' lie on the same line through p. Also note that the centres of the two circles also lie on this same line. The given circles D and E have centres $\left(\frac{3}{2}, \frac{3}{2}\right)$ and $\left(\frac{5}{2}, \frac{5}{2}\right)$ and so the points u, v, u', v' and p must all lie on the line through these two centres, namely the line $y = x$. The line $y = x$ meets the circle D at the points $(1, 1)$ and $(2, 2)$ so we can take $u = (1, 1)$ and $v = (2, 2)$. The line $y = x$ meets the circle E at the points $(0, 0)$ and $(5, 5)$ so either we must take $u' = (0, 0)$ and $v' = (5, 5)$ or we must take $u' = (5, 5)$ and $v' = (0, 0)$. In order to have u and u' lie on the same ray through p and to have v and v' lie on the same ray through p we must choose $u' = (0, 0)$ and $v' = (5, 5)$ with p on the line $y = x$ between $u = (1, 1)$ and $v = (2, 2)$.

Let $p = t(1, 1)$ with $1 < t < 2$. Then to get $F_C(u) = u'$ and $F_C(v) = v'$ we need $|u - p||u' - p| = r^2$ and $|v - p||v' - p| = r^2$ so

$$\begin{aligned} |u-p||u'-p| = |v-p||v'-p| &\implies \big|(1,1) - t(1,1)\big|\big|(0,0) - t(1,1)\big| = \big|2(1,1) - t(1,1)\big|\big|5(1,1) - t(1,1)\big| \\ &\implies (t-1)\sqrt{2} \cdot t\sqrt{2} = (2-t)\sqrt{2} \cdot (5-t)\sqrt{2} \implies t(t-1) = (t-2)(t-5) \\ &\implies t^2 - t = t^2 - 7t + 10 \implies 6t = 10 \implies t = \tfrac{5}{3} \end{aligned}$$

Thus we must take $p = t(1, 1) = \frac{5}{3}(1, 1)$ and $r^2 = |u - p||u' - p| = \left|\frac{2}{3}(1, 1)\right|\left|\frac{5}{3}(1, 1)\right| = \frac{20}{9}$, so $r = \frac{2\sqrt{5}}{3}$.

3: (a) Let $0 < a < \frac{\sqrt{3}}{2}$. Find the hyperbolic length of the Euclidean line segment given by $(x, y) = \alpha(t) = \left(\frac{1}{2}, t\right)$ for $0 \le t \le a$.

Solution: We have $\alpha'(t) = (0, 1)$ so the hyperbolic arclength is

$$\begin{aligned} L &= \int_{t=0}^{a} \frac{2\,|\alpha'(t)|}{1 - |\alpha(t)|^2}\,dt = \int_{t=0}^{a} \frac{2\,dt}{1 - \left(\frac{1}{4} + t^2\right)} = \int_{t=0}^{a} \frac{8\,dt}{3 - 4t^2} = \int_{t=0}^{a} \frac{\frac{4}{\sqrt{3}}}{\sqrt{3} - 2t} + \frac{\frac{4}{\sqrt{3}}}{\sqrt{3} - 2t}\,dt \\ &= \left[\frac{2}{\sqrt{3}} \ln\left(\frac{\sqrt{3} + 2t}{\sqrt{3} - 2t}\right)\right]_{t=0}^{a} = \frac{2}{\sqrt{3}} \ln\left(\frac{\sqrt{3} + 2a}{\sqrt{3} - 2a}\right). \end{aligned}$$

(b) Let $0 < a < 1$. Find the hyperbolic area of the circle given by $x^2 + y^2 = ax$.

Solution: We give two solutions. For a short solution, note that the hyperbolic diameter d of this circle is the hyperbolic length of the straight line segment from $(0, 0)$ to $(a, 0)$, which is equal to $d = \ln \frac{1+a}{1-a}$, so the radius is $r = \frac{1}{2} \ln \frac{1+a}{1-a} = \ln \sqrt{\frac{1-a}{1+a}}$. Thus the area is

$$A = 2\pi(\cosh r - 1) = 2\pi\Big(\cosh\left(\ln\sqrt{\tfrac{1+a}{1-a}}\right) - 1\Big) = 2\pi\Big(\tfrac{1}{2}\Big(\sqrt{\tfrac{1+a}{1-a}} + \sqrt{\tfrac{1-a}{1+a}}\Big) - 1\Big) = 2\pi\Big(\tfrac{1}{\sqrt{1-a^2}} - 1\Big).$$

Here is a second solution. The area is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a\cos\theta} \frac{4r}{(1 - r^2)^2}\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{2}{1 - r^2}\right]_{r=0}^{a\cos\theta} d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \frac{2}{1 - a^2\cos^2\theta} - 2\,d\theta = -2\pi + \int_{\theta=0}^{\pi/2} \frac{4\,d\theta}{1 - a^2\cos^2\theta} \\ &= -2\pi + \int_{\theta=0}^{\frac{\pi}{2}} \frac{4\,d\theta}{1 - a^2\left(\frac{1+\cos 2\theta}{2}\right)} = -2\pi + \int_{\theta=0}^{\pi/2} \frac{8\,d\theta}{(2 - a^2) - a^2\cos 2\theta} \\ &= -2\pi + \int_{\phi=0}^{\pi} \frac{4\,d\phi}{(2 - a^2) - a^2\cos\phi}\ \text{, where } \phi = 2\theta \\ &= -2\pi + \int_{u=0}^{\infty} \frac{\frac{8}{1+u^2}\,du}{(2 - a^2) - a^2\,\frac{1-u^2}{1+u^2}}\ \text{, where } u = \tan\tfrac{\phi}{2}\text{ , } \cos\phi = \tfrac{1-u^2}{1+u^2}\text{ , } d\phi = \tfrac{2}{1-u^2}\,du \\ &= -2\pi + \int_{u=0}^{\infty} \frac{8\,du}{(2 - a^2)(1 + u^2) - a^2(1 - u^2)} \\ &= -2\pi + \int_{u=0}^{\infty} \frac{8\,du}{2 + 2u^2 - 2a^2} = -2\pi + \int_{u=0}^{\infty} \frac{4\,du}{u^2 + (1 - a^2)} \\ &= -2\pi + \int_{\psi=0}^{\pi/2} \frac{4\sqrt{1 - a^2}\,\sec^2\psi\,d\psi}{\left(\sqrt{1 - a^2}\sec\psi\right)^2}\ \text{, where } \tan\psi = \tfrac{u}{\sqrt{1-a^2}} \\ &= -2\pi + \left[\tfrac{4}{\sqrt{1-a^2}}\,\psi\right]_{\psi=0}^{\pi/2} = -2\pi + \tfrac{4}{\sqrt{1-a^2}} \cdot \tfrac{\pi}{2} = 2\pi\big(\tfrac{1}{\sqrt{1-a^2}} - 1\big). \end{aligned}$$

4: (a) Let $u = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $v = \left(\frac{3}{4}, \frac{1}{4}\right)$. Find the centre $p \in \mathbb{R}^2$ and the radius $r > 0$ of the Euclidean circle C in $\mathbb{R}^2$ such that $L = C \cap \mathbb{H}^2$ is the hyperbolic line in $\mathbb{H}^2$ through u and v.

