PMATH 321 Non-Euclidean Geometry, Solutions to the Exercises for Chapter 3

1: (a) Let $x=(-2,1,3)$ and $y=(5,1,-4)$. Find $d_P\big([x],[y]\big)$.

Solution: We have
$$d_H\big([x],[y]\big)=\cos^{-1}\tfrac{|x\,\bullet\,y|}{|x||y|}=\cos^{-1}\tfrac{|-21|}{\sqrt{14}\sqrt{42}}=\cos^{-1}\tfrac{\sqrt{3}}{2}=\tfrac{\pi}{6}.$$

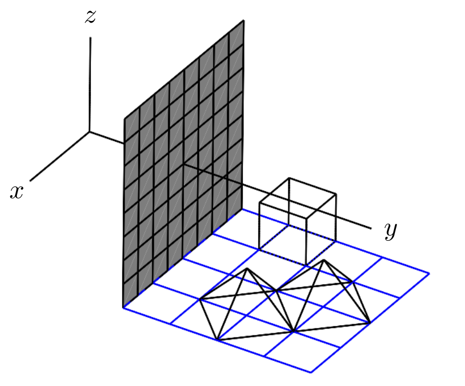
(b) What fraction of the area of $\mathbb{P}^2$ is covered by a triangle with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$?

Solution: The area of the triangle is equal to $\left(\frac{\pi}{3}+\frac{\pi}{3}+\frac{\pi}{2}\right)-\pi=\frac{\pi}{6}$, and the total area of $\mathbb{P}^2$ is equal to 2π, and so the required fraction is $\frac{1}{12}$.

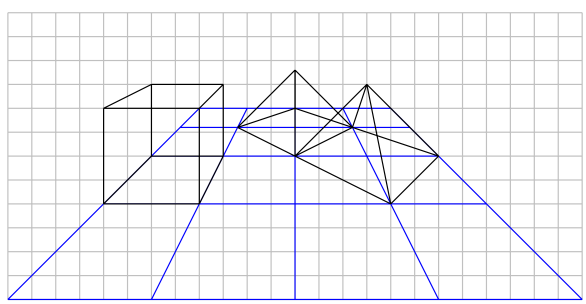
(c) Find the area of the circle on $\mathbb{P}^2$ which is circumscribed around a square on $\mathbb{P}^2$ with sides of length $\frac{\pi}{3}$.

Solution: The square can be cut into 4 congruent triangles meeting at the centre. In each triangle the angle at the centre is $\alpha=\frac{\pi}{4}$, the other two angles β and γ are equal, the side opposite α has length $a=\frac{\pi}{3}$, and the other two side lengths b and c are equal. By the First Law of Cosines we have
$$0=\cos\alpha=\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{\frac{1}{2}-\cos^2 b}{\sin^2 b}$$
and so we have $\cos^2 b=\frac{1}{2}$ hence $b=\frac{\pi}{4}$. The radius of the circumscribed circle is $r=b$ and so the area is $A=2\pi(1-\cos b)=2\pi\big(1-\frac{1}{\sqrt{2}}\big)=\pi(2-\sqrt{2})$.

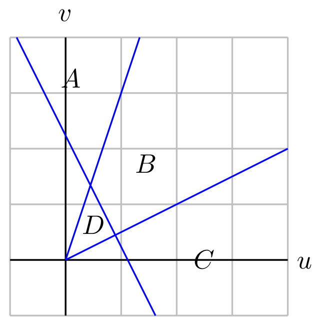
2: Make an accurate sketch of the artist's perspective drawing (the image under the gnomic projection) of the scene shown below. The artist's eye is at the origin, the (transparent) canvas is at the plane $y=1$, the floor lies along the plane $z=-1$, the blue grid lines on the floor are spaced $\frac{1}{2}$ units apart, the cube has sides of length $\frac{1}{2}$ and the two pyramids have height $\frac{1}{2}$.



