PMATH 321 Non-Euclidean Geometry, Exercises for Chapter 4: Hyperbolic Geometry

1: (a) Let C be the Euclidean circle in $\mathbb{R}^2$ with diameter from $a=(1,4)$ to $b=(3,-2)$, and let D be the Euclidean circle in $\mathbb{R}^2$ with diameter from $c=(4,2)$ to $d=(3,3)$. Find the (Euclidean) area of the image of D under the reflection F_C.

(b) Let C be the Euclidean circle in $\mathbb{R}^2$ of radius $r=5$ centred at $p=(-1,-1)$ and let T be the Euclidean triangle in $\mathbb{R}^2$ with vertices at $a=(1,0)$, $b=(3,1)$ and $c=(0,2)$. Find the (Euclidean) area of the image of T under the reflection F_C.

2: (a) Find the centre $p\in\mathbb{R}^2$ and radius $r>0$ of the Euclidean circle C in $\mathbb{R}^2$ such that the reflection F_C sends the line L with equation $2x+y=8$ to the circle D with equation $(x+1)^2+y^2=5$ (with one point removed).

(b) Find the centre $p\in\mathbb{R}^2$ and the radius $r>0$ of the Euclidean circle C in $\mathbb{R}^2$ such that the reflection F_C sends the Euclidean circle D with diameter from $a=(2,1)$ to $b=(1,2)$ to the Euclidean circle E with diameter from $c=(-1,3)$ to $d=(6,2)$.

3: (a) Let $0<a<\frac{\sqrt{3}}{2}$. Find the hyperbolic length of the Euclidean line segment given by $(x,y)=\alpha(t)=\left(\frac{1}{2},t\right)$ for $0\leq t\leq a$.

(b) Let $0<a<1$. Find the hyperbolic area of the circle given by $x^2+y^2=ax$.

4: (a) Let $u=\left(\frac{1}{2},\frac{1}{2}\right)$ and $v=\left(\frac{3}{4},\frac{1}{4}\right)$. Find the centre $p\in\mathbb{R}^2$ and the radius $r>0$ of the Euclidean circle C in $\mathbb{R}^2$ such that $L=C\cap\mathbb{H}^2$ is the hyperbolic line in $\mathbb{H}^2$ through u and v.

(b) Let $u=\left(-\frac{1}{5},\frac{3}{5}\right)$ and $v=\left(\frac{4}{5},-\frac{2}{5}\right)$. Find the centre $p\in\mathbb{R}^2$ and the radius $r>0$ of the Euclidean circle C in $\mathbb{R}^2$ such that the $L=C\cap\mathbb{H}^2$ is the hyperbolic line such that $F_C(u)=v$.

(c) Let $u=\left(-\frac{3}{5},\frac{4}{5}\right)$, let $b=\left(\frac{6}{5},\frac{2}{5}\right)$, let $s=\sqrt{|b|^2-1}$, let C be the Euclidean circle in $\mathbb{R}^2$ centred at b of radius s, and let $L=C\cap\mathbb{H}^2$. Find the centre $a\in\mathbb{R}^2$ and the radius $r>0$ of a Euclidean circle D in $\mathbb{R}^2$ such that $M=D\cap\mathbb{H}^2$ is the hyperbolic line which is asymptotic to u and intersects orthogonally with L.

5: (a) Let $a=\left(\frac{1}{5},\frac{6}{5}\right)$ and $b=\left(\frac{4}{5},-\frac{6}{5}\right)$, let C and D be the Euclidean circles centred at a and b, respectively, which intersect orthogonally with $\mathbb{S}^1$, and let $L=C\cap\mathbb{H}^2$ and $M=D\cap\mathbb{H}^2$. Find the centre p and radius r of the Euclidean circle E such that $N=E\cap\mathbb{H}^2$ is the hyperbolic line which intersects both L and M orthogonally.

(b) Let $u\in\mathbb{H}$ and let L by a hyperbolic line in $\mathbb{H}$. Prove that there exists a unique hyperbolic line M which contains u and intersects orthogonally with L.

6: (a) Let $u=(0,0)$, $v=\left(\frac{1}{2},0\right)$ and $w=\left(\frac{1}{2},\frac{1}{2}\right)$. Find the hyperbolic area of the triangle $[u,v,w]$ in $\mathbb{H}^2$.

(b) Let $u=(1,0)$, $v=(0,1)$ and $w=(0,-1)$. Find the hyperbolic area of the circle inscribed in the triply asymptotic triangle $[u,v,w]$ in $\mathbb{H}^2$.