Solution: Let $L = C_E(p, r) \cap \mathbb{H}^2$ with $p = (x, y)$. As shown in the proof of Theorem 4.18, in order to have $u \in L$ we need $p \cdot u = \frac{|u|^2+1}{2}$, that is $\frac{1}{2}x + \frac{1}{2}y = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$, or equivalently $x + y = \frac{3}{2}$ (1), and in order to have $v \in L$ we need $p \cdot v = \frac{|v^2|+1}{2}$, that is $\frac{3}{4}x + \frac{1}{4}y = \frac{\frac{5}{8}+1}{2} = \frac{13}{16}$, or equivalently $3x + y = \frac{13}{4}$ (2). Subtract Equation (1) from Equation (2) to get $2x = \frac{7}{4}$ so that $x = \frac{7}{8}$, then put this into Equation (1) to get $y = \frac{3}{2} - x = \frac{3}{2} - \frac{7}{8} = \frac{5}{8}$. Thus we must take $p = (x, y) = \left(\frac{7}{8}, \frac{5}{8}\right)$ and, by Note 4.14, $r = \sqrt{|p|^2 - 1} = \sqrt{\frac{49+25-64}{64}} = \frac{\sqrt{10}}{8}$.

(b) Let $u = \left(-\frac{1}{5}, \frac{3}{5}\right)$ and $v = \left(\frac{4}{5}, -\frac{2}{5}\right)$. Find the centre $p \in \mathbb{R}^2$ and the radius $r > 0$ of the Euclidean circle C in $\mathbb{R}^2$ such that the $L = C \cap \mathbb{H}^2$ is the hyperbolic line such that $F_C(u) = v$.

Solution: By the proof of Theorem 4.20 we can take $p = u + t(v - u)$ with $t = \frac{1-|u|^2}{|v|^2-|u|^2}$. Thus we let

$$t = \frac{1 - |u|^2}{|v|^2 - |u|^2} = \frac{1 - \frac{2}{5}}{\frac{4}{5} - \frac{2}{5}} = \frac{3}{2},$$
$$p = u + t(v - u) = \left(-\frac{1}{5}, \frac{3}{5}\right) + \frac{3}{2}(1, -1) = \left(\frac{13}{10}, -\frac{9}{10}\right) \text{ and}$$
$$r = \sqrt{|p|^2 - 1} = \sqrt{\frac{169+81-100}{100}} = \frac{\sqrt{150}}{10} = \frac{\sqrt{6}}{2}.$$

(c) Let $u = \left(-\frac{3}{5}, \frac{4}{5}\right)$, let $b = \left(\frac{6}{5}, \frac{2}{5}\right)$, let $s = \sqrt{|b|^2 - 1}$, let C be the Euclidean circle in $\mathbb{R}^2$ centred at b of radius s, and let $L = C \cap \mathbb{H}^2$. Find the centre $a \in \mathbb{R}^2$ and the radius $r > 0$ of a Euclidean circle D in $\mathbb{R}^2$ such that $M = D \cap \mathbb{H}^2$ is the hyperbolic line which is asymptotic to u and intersects orthogonally with L.

Solution: Let $a = (x, y)$. In order for D to intersect orthogonally with $\mathbb{S}^1$, by Note 4.14 we need $r = \sqrt{|a|^2 - 1}$. In order to have $u \in D$, by the proof of Theorem 4.18 we need $a \cdot u = \frac{|u|^2+1}{2}$, that is $-\frac{3}{5}x + \frac{4}{5}y = 1$ or equivalently $-3x + 4y = 5$ (1). In order for D to intersect orthogonally with C at say the point q, the points a, b and q must form a right-angled Euclidean triangle with side lengths s, r and $|a - b|$ so that, by Pythagoras' Theorem, we need $s^2 + r^2 = |a - b|^2$ hence $|b|^2 - 1 + |a|^2 - 1 = |a|^2 - 2a \cdot b + |b|^2$ which simplifies to $a \cdot b = 1$, that is $\frac{6}{5}x + \frac{2}{5}y = 1$, or equivalently $6x + 2y = 5$ (2). Solve Equations (1) and (2) to get $a = (x, y) = \left(\frac{1}{3}, \frac{3}{2}\right)$ and hence $r = \sqrt{|a|^2 - 1} = \frac{7}{6}$.

5: (a) Let $a = \left(\frac{1}{5}, \frac{6}{5}\right)$ and $b = \left(\frac{4}{5}, -\frac{6}{5}\right)$, let C and D be the Euclidean circles centred at a and b, respectively, which intersect orthogonally with $\mathbb{S}^1$, and let $L = C \cap \mathbb{H}^2$ and $M = D \cap \mathbb{H}^2$. Find the centre p and radius r of the Euclidean circle E such that $N = E \cap \mathbb{H}^2$ is the hyperbolic line which intersects both L and M orthogonally.

Solution: Let s and t be the radii of the circles C and D respectively. Since C and D intersect orthogonally with $\mathbb{S}^1$ we have $s^2 = |a|^2 - 1$ and $t^2 = |b|^2 - 1$. In order that E intersects orthogonally with all three circles $\mathbb{S}^1$, C and D, by Pythagoras' Theorem we need $|p|^2 = r^2 + 1$, $|p-a|^2 = r^2 + s^2$ and $|p-b|^2 = r^2 + t^2$. When $r^2 = |p|^2 - 1$ we have

$$|p-a|^2 = r^2 + s^2 \iff |p|^2 - 2p \bullet a + |a|^2 = (|p|^2 - 1) + (|a|^2 - 1) \iff p \bullet a = 1$$

and similarly $|p-b|^2 = r^2 + t^2 \iff p \bullet b = 1$. Thus we must take p to be the (unique) point such that $p \bullet a = 1$ and $p \bullet b = 1$. For $p = (x, y)$ we have

$$\begin{aligned} p \bullet a = 1 &\iff \tfrac{1}{5}x + \tfrac{6}{5}y = 1 \iff x + 6y = 5 \quad (1) \\ p \bullet b = 1 &\iff \tfrac{4}{5}x - \tfrac{6}{5}y = 1 \iff 4x - 6y = 5 \quad (2) \end{aligned}$$

Add (1) and (2) to get $5x = 10$ so that $x = 2$, then put this into (1) to get $6y = 5 - x = 5 - 2 = 3$ so that $y = \frac{1}{2}$. Thus we must take $p = (x, y) = \left(2, \frac{1}{2}\right)$ and $r = \sqrt{|p|^2 - 1} = \sqrt{\frac{16+1-4}{4}} = \frac{\sqrt{13}}{2}$.

(b) Let $u \in \mathbb{H}$ and let L by a hyperbolic line in $\mathbb{H}$. Prove that there exists a unique hyperbolic line M which contains u and intersects orthogonally with L.