Solution: The sketch is shown below. The sketch shows the portion of the canvas in the rectangle $-1 \le u \le 1$, $-1 \le v \le 0$ in the uv-plane, and the grey grid lines are spaced $\frac{1}{12}$ units apart. The sketch is produced as follows. We find the coordinates (x,y,z) of various points in the scene and sketch the corresponding points on the canvas at position $(u,v)=(x/y,z/y)$. For example, the top 4 vertices of the cube are at positions $\left(-1,\frac{3}{2},-\frac{1}{2}\right)$, $\left(-\frac{1}{2},\frac{3}{2},-\frac{1}{2}\right)$, $\left(-\frac{1}{2},2,-\frac{1}{2}\right)$ and $\left(-1,2,-\frac{1}{2}\right)$ and the corresponding points on the canvas are at $(u,v)=\left(-\frac{2}{3},-\frac{1}{3}\right)$, $\left(-\frac{1}{3},-\frac{1}{3}\right)$, $\left(-\frac{1}{4},-\frac{1}{4}\right)$ and $\left(-\frac{1}{2},-\frac{1}{4}\right)$. As another example, the top vertices of the 2 pyramids are at $(x,y,z)=\left(0,\frac{5}{2},-\frac{1}{2}\right)$ and $\left(\frac{1}{2},2,-\frac{1}{2}\right)$ corresponding to $(u,v)=\left(0,-\frac{1}{5}\right)$ and $\left(\frac{1}{4},-\frac{1}{4}\right)$.



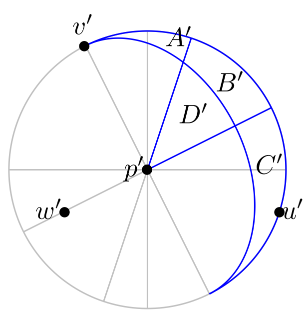
3: Find the areas of the inverse images in $U_3 \subseteq \mathbb{P}^2$ under the gnomic projection ϕ_3, given by $\phi_3\big([x,y,z]\big) = \left(\frac{x}{z}, \frac{y}{z}\right)$, of the regions A, B, C and D in the uv-plane as shown below. The regions are bounded by the lines $v = 3u$, $u = 2v$ and $2u + v = \sqrt{5}$.



Solution: By letting $u = \frac{x}{z}$ and $v = \frac{y}{z}$ we see that inverse image of the lines $v = 3u$, $u = 2v$ and $2u + v = \sqrt{5}$ are the lines in $\mathbb{S}^2$ given by $y = 3x$, $x = 2y$ and $2x + y = \sqrt{5}\, z$, that is the lines with poles $u' = \frac{1}{\sqrt{10}}(3, -1, 0)$, $v' = \frac{1}{\sqrt{5}}(-1, 2, 0)$ and $w' = \frac{1}{\sqrt{10}}(-2, -1, \sqrt{5})$, respectively. The line at infinity is the line $z = 0$ with pole $p' = (0, 0, 1)$. A picture of these lines and the inverse images A', B', C' and D' of the 4 given regions is shown below. The picture shows the upper hemisphere with the z-axis pointing towards us and the x-axis pointing to the right. To find the areas of the various regions, we need to determine the angles. Notice that triangle D' has polar triangle $[u', v', w']$. The edge lengths of the polar triangle are $a' = \cos^{-1}(v' \bullet w') = \cos^{-1}(0) = \frac{\pi}{2}$, $b' = \cos^{-1}(w' \bullet u') = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ and $c' = \cos^{-1}(u' \bullet v') = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$, and so the angles in triangle D' are $\alpha = \pi - a' = \frac{\pi}{2}$, $\beta = \pi - b' = \frac{\pi}{3}$ and $\gamma = \pi - c' = \frac{\pi}{4}$. Similarly, triangle A' has polar triangle $[p', -w', -u']$ and so the angle in A' at vertex v' is $\pi - \cos^{-1}(p' \bullet (-w')) = \pi - \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$. All of the other angles in the figure can be determined without calculation. The angles in triangle A' are $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$, the angles in the quadrilateral B' are $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{2\pi}{3}$, and the angles in triangle C' are $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{4}$. It follows that the areas of the various regions are

$$\begin{aligned}
\text{Area}(A') &= \tfrac{\pi}{4} + \tfrac{\pi}{3} + \tfrac{\pi}{2} - \pi = \tfrac{\pi}{12}, \\
\text{Area}(B') &= \tfrac{\pi}{2} + \tfrac{\pi}{2} + \tfrac{\pi}{2} + \tfrac{2\pi}{3} - 2\pi = \tfrac{\pi}{6}, \\
\text{Area}(C') &= \tfrac{\pi}{2} + \tfrac{\pi}{2} + \tfrac{\pi}{4} - \pi = \tfrac{\pi}{4} \text{ , and} \\
\text{Area}(D') &= \tfrac{\pi}{2} + \tfrac{\pi}{3} + \tfrac{\pi}{4} - \pi = \tfrac{\pi}{12}.
\end{aligned}$$

We remark that it is possible to solve this problem with almost no calculations by noticing that the areas of the various regions will not be changed if we rotate the given figure clockwise by the angle $\theta = \tan^{-1}\frac{1}{2}$ which has the effect of replacing the three given lines to the lines $v = u$, $v = 0$ and $u = 1$.