7: (a) Find the hyperbolic area and perimeter of the regular hexagon in $\mathbb{H}^2$ with interior angles $\frac{\pi}{2}$.

(b) Find $a>0$ such that the regular hexagon in $\mathbb{H}^2$ with vertices at $(\pm a,0)$, $\left(\pm\frac{a}{2},\pm\frac{\sqrt{3}\,a}{2}\right)$ has interior angles equal to $\frac{\pi}{6}$.

8: (a) Find the hyperbolic circumference and the area of the circle in $\mathbb{H}^2$ which is inscribed in the hyperbolic square with interior angles $\frac{\pi}{3}$.

(b) Find the hyperbolic perimeter and area of the square in $\mathbb{H}^2$ with edges along the lines K, L, M and N such that $F_K(0)=\left(\frac{1}{2},0\right)$, $F_L(0)=\left(0,\frac{1}{2}\right)$, $F_M(0)=\left(-\frac{1}{2},0\right)$ and $F_N(0)=\left(0,-\frac{1}{2}\right)$.

9: (a) Let $p = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{2}$ and $a = \left(\frac{1}{3}, \frac{2}{3}\right)$. Find $R_{p,\theta}(a)$ in $\mathbb{H}^2$.

(b) Let $u = \left(\frac{3}{5}, -\frac{4}{5}\right)$ and $v = (1,0)$, and let P be the parallel displacement such that $P(u) = u$ and $P(-u) = v$. Find $a \in \mathbb{H}^2$ such that $P(a) = 0$.

10: (a) Let $a = \left(\frac{5}{4}, \frac{1}{4}\right)$ and $r = \frac{\sqrt{10}}{4}$, and let $L = C_E(a,r) \cap \mathbb{H}^2$. Let $u = \left(\frac{5}{8}, -\frac{5}{8}\right)$ and $v = \left(\frac{3}{4}, 0\right)$. Find the hyperbolic line M such that $F_L F_M(u) = v$ and determine whether the isometry $F_M F_L$ is a rotation, a translation, or a parallel displacement.

(b) Let $u = (0,0)$ and $v = \left(\frac{2}{5}, \frac{1}{5}\right)$. Find the point $p \in \mathbb{H}^2$ such that $R_{v,\frac{\pi}{2}} R_{u,\frac{\pi}{2}} = R_{p,\theta}$ for some $\theta \in [0, 2\pi)$.

11: Let $0 \neq u \in \mathbb{R}^2$. Let K, L and M be the lines in $\mathbb{H}^2$ such that $F_K(0) = -u$, $F_L(-u) = u$ and $F_M(0) = u$. Let T_u denote the translation $T_u = F_M F_L$.

(a) Show that $T_u = F_L F_K$

(b) Find $p \in \mathbb{H}^2$ and $\theta \in \mathbb{R}$ such that $T_u R_{0,\pi} T_u = R_{p,\theta}$.

(c) Let $u = \left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$, $v = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ and $w = \left(-\frac{1}{2}, 0\right)$. Let L_1, L_2, L_3, L_4, L_5 and L_6 be the lines in $\mathbb{H}^2$ containing the hyperbolic line segments $[u,v]$, $[0,v]$, $[v,w]$, $[0,w]$, $[w,u]$ and $[0,u]$ respectively. Find $p \in \mathbb{H}^2$ and $\theta \in \mathbb{R}$ such that $F_{L_6} F_{L_5} F_{L_4} F_{L_3} F_{L_2} F_{L_1} = R_{p,\theta}$.

12: In this problem we find formulas for isometries on $\mathbb{H}^2$ using complex number notation.

(a) Let $p \in \mathbb{C}$, let $0 \neq u \in \mathbb{C}$ and let L be the line in $\mathbb{C}$ through p perpendicular to u. Show that for $z \in \mathbb{C}$ we have $F_L(z) = p - \dfrac{u^2}{|u|^2}\,(\overline{z} - \overline{p})$.

(b) Let $p \in \mathbb{C}$, let $r > 0$ and let C be the circle centred at p of radius r. Show that for $z \in \mathbb{C}$ we have $F_C(z) = p + \dfrac{r^2}{\overline{z} - \overline{p}}$.

(c) Let $0 \neq u \in \mathbb{R}^2$ and let T_u be as in Problem 11. Show that $T_u(z) = \dfrac{z + u}{\overline{u}\,z + 1}$.