Solution: First let us consider the case that $L = N \cap \mathbb{H}^2$ where N is a line in $\mathbb{R}^2$ through 0. Let $p \in \mathbb{R}^2$ with $|p| > 1$ and let C be the circle centred at p of radius $r = \sqrt{|p|^2 - 1}$. Note that C intersects orthogonally with N if and only if $p \in N$ and note that

$$u \in C \iff |u - p|^2 = r^2 \iff |u|^2 - 2p \bullet u + |p|^2 = |p|^2 - 1 \iff p \bullet u = \tfrac{|u|^2+1}{2}.$$

We also remark that, when $u \neq 0$, the point on the line $x \bullet u = \frac{|u|^2+1}{2}$ which is nearest the origin is the point $x = \frac{|u|^2+1}{2|u|^2}\, u$ which has norm $|x| = \frac{|u|^2+1}{2} > 1$, so any point p which lies on this line satisfies $|p| > 1$.

When $u = 0$, the (unique) line in $\mathbb{R}^2$ through u perpendicular to N passes through 0 (hence determines a hyperbolic line) but there is no point $p \in \mathbb{R}^2$ for which $p \bullet u = \frac{|u|^2+1}{2}$. When $u \neq 0$ and u is orthogonal to N, the (unique) line in $\mathbb{R}^2$ through u perpendicular to N passes through 0 (hence determines a hyperbolic line) but there is no point $p \in M$ for which $p \bullet u = \frac{|u|^2+1}{2}$ since the line $x \bullet u = \frac{|u|^2+1}{2}$ is parallel to N. When $u \neq 0$ and u is not orthogonal to N, the line in $\mathbb{R}^2$ through u perpendicular to N does not pass through 0, and there exists a unique point $p \in N$ with $p \bullet u = \frac{|u|^2+1}{2}$ because the line $x \bullet u = \frac{|u|^2+1}{2}$ is not parallel to N. In all cases we find that there is a unique hyperbolic line through u orthogonal to N.

Now let us consider the case that $L = D \cap \mathbb{H}^2$ where D is the circle in $\mathbb{R}^2$ centred at $a \in \mathbb{R}^2$ with $|a| > 1$ of radius $s = \sqrt{|a|^2 - 1}$. Again, let $p \in \mathbb{R}^2$ and let C be the circle in $\mathbb{R}^2$ centred at p with radius $r = \sqrt{|p|^2 - 1}$. As above, we have $u \in C$ if and only if $p \bullet u = \frac{|u|^2+1}{2}$. As in Part (a), C intersects orthogonally with D, say at q, if and only if the Euclidean triangle with vertices at a, p, q is right-angled, if and only if $|p-a|^2 = r^2 + s^2 = (|p|^2 - 1) + (|a|^2 - 1)$, if and only if $p \bullet a = 1$.

When $\{u, a\}$ is linearly independent, the line through 0 and a (which is the unique line in $\mathbb{R}^2$ through 0 which is orthogonal to D) does not pass through u, but there is a unique point p for which the above circle C passes through u and intersects orthogonally with p, namely the (unique) point of intersection of the non-parallel lines $x \bullet u = \frac{|u|^2+1}{2}$ and $x \bullet a = 1$.

Suppose that $\{u, a\}$ is linearly dependent, say $u = ta$ with $t \in \mathbb{R}$. Then the line through 0 and a passes through u. We claim that there is no point $p \in \mathbb{R}^2$ for which the above circle C passes through u and intersects orthogonally with D. Suppose, for a contradiction, that p is such a point. Then we have $p \bullet u = \frac{|u|^2+1}{2}$ and $p \bullet a = 1$. It follows that

$$t = t(p \bullet a) = p \bullet (ta) = p \bullet u = \tfrac{|u|^2+1}{2} = \tfrac{|ta|^2+1}{2} = \tfrac{t^2|a|^2+1}{2}.$$

But this is not possible since $|a| > 1$ so that

$$\tfrac{t^2|a|^2+1}{2} - t > \tfrac{t^2+1}{2} - t = \tfrac{(t-1)^2}{2} \geq 0.$$

6: (a) Let $u=(0,0)$, $v=\left(\frac{1}{2},0\right)$ and $w=\left(\frac{1}{2},\frac{1}{2}\right)$. Find the hyperbolic area of the triangle $[u,v,w]$ in $\mathbb{H}^2$.

Solution: We provide two solutions. The first solution you might find it useful to draw an accompanying picture. The hyperbolic line segments $[u,v]$ and $[u,w]$ are equal to the Euclidean line segments $[u,v]$, $u,w]$ (since they lie along lines through 0) so we have $\alpha=\frac{\pi}{4}$ (α is the angle at u between $[u,v]$ and $[u,w]$). Let us find the hyperbolic line L through v and w. Say $L=C\cap\mathbb{H}^2$ where $C_E(p,r)$ and write $p=(x,y)$. To have $u\in L$ we need $p\bullet u=\frac{|u|^2+1}{2}$ that is $\frac{1}{2}x=\frac{\frac{1}{4}+1}{2}=\frac{5}{8}$ and so $x=\frac{5}{4}$. To have $w\in L$ we need $p\bullet w=\frac{|w|^2+1}{2}$, that is $\frac{1}{2}x+\frac{1}{2}y=\frac{\frac{1}{4}+\frac{1}{4}+1}{2}=\frac{3}{4}$, so $y=\frac{3}{2}-x=\frac{3}{2}-\frac{5}{4}=\frac{1}{4}$. Thus we obtain $p=(x,y)=\left(\frac{5}{4},\frac{1}{4}\right)$. Since the radius of C from p to v has slope $\frac{1}{3}$, the tangent line to L at at v has slope -3 and so we have $\beta=\tan^{-1}3$. Since the radius of C from p to w has slope $-\frac{1}{3}$, the tangent line to L at w has slope 3 and so we have $\gamma=\frac{\pi}{4}-\tan^{-1}\frac{1}{3}$. Thus the hyperbolic area of the hyperbolic triangle $[u,v,w]$ is

$$A=\pi-(\alpha+\beta+\gamma)=\pi-\left(\tfrac{\pi}{4}+\tan^{-1}3+\tfrac{\pi}{4}-\tan^{-1}\tfrac{1}{3}\right)=\tfrac{\pi}{2}+\tan^{-1}\tfrac{1}{3}-\tan^{-1}3.$$

Since $\tan^{-1}3+\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$, we can also write this as $A=2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}$.