4: (a) Let $u = \frac{1}{\sqrt{2}}(0,1,1)$, $v = \frac{1}{\sqrt{3}}(1,1,1)$ and $\beta = \frac{2\pi}{3}$. Find $p \in \mathbb{S}^2$ and $\theta \in [0,\pi]$ so that $F_u R_{v,\beta} = R_{p,\theta}$ as an isometry on $\mathbb{P}^2$.

Solution: Let J, K, L, M and N be the lines with poles j, k, ℓ, m and n where $j = \frac{1}{\sqrt{2}}(1,0,-1)$, $k = \frac{1}{\sqrt{2}}(0,1,-1)$, $\ell = u = \frac{1}{\sqrt{2}}(0,1,1)$, $m = (0,0,1)$ and $n = (0,1,0)$. Also, let $q = (1,0,0)$. We can determine the points of intersection and the angles between these lines using a picture (or by analytic calculations) to see that $R_{v,\beta} = F_K F_J$ and $R_{q,\pi} = F_L F_K = F_N F_M$ and $R_{n,\frac{\pi}{2}} = F_M F_J$, and so on $\mathbb{S}^2$ we have

$$F_u R_{v,\beta} = F_L R_{v,\beta} = F_L F_K F_J = R_{q,\pi} F_J = F_N F_M F_J = F_N R_{n,\frac{\pi}{2}} = -R_{n,\frac{3\pi}{2}} = -R_{-n,\frac{\pi}{2}}$$

and so on $\mathbb{P}^2$ we have

$$F_u R_{v,\theta} = -R_{-n,\frac{\pi}{2}} = R_{-n,\frac{\pi}{2}} = R_{p,\theta}$$

where $p = -n = (0,-1,0)$ and $\theta = \frac{\pi}{2}$.

(b) Let $u = (0,0,1)$, $v = \frac{1}{\sqrt{2}}(1,1,0)$ and $w = \frac{1}{\sqrt{2}}(0,1,1)$. Find $p \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ such that $F_u F_v F_w = R_{p,\theta}$, as an isometry on $\mathbb{P}^2$.

Solution 1: You should draw a picture to accompany this solution. Let $L = L_u$, $M = L_v$ and $N = L_w$. Notice that L and M intersect orthogonally at the point $q = \frac{1}{\sqrt{2}}(1,-1,0)$ so that $F_L F_M = R_{q,\pi}$. Let $J = L_r$ with $r = \frac{1}{\sqrt{6}}(1,1,2)$ and let $K = L_p$ with $p = \frac{1}{\sqrt{3}}(1,1,-1$ so that J and K intersect orthogonally at q and K intersects orthogonally with N. Notice that N and J intersect at p and the oriented angle from N to J at p is $\varphi = \frac{\pi}{6}$. Thus we have

$$F_u F_v F_w = F_L F_M F_N = R_{q,\pi} F_N = F_K F_J F_N = F_K R_{p,2\varphi} = F_p R_{p,\frac{\pi}{3}} = -R_{p,\frac{4\pi}{3}}.$$

As an isometry on $\mathbb{P}^2$ we have $-R_{p,\theta} = R_{p,\theta}$, and so $F_u F_v F_w = R_{p,\theta}$ with $p = \frac{1}{\sqrt{3}}(1,1,-1)$ and $\theta = \frac{4\pi}{3}$.

Solution 2: By the classification of isometries, we know that the matrix $F_u F_v F_w$ is of the form $-R$ for some rotation matrix R, and as an isometry on $\mathbb{P}^2$ we have $-R = R$. Let

$$\begin{aligned} R = -F_u F_v F_w &= -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

To find the axis of rotation we find the eigenspace of 1. We have

$$R - I = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so a basis for the eigenspce is $\{(1,1,-1)\}$ and so we can take $p = \frac{1}{\sqrt{3}}(1,1,-1)$. Perhaps the easiest way to determine the angle of rotation is to consider a picture of the sphere. A picture shows that in order to have $R_{p,\theta}(1,0,0) = (0,1,0)$ we need to choose $\theta = \frac{4\pi}{3}$.