The second solution is purely algebraic. For the hyperbolic triangle $[u,v,w]$ we have

$$\begin{aligned}a&=d_H(v,w)=\cosh^{-1}\left(1+\tfrac{2\cdot\frac{1}{4}}{\frac{3}{4}\cdot\frac{1}{2}}\right)=\cosh^{-1}\tfrac{7}{3}\\ b&=d_H(w,u)=\cosh^{-1}\left(1+\tfrac{2\cdot\frac{1}{2}}{\frac{1}{2}\cdot 1}\right)=\cosh^{-1}3\\ c&=d_H(u,v)=\cosh^{-1}\left(1+\tfrac{2\cdot\frac{1}{4}}{1\cdot\frac{3}{4}}\right)=\cosh^{-1}\tfrac{5}{3}\end{aligned}$$

hence also $\sinh a=\sqrt{\cosh^2 a-1}=\sqrt{\frac{49}{9}-1}=\frac{2\sqrt{10}}{3}$, $\sinh b=\sqrt{\cosh^2 b-1}=\sqrt{9-1}=2\sqrt{2}$ and $\sinh c=\sqrt{\cosh^2 c-1}=\sqrt{\frac{25}{9}-1}=\frac{4}{3}$. By the First Law of Cosines, we have

$$\begin{aligned}\cos\alpha&=\frac{\cosh b\cosh c-\cosh a}{\sin hb\sin ha}=\frac{3\cdot\frac{5}{3}-\frac{7}{3}}{2\sqrt{2}\cdot\frac{4}{3}}=\tfrac{1}{\sqrt{2}}\\ \cos\beta&=\frac{\cosh a\cosh c-\cosh b}{\sinh a\sinh c}=\frac{\frac{7}{3}\cdot\frac{5}{3}-3}{\frac{2\sqrt{10}}{3}\cdot\frac{4}{3}}=\tfrac{1}{\sqrt{10}}\\ \cos\gamma&=\frac{\cosh a\cosh b-\cosh c}{\sinh a\sinh c}=\frac{\frac{7}{3}\cdot 3-\frac{5}{3}}{\frac{2\sqrt{10}}{3}\cdot 2\sqrt{2}}=\tfrac{2}{\sqrt{5}}.\end{aligned}$$

Thus the hyperbolic area of the hyperbolic triangle $[u,v,w]$ is

$$A=\pi-(\alpha+\beta+\gamma)=\pi-\left(\tfrac{\pi}{4}+\cos^{-1}\tfrac{1}{\sqrt{10}}+\cos^{-1}\tfrac{2}{\sqrt{5}}\right)=\tfrac{3\pi}{4}-\cos^{-1}\tfrac{1}{\sqrt{10}}-\cos^{-1}\tfrac{2}{\sqrt{5}}.$$

If you want, you can show that this simplifies to $A=\tan^{-1}\frac{3}{4}$.

(b) Let $u=(1,0)$, $v=(0,1)$ and $w=(0,-1)$. Find the hyperbolic area of the circle inscribed in the triply asymptotic triangle $[u,v,w]$ in $\mathbb{H}^2$.

Solution: It helps to draw a picture to accompany the solution. The hyperbolic line through u and v is $C_E(a,1)$ where $a=(1,1)$ and the hyperbolic line through v and w is $C_E(b,1)\cap\mathbb{H}^2$ where $b=(-1,1)$. By symmetry, both the Euclidean centre and the hyperbolic centre of the inscribed circle lie along the y-axis. To find the Euclidean centre $p=(0,y)$, note that the points $(0,y)$, $(1,y)$ and $(1,1)$ form a Euclidean right-angled triangle with edge lengths 1, $1-y$ and $1+y$, and so we must have $(1+y)^2=1^2+(1-y)^2$, that is $1+2y+y^2=2-2y+y^2$, hence $4y=1$. Thus the Euclidean centre of the inscribed circle is at $p=\left(0,\frac{1}{4}\right)$ and the Euclidean radius is $\frac{1}{4}$. The line segment from $0=(0,0)$ to $q=\left(0,\frac{1}{2}\right)$ is both a Euclidean and hyperbolic diameter for the inscribed circle. The hyperbolic length of the diameter is $d=d_H(0,q)=\cosh^{-1}\left(1+\frac{2\cdot\frac{1}{4}}{1\cdot\frac{3}{4}}\right)=\cosh^{-1}\frac{5}{3}$, so the hyperbolic radius is $r=\frac{1}{2}d$ so that $\cosh 2r=\cosh d=\frac{5}{3}$. We have

$$\cosh^2 r=\left(\tfrac{e^r+e^{-r}}{2}\right)^2=\tfrac{e^{2r}+2+e^{-2r}}{4}=\tfrac{1}{2}\left(1+\cosh 2r\right)=\tfrac{1}{2}\left(1+\tfrac{5}{3}\right)=\tfrac{4}{3}$$

so that $\cosh r=\frac{2}{\sqrt{3}}$, and so the hyperbolic area of the inscribed circle is

$$A=2\pi\left(\cosh r-1\right)=2\pi\left(\tfrac{2}{\sqrt{3}}-1\right).$$

7: (a) Find the hyperbolic area and perimeter of the regular hexagon in $\mathbb{H}^2$ with interior angles $\frac{\pi}{2}$.

Solution: The hexagon can be cut into 6 triangles, meeting at 0, each of which is congruent to a triangle $[u,v,w]$ with $\alpha=\frac{\pi}{3}$ and $\beta=\gamma=\frac{\pi}{4}$. The area of the hexagon is

$$A=6\big(\pi-(\alpha+\beta+\gamma)\big)=6\big(\pi-\left(\tfrac{\pi}{3}+\tfrac{\pi}{4}+\tfrac{\pi}{4}\right)\big)=6\cdot\tfrac{\pi}{6}=\pi.$$

By the Second Hyperbolic Law of Cosines, the length $\ell=a$ of the side opposite to u is given by

$$\cosh(\ell)=\frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}=\frac{\frac{1}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}=2$$

and so the perimeter of the hexagon is $L=6\ell=6\cosh^{-1}(2)$. We remark that $\cosh^{-1}2=\ln(2+\sqrt{3})$.

(b) Find $a>0$ such that the regular hexagon in $\mathbb{H}^2$ with vertices at $(\pm a,0)$, $\left(\pm\frac{a}{2},\pm\frac{\sqrt{3}\,a}{2}\right)$ has interior angles equal to $\frac{\pi}{6}$.

Solution: The hexagon can be cut into 6 triangles meeting at 0 each of which is congruent to the triangle $[u,v,w]$ where $u=(0,0)$, $v=(a,0)$ and $w=\left(\frac{a}{2},\frac{\sqrt{3}a}{2}\right)$ with interior angles $\alpha=\frac{\pi}{3}$ and $\beta=\gamma=\frac{\pi}{12}$. The length ℓ of the side opposite to u is given by

$$\begin{aligned}\cosh(\ell)&=\frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}=\frac{\cos\frac{\pi}{3}+\cos^2\frac{\pi}{12}}{\sin^2\frac{\pi}{12}}=\frac{\frac{1}{2}+\frac{1+\cos\frac{\pi}{6}}{2}}{\frac{1-\cos\frac{\pi}{6}}{2}}=\frac{2+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\\&=\tfrac{4+\sqrt{3}}{2-\sqrt{3}}=\tfrac{(4+\sqrt{3})(2+\sqrt{3})}{4-3}=11+6\sqrt{3}.\end{aligned}$$

By the formula for the hyperbolic distance between two points, we also have

$$\cosh\ell=\cosh d_H(u,v)=1+\tfrac{2|v-u|^2}{(1-|u|^2)(1-|v|^2)}=1+\tfrac{2a^2}{(1-a^2)^2}$$

so we have

$$\begin{aligned}1+\tfrac{2a^2}{(1-a^2)^2}=11+6\sqrt{3}&\implies\frac{2a^2}{(1-a^2)^2}=10+6\sqrt{3}\implies\frac{a^2}{(1-a^2)^2}=5+3\sqrt{3}\implies\frac{a}{1-a^2}=\sqrt{5+3\sqrt{3}}\\&\implies a=\sqrt{5+3\sqrt{3}}(1-a^2)\implies\sqrt{5+3\sqrt{3}}\,a^2+a-\sqrt{5+3\sqrt{3}}=0\\&\implies a=\tfrac{-1\pm\sqrt{1+4(5+3\sqrt{3})}}{2\sqrt{5+3\sqrt{3}}}=\tfrac{-1+\sqrt{21+12\sqrt{3}}}{2\sqrt{5+3\sqrt{3}}}=\tfrac{-1+(3+2\sqrt{3})}{2\sqrt{5+3\sqrt{3}}}=\tfrac{1+\sqrt{3}}{\sqrt{5+3\sqrt{3}}}.\end{aligned}$$

8: (a) Find the hyperbolic circumference and the area of the circle in $\mathbb{H}^2$ which is inscribed in the hyperbolic square with interior angles $\frac{\pi}{3}$.

Solution: We can place the vertices of the square at positions $(\pm k, 0)$ and $(0, \pm k)$. The square can be cut into 8 triangles meeting at 0 so that each triangle is congruent to the triangle $[u, v, w]$ where $u = (0,0)$, $v = (k, 0)$ and w is the midpoint of the hyperbolic line segment from $(k, 0)$ to $(0, k)$, which has interior angles $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{6}$ and $\gamma = \frac{\pi}{2}$. The hyperbolic radius of the inscribed circle is the length $r = b$ of the side opposite the angle $\beta = \frac{\pi}{6}$ at vertex v. By the Second Hyperbolic Law of Cosines, we have

$$\cosh(r) = \frac{\cos\beta + \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha} = \frac{\frac{\sqrt{3}}{2} + 0 \cdot \frac{\sqrt{2}}{2}}{1 \cdot \frac{\sqrt{2}}{2}} = \frac{\sqrt{3}}{\sqrt{2}}.$$

Thus the area A and the circumference L of the circle are given by

$$A = 2\pi\big(\cosh(r) - 1\big) = 2\pi\big(\tfrac{\sqrt{3}}{\sqrt{2}} - 1\big) = \pi(\sqrt{6} - 2), \text{ and}$$
$$L = 2\pi\sinh(r) = 2\pi\sqrt{\cosh^2(r) - 1} = 2\pi\sqrt{\tfrac{3}{2} - 1\big)} = 2\pi \cdot \tfrac{1}{\sqrt{2}} = \sqrt{2}\,\pi.$$

(b) Find the hyperbolic perimeter and area of the square in $\mathbb{H}^2$ with edges along the lines K, L, M and N such that $F_K(0) = \big(\frac{1}{2}, 0\big)$, $F_L(0) = \big(0, \frac{1}{2}\big)$, $F_M(0) = \big(-\frac{1}{2}, 0\big)$ and $F_N(0) = \big(0, -\frac{1}{2}\big)$.

Solution: Using the formula in the proof of Theorem 10.10 (or the formula from Example 10.11), the circle C for which $K = C \cap \mathbb{H}^2$ is centred at the point $p = \frac{\left(\frac{1}{2}, 0\right)}{\left|\left(\frac{1}{2}, 0\right)\right|^2} = (2, 0)$ and has radius $r = \sqrt{|p|^2 - 1} = \sqrt{3}$, so C is the circle $(x-2)^2 + y^2 = 3$. The intersection of $K = C \cap \mathbb{H}^2$ with the x-axis is at $v = (2 - \sqrt{3}, 0)$. By symmetry, the point of intersection of K with L lies on the line $y = x$, so we put $y = x$ into the equation $(x-2)^2 + y^2 = 3$ to get $(x-2)^2 + x^2 = 3 \implies 2x^2 - 4x + 1 = 0 \implies x = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{1}{\sqrt{2}}$ so that the intersection point is $w = (x, y) = (1 - \frac{1}{\sqrt{2}})(1, 1)$. Thus the square can be cut into 8 congruent triangles each of which is congruent to the square with vertices at $u = (0, 0)$, $v = \big(2 - \sqrt{3}\,, 0\big)$ and $w = \big(1 - \frac{1}{\sqrt{2}}\,, 1 - \frac{1}{\sqrt{2}}\big)$.

In the triangle $[u, v, w]$ we have $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{2}$. By the formula for the hyperbolic distance between two points, we have

$$\begin{aligned}\cosh(c) &= \cosh d_H(u, v) = \cosh d_H(0, v) = 1 + \tfrac{2|v|^2}{1 - |v|^2} \\ &= 1 + \tfrac{2(7 - 4\sqrt{3})}{1 - (7 - 4\sqrt{3}} = 1 + \tfrac{7 - 4\sqrt{3})}{-3 + 2\sqrt{3}} = 1 + \tfrac{-3 + 2\sqrt{3}}{3} = \tfrac{2\sqrt{3}}{3} = \tfrac{2}{\sqrt{3}}.\end{aligned}$$

By the Second Hyperbolic Cosine Law,we have

$$\cos\gamma = \sin\alpha\sin\beta\cosh c - \cos\alpha\cos\beta = \tfrac{\sqrt{2}}{2} \cdot 1 \cdot \tfrac{2}{\sqrt{3}} - \tfrac{\sqrt{2}}{2} \cdot 0 = \tfrac{\sqrt{2}}{\sqrt{3}}, \text{ and}$$
$$\cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma} = \frac{\frac{\sqrt{2}}{2} + 0 \cdot \frac{\sqrt{2}}{\sqrt{3}}}{1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{2}}.$$

Thus the perimeter L and the area A of the square are

$$L = 8a = 8\cosh^{-1}\tfrac{\sqrt{3}}{\sqrt{2}}$$
$$A = 8\big(\pi - (\alpha + \beta + \gamma)\big) = 8\big(\pi - \big(\tfrac{\pi}{4} + \tfrac{\pi}{2} + \cos^{-1}\tfrac{\sqrt{2}}{\sqrt{3}}\big)\big) = 2\pi - 8\cos^{-1}\tfrac{\sqrt{2}}{\sqrt{3}}.$$

9: (a) Let $p=\left(\frac{1}{2},\frac{1}{2}\right)$, $\theta=\frac{\pi}{2}$ and $a=\left(\frac{1}{3},\frac{2}{3}\right)$. Find $R_{p,\theta}(a)$ in $\mathbb{H}^2$.

Solution: We find two hyperbolic lines L and M through p such that $R_{p,\theta}=F_MF_L$. We take L to be the line $x=y$ (the through 0 and p). We want to find a line M through p such that the angle from L to M is $\frac{\pi}{4}$, say $M=C_E(q,s)\cap\mathbb{H}^2$ with $q=(x,y)$. To have $p\in M$, we need $q\bullet p=\frac{|p|^2+1}{2}$, that is $\frac{1}{2}x+\frac{1}{2}y=\frac{\frac{1}{4}+\frac{1}{4}+1}{2}=\frac{3}{4}$, or equivalently $x+y=\frac{3}{2}$ (1). In order that the angle from L to M is $\frac{\pi}{4}$, we want M to have a vertical tangent line at the point p, so q must lie on the horizontal line through p. that is the line $y=\frac{1}{2}$. We put $y=\frac{1}{2}$ into Equation (1) to get $x=1$, and so we obtain $q=(x,y)=\left(1,\frac{1}{2}\right)$. And we need $s^2=|q|^2-1=1+\frac{1}{4}-1=\frac{1}{4}$ and so $s=\frac{1}{2}$. Since L is the line $y=x$ we have $b=F_L(a)=F_{y=x}\left(\frac{1}{3},\frac{2}{3}\right)=\left(\frac{2}{3},\frac{1}{3}\right)$, and since $F_M=F_C$ where $C=C_E(q,s)$ we have

$$R_{p,\theta}(a)=F_MF_L(a)=F_C(b)=q+\frac{s^2}{|b-q|^2}(b-q)=\left(1,\tfrac{1}{2}\right)+\frac{\frac{1}{4}}{\frac{1}{9}+\frac{1}{36}}\left(-\tfrac{1}{3},-\tfrac{1}{6}\right)=\left(1,\tfrac{1}{2}\right)-\tfrac{3}{10}(2,1)=\left(\tfrac{2}{5},\tfrac{1}{5}\right).$$

(b) Let $u=\left(\frac{3}{5},-\frac{4}{5}\right)$ and $v=(1,0)$, and let P be the parallel displacement such that $P(u)=u$ and $P(-u)=v$. Find $a\in\mathbb{H}^2$ such that $P(a)=0$.

Solution: Let L be the line through u and $-u$, that is the line $4x+3y=0$, and note that $F_L(u)=u$ and $F_L(-u)=-u$. We want to find the line M through u such that $F_M(-u)=v$ and then we can take $P=F_MF_L$ to get $P(u)=u$ and $P(-u)=v$. Say $M=C_E(p,r)$ with $p=(x,y)$. To have $u\in M$ we need $p\bullet u=\frac{|u|^2+1}{2}$, that is $\frac{3}{5}x-\frac{4}{5}y=1$, or equivalently $3x-4y=5$ (1). To get $F(-u)=p$ we need p to lie on the line through $-u$ and v, that is the line $x+2y=1$ (2). Solve Equations (1) and (2) to get $p=(x,y)=\left(\frac{7}{5},-\frac{1}{5}\right)$. And we need $r^2=|p|^2-1=\frac{49}{25}+\frac{1}{25}-1=1$ so that $r=1$. To summarize, we have $P=F_MF_L$ where L is the line $4x+3y=0$ and $M=C\cap\mathbb{H}^2$ where $C=C_E(p,r)$ with $p=\left(\frac{7}{5},-\frac{1}{5}\right)$ and $r=1$. To get $P(a)=0$, that is $F_MF_L(a)=0$, we need $a=(F_MF_L)^{-1}(0)=F_LF_M(0)$. We have $F_M(0)=F_C(0)=p+\frac{r^2}{|p|^2}(-p)=p-\frac{1}{2}p=\frac{1}{2}p=\left(\frac{7}{10},-\frac{1}{10}\right)$ and so (using a formula from Part (4) of Example 1.82) we have

$$a=F_LF_M(0)=F_{4x+3y=0}\left(\tfrac{7}{10},-\tfrac{1}{10}\right)=\left(\tfrac{7}{10},-\tfrac{1}{10}\right)-\frac{2\left(4\cdot\frac{7}{10}-3\cdot\frac{1}{10}\right)}{4^2+3^2}(4,3)=\left(\tfrac{7}{10},-\tfrac{1}{10}\right)-\left(\tfrac{4}{5},\tfrac{3}{5}\right)=\left(-\tfrac{1}{10},-\tfrac{7}{10}\right).$$

.

10: (a) Let $a=\left(\frac{5}{4},\frac{1}{4}\right)$ and $r=\frac{\sqrt{10}}{4}$, and let $L=C_E(a,r)\cap\mathbb{H}^2$. Let $u=\left(\frac{5}{8},-\frac{5}{8}\right)$ and $v=\left(\frac{3}{4},0\right)$. Find the hyperbolic line M such that $F_LF_M(u)=v$ and determine whether the isometry F_MF_L is a rotation, a translation, or a parallel displacement.

Solution: To get $F_LF_M(u)=v$ we need $F_M(u)=F_L(v)$. We have

$$F_L(v)=a+\tfrac{r^2}{|v-a|^2}(v-a)=\left(\tfrac{5}{4},\tfrac{1}{4}\right)+\tfrac{\frac{5}{8}}{\frac{1}{4}+\frac{1}{16}}\left(-\tfrac{1}{2},-\tfrac{1}{4}\right)=\left(\tfrac{5}{4},\tfrac{1}{4}\right)-2\left(\tfrac{1}{2},\tfrac{1}{4}\right)=\left(\tfrac{1}{4},-\tfrac{1}{4}\right).$$

Let $w=F_L(v)=\left(\frac{1}{4},-\frac{1}{4}\right)$ and let $M=C_E(b,s)\cap\mathbb{H}^2$. From the definition of F_M, in order to have $F_M(u)=w$, the point b must lie below and to the right of u on the line through u and w, that is the line $y=-x$, and we need $|u-b|\cdot|w-b|=s^2=|b|^2-1$. Say $b=(t,-t)$ with $t>\frac{5}{8}$ (so that b lies below and to the right of u). We have $u=\frac{5}{8}(1,-1)$ and $w=\frac{1}{4}(1,-1)$ and $b=t(1,-1)$ and so

$$\begin{aligned}|u-b|\cdot|w-b|=s^2=|b|^2-1&\Longrightarrow\left(t-\tfrac{5}{8}\right)\sqrt{2}\cdot\left(t-\tfrac{1}{4}\right)\sqrt{2}=2t^2-1\Longrightarrow 2\left(t^2-\tfrac{7}{8}t+\tfrac{5}{32}\right)-2t^2-1\\&\Longrightarrow\tfrac{7}{4}t=\tfrac{21}{16}\Longrightarrow t=\tfrac{3}{4}.\end{aligned}$$

Thus we have $b=\left(\frac{3}{4},-\frac{3}{4}\right)$ and we have $s^2=|b|^2-1=\frac{9}{8}-1=\frac{1}{8}$ so that $s=\frac{\sqrt{2}}{4}$.

We have found the hyperbolic line M, and it remains to determine whether F_MF_L is a translation, a parallel displacement, or a rotation. This depends on whether the two circles $C_E(a,r)$ and $C_E(b,s)$ have 0 points of intersection (giving a translation), or 1 point of intersection (giving a parallel displacement about that point), or 2 points of intersection (giving a rotation about the point that lies in $\mathbb{H}^2$). The two circles have 0k, 1 or 2 points of intersection according to whether $|a-b|>r+s$ or $|a-b|=r+s$ or $|a-b|<r+s$. In this case we have $|a-b|^2=\left|\left(\frac{1}{2},1\right)\right|^2=\frac{5}{4}$ and we have $(r+s)^2=\left(\frac{\sqrt{10}}{4}+\frac{\sqrt{2}}{4}\right)^2=\frac{3+\sqrt{5}}{4}>\frac{5}{4}$ so that the two circles have two points of intersection, hence the composite F_MF_L is a rotation.

Another way to see that the two circles have two points of intersection is to notice, by inspection using a picture, that $\left(1,-\frac{1}{2}\right)$ is a point of intersection, and it does not lie on the line containing a and b, so there must be another point of intersection. Yet another way is to find the two points of intersection algebraically from the equations of the two circles.

(b) Let $u=(0,0)$ and $v=\left(\frac{2}{5},\frac{1}{5}\right)$. Find the point $p\in\mathbb{H}^2$ such that $R_{v,\frac{\pi}{2}}R_{u,\frac{\pi}{2}}=R_{p,\theta}$ for some $\theta\in[0,2\pi)$.

Solution: Let L be the line $x+3y=0$ and let M be the line $x-2y=0$. Note that L and M pass through $u=(0,0)$ with $\theta_o(L,M)=\frac{\pi}{4}$ so that $F_MF_L=R_{u,\frac{\pi}{2}}$ and that M also passes through v (to see that $\theta_o(L,M)=\frac{\pi}{4}$, consider the square with vertices at $(0,0)$. $(1,-2)$, $(3,-1)$ and $(2,1)$). We want to find the hyperbolic line N which passes through v, and whose tangent line at v has slope 3 so that $\theta_o(M,N)=\frac{\pi}{4}$ (consider the above square again to se why the slope should be 3). Let $N=C_E(q,s)$ with $q=(x,y)$. To have $v\in N$ we need $q\cdot v=\frac{|q|^2+1}{2}$, that is $\frac{2}{5}x+\frac{1}{5}y=\frac{\frac{4}{25}+\frac{1}{25}+1}{2}=\frac{3}{5}$, or equivalently $2x+y=3$ (1). For the tangent line at v to have slope 3, we need q to lie on the line through v with slope $-\frac{1}{3}$, that is the line $x+3y=1$ (2). Solve Equations (1) and (2) to get $q=(x,y)=\left(\frac{8}{5},-\frac{1}{5}\right)$. And we need $s^2=|q|^2-1=\frac{64}{25}+\frac{1}{25}-1=\frac{8}{5}$ so that $s=\frac{2\sqrt{10}}{5}$. With this choice for the lines L, M and N we have

$$R_{v,\frac{\pi}{2}}R_{u,\frac{\pi}{2}}=F_NF_MF_MF_L=F_NF_L=R_{p,\theta}$$

where p is the point of intersection of L and M in $\mathbb{H}^2$ and $\theta=\theta_o(L,N)$. Thee line L has Equation $x+3y=0$ (3) and we have $M=C_E(q,s)\cap\mathbb{H}^2$ where the circle $C_E(q,s)$ has equation $\left(x-\frac{8}{5}\right)^2+\left(y+\frac{1}{5}\right)^2=\frac{8}{5}$ (4). We solve Equations (3) and (4): From (3) we have $x=-3y$, and we put this in (4) to get

$$\left(-3y-\tfrac{8}{5}\right)^2+\left(y+\tfrac{1}{5}\right)^2=\tfrac{8}{5}\Longrightarrow 9y^2+\tfrac{48}{5}y+\tfrac{64}{25}+y^2+\tfrac{2}{5}y+\tfrac{1}{25}=\tfrac{8}{5}\Longrightarrow 10y^2+10y+1=0\Longrightarrow y=\tfrac{-10\pm\sqrt{60}}{20}$$

For the point in $\mathbb{H}^2$ we need $|y|<1$ so we choose $y=\frac{-10+\sqrt{60}}{20}=\frac{-5+\sqrt{15}}{10}$, and take $x=-3y=\frac{15-3\sqrt{15}}{10}$. Thus the desired point p is the point $p=\left(\frac{15-3\sqrt{15}}{10},\frac{-5+\sqrt{15}}{10}\right)$.

11: Let $0 \neq u \in \mathbb{R}^2$. Let K, L and M be the lines in $\mathbb{H}^2$ such that $F_K(0) = -u$, $F_L(-u) = u$ and $F_M(0) = u$. Let T_u denote the translation $T_u = F_M F_L$.

(a) Show that $T_u = F_L F_K$

Solution: Note that L is (the intersection with $\mathbb{H}^2$ of) the line through 0 with normal vector u and that, by symmetry, F_L sends the line K to the line M. Let v be a point on K which does not lie on the line through 0 and u, and let $w = F_L(v) \in M$. Consider the hyperbolic triangle $[-u, 0, v]$. We have

$$F_M\big(F_L(-u)\big) = F_M(u) = 0 \ , \ F_M\big(F_L(0)\big) = F_M(0) = u \ , \text{ and } F_M\big(F_L(v)\big) = F_M(w) = w$$

and we have

$$F_L\big(F_K(-u)\big) = F_L(0) = 0 \ , \ F_L\big(F_K(0)\big) = F_L(-u) = u \ , \text{ and } F_L\big(F_K(v)\big) = F_L(v) = w.$$

It follows from Theorem 13.6 that $T_u = F_L F_K = F_L F_K$.

(b) Find $p \in \mathbb{H}^2$ and $\theta \in \mathbb{R}$ such that $T_u R_{0,\pi} T_u = R_{p,\theta}$.

Solution: Let N be (the intersection with $\mathbb{H}^2$ of) the line through 0 and u, and let a be the point where K meets N (in other words let a be the hyperbolic midpoint between $-u$ and 0). Then

$$T_u R_{0,\pi} T_u = T_u F_N F_L F_L F_K = T_u F_N F_K = T_u R_{a,\pi} = F_L F_K F_K F_N = F_L F_N = R_{0,\pi}.$$

Thus we must take $p = 0$ and $\theta = \pi$.

(c) Let $u = \left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$, $v = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ and $w = \left(-\frac{1}{2}, 0\right)$. Let L_1, L_2, L_3, L_4, L_5 and L_6 be the lines in $\mathbb{H}^2$ containing the hyperbolic line segments $[u, v]$, $[0, v]$, $[v, w]$, $[0, w]$, $[w, u]$ and $[0, u]$ respectively. Find $p \in \mathbb{H}^2$ and $\theta \in \mathbb{R}$ such that $F_{L_6} F_{L_5} F_{L_4} F_{L_3} F_{L_2} F_{L_1} = R_{p,\theta}$.

Solution: Let α be the interior angle at each vertex in the equilateral hyperbolic triangle $[u, v, w]$. Then we have

$$\begin{aligned} F_{L_6} F_{L_5} F_{L_4} F_{L_3} F_{L_2} F_{L_1} &= F_{L_6} F_{L_5} F_{L_4} F_{L_3} R_{v,-\alpha} = F_{L_6} F_{L_5} F_{L_4} F_{L_3} F_{L_3} F_{L_2} = F_{L_6} F_{L_5} F_{L_4} F_{L_2} \\ &= F_{L_6} F_{L_5} R_{0,-\frac{4\pi}{3}} = R_{u,-\alpha} R_{0,-\frac{4\pi}{3}} = F_{L_1} F_{L_6} F_{L_6} F_{L_4} = F_{L_1} F_{L_4} = R_{p,\theta} \end{aligned}$$

where p is the point of intersection of lines L_1 and L_4 (that is the hyperbolic midpoint of u and v) and $\theta = \pi$.

To finish our solution, let us calculate the coordinates of the point p. Let C be the circle in $\mathbb{R}^2$ such that $L_1 = C \cap \mathbb{H}^2$. Let $a = (x, y)$ be the centre of C. Note that

$$u \in C \Longrightarrow a \bullet u = \tfrac{|u|^2+1}{2} \Longrightarrow (x, y) \bullet \left(\tfrac{1}{4}, \tfrac{\sqrt{3}}{4}\right) = \tfrac{\frac{1}{4}+1}{2} \Longrightarrow \tfrac{1}{4}x + \tfrac{\sqrt{3}}{4}y = \tfrac{5}{8} \Longrightarrow x + \sqrt{3}y = \tfrac{5}{2}.$$

Also, by symmetry, note that $y = 0$ so that $a = (x, y) = \left(\frac{5}{2}, 0\right)$. Since the centre of C is at $a = \left(\frac{5}{2}, 0\right)$ and the radius of C is $r = \sqrt{|a|^2 - 1} = \sqrt{\frac{25}{4} - 1} = \frac{\sqrt{21}}{2}$ it follows that $p = \left(\frac{5-\sqrt{21}}{2}, 0\right)$.

12: In this problem we find formulas for isometries on $\mathbb{H}^2$ using complex number notation.

(a) Let $p \in \mathbb{C}$, let $0 \neq u \in \mathbb{C}$ and let L be the line in $\mathbb{C}$ through p perpendicular to u. Show that for $z \in \mathbb{C}$ we have $F_L(z) = p - \dfrac{u^2}{|u|^2}\,(\overline{z} - \overline{p})$.

Solution: We know that $F_L(z) = z - \dfrac{2\,(z-p) \bullet u}{|u|^2}\,u$. Let $G_L(z) = p - \dfrac{u^2}{|u|^2}(\overline{z} - \overline{p})$. Let M be the line through 0 perpendicular to u. Then for $w = x + iy = (x, y)$ and $u = k + il = (k, l)$ we have

$$\begin{aligned} F_M(w) &= w - \frac{2\,w \bullet u}{|u|^2}\,u = (x,y) - \frac{2\,(x,y)\bullet(k,l)}{k^2+l^2}\,(k,l)\ \text{, so that} \\ (k^2+l^2)F_M(w) &= \big((k^2+l^2)x\,,\,(k^2+l^2)y\big) - 2(kx+ly)(k,l) \\ &= \Big((l^2-k^2)x - 2kly\,,\,(k^2-l^2)y - 2klx\Big) \end{aligned}$$

and

$$\begin{aligned} G_M(w) &= -\frac{u^2}{|u|^2}\,\overline{w} = -\frac{(k+il)^2(x-iy)}{k^2+l^2}\ \text{, so that} \\ (k^2+l^2)G_M(w) &= -\big((k^2-l^2) + i\,2kl\big)(x-iy) \\ &= -\Big(\big((k^2-l^2)x + 2kly\big) - i\big((k^2-l^2)y - 2klx\big)\Big) \\ &= \big((l^2-k^2)x - 2kly\big) + i\big((k^2-l^2)y - 2klx\big). \end{aligned}$$

Thus we have $G_M(w) = F_M(w)$ for all $w \in \mathbb{C}$. It follows that for all $p \in \mathbb{C}$ and $z \in \mathbb{C}$ we have

$$\begin{aligned} G_L(z) &= p - \frac{u^2}{|u|^2}(\overline{z} - \overline{p}) = p + G_M(z-p) = p + F_M(z-p) \\ &= p + \Big((z-p) - \frac{2\,(z-p)\bullet u}{|u|^2}\,u\Big) = z - \frac{2(z-p)\bullet u}{|u|^2}\,u = F_L(z). \end{aligned}$$

(b) Let $p \in \mathbb{C}$, let $r > 0$ and let C be the circle centred at p of radius r. Show that for $z \in \mathbb{C}$ we have $F_C(z) = p + \dfrac{r^2}{\overline{z} - \overline{p}}$.

Solution: For $z \in \mathbb{C}$ we have

$$F_C(z) = p + \frac{r^2}{|z-p|^2}\,(z-p) = p + \frac{r^2}{(z-p)(\overline{z}-\overline{p})}(z-p) = p + \frac{r^2}{\overline{z}-\overline{p}},$$

as required.

(c) Let $0 \neq u \in \mathbb{R}^2$ and let T_u be as in Problem 11. Show that $T_u(z) = \dfrac{z+u}{\overline{u}\,z + 1}$.

Solution: In Problem 11, the line L is the line through 0 perpendicular to u, and the line M is the line such that $F_M(0) = u$, which is given by $M = C \cap \mathbb{H}^2$ where C is the circle in $\mathbb{R}^2$ centred at $p = \frac{u}{|u|^2}$ of radius $r = \sqrt{|p|^2 - 1} = \sqrt{\frac{1}{|u|^2} - 1}$. By Parts (a) and (b), for $z, w \in \mathbb{C}$ we have

$$F_L(z) = -\frac{u^2}{|u|^2}\,\overline{z}\ \text{ and }\ F_M(w) = F_C(w) = p + \frac{r^2}{\overline{w} - \overline{p}} = \frac{u}{|u|^2} + \frac{\frac{1}{|u|^2} - 1}{\overline{w} - \frac{\overline{u}}{|u|^2}}.$$

Thus for $z \in \mathbb{C}$ we have

$$\begin{aligned} T_u(z) &= F_M\big(F_L(z)\big) = F_M\big(-\frac{u^2}{|u|^2}\,\overline{z}\big) = \frac{u}{|u|^2} + \frac{\frac{1}{|u|^2} - 1}{-\frac{\overline{u}^2}{|u|^2}\,z - \frac{\overline{u}}{|u|^2}} = \frac{u}{u\,\overline{u}} + \frac{1 - |u|^2}{-\overline{u}^2\,z - \overline{u}} \\ &= \frac{1}{\overline{u}}\big(1 - \frac{1 - u\,\overline{u}}{\overline{u}\,z + 1}\big) = \frac{1}{\overline{u}}\big(\frac{\overline{u}\,z + u\,\overline{u}}{\overline{u}\,z + 1}\big) = \frac{z+u}{\overline{u}\,z + 1}, \end{aligned}$$

as required.