5: (a) Let $u = (1,1,-1)$, $v = (0,1,1)$, $w = (1,-1,0)$, $x = (1,1,1)$, $y = (0,1,-1)$ and $z = w$. There are two isometries F on $\mathbb{P}^2$ such that $F([u]) = [x]$, $F([v]) = [y]$ and $F([w]) = [z]$. Express these two isometries in the form $R_{p,\theta}$ and $R_{q,\phi}$ with $p, q \in \mathbb{S}^2$ and $\theta, \phi \in [0, \pi]$.

Solution: You should draw a picture to accompany this solution. Let $\hat{u} = \frac{u}{|u|} = \frac{1}{\sqrt{3}}(1,1,-1)$ and similarly for $\hat{v}$, $\hat{w}$, $\hat{x}$, $\hat{y}$ and $\hat{z}$. Notice that the triangles $[\hat{u},\hat{v},\hat{w}]$ and $[\hat{x},\hat{y},\hat{z}]$ are congruent, each with edge lengths $a = \frac{2\pi}{3}$, $b = c = \frac{\pi}{2}$ and angles $\alpha = \frac{2\pi}{3}$ and $\beta = \gamma = \frac{\pi}{2}$. Note that $[\hat{u},\hat{v},\hat{w}]$ is positively oriented while $[\hat{x},\hat{y},\hat{z}]$ is negatively oriented. Note that, because there $b = c = \frac{\pi}{2}$, if we replace $\hat{x}$ by $-\hat{x}$ then we obtain a congruent triangle $[-\hat{x},\hat{y},\hat{z}]$. Of the 8 triangles $[\pm\hat{x},\pm\hat{y},\pm\hat{z}]$, only the 4 triangles $[\pm\hat{x},\hat{y},\hat{z}]$ and $[\pm\hat{x},-\hat{y},-\hat{z}]$ are congruent and, of these 4 triangles, only the 2 triangles $[-\hat{x},\hat{y},\hat{z}]$ and $[-\hat{x},-\hat{y},-\hat{z}]$ are positively oriented. It follows that the two isometries on $\mathbb{P}^2$ which send $[u]$, $[v]$ and $[w]$ to $[x]$, $[y]$ and $[z]$ are induced by the two rotations on $\mathbb{S}^2$ which send the triangle $[\hat{u},\hat{v},\hat{w}]$ to each of the triangles $[-\hat{x},\hat{y},\hat{z}]$ and $[-\hat{x},-\hat{y},-\hat{z}]$.

The rotation R_1 on $\mathbb{S}^2$ which sends $[\hat{u},\hat{v},\hat{w}]$ to $[-\hat{x},\hat{y},\hat{z}]$ has axis vector $\hat{w}$ (since $\hat{w} = \hat{z}$). We have $\theta = \theta(\hat{w}_{\hat{v}}, \hat{w}_{\hat{y}}) = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$, and the rotation about $\hat{w}$ clockwise by θ sends $\hat{v}$ to $\hat{y}$, and so

$$R_1 = R_{\hat{w},-\theta} = R_{p,\theta} \quad \text{, with } p = -\hat{w} = \tfrac{1}{\sqrt{2}}(-1,1,0) \text{ and } \theta = \cos^{-1}\left(-\tfrac{1}{\sqrt{3}}\right).$$

By inspection (with the help of a picture) we see that the isometry on $\mathbb{S}^2$ which sends $[\hat{u},\hat{v},\hat{w}]$ to $[\hat{x},\hat{y},\hat{z}]$ is the reflection F_n where $n = (0,0,1)$, and so the rotation R_2 on $\mathbb{S}^2$ which sends $[\hat{u},\hat{v},\hat{w}]$ to $[-\hat{x},-\hat{y},-\hat{z}]$ is given by

$$R_2 = -F_n = R_{q,\phi} \text{ with } q = n = (0,0,1) \text{ and } \phi = \pi.$$

(b) Let $x = (1,1,1)$, $y = (1,-1,1)$, $z = (1,0,-1)$, $u = x$, $v = (1,1,-1)$ and $w = (1,-1,0)$. There are two isometries F on $\mathbb{P}^2$ such that $F([x]) = [u]$, $F([y]) = [v]$ and $F([z]) = [w]$. Express these two isometries in the form $R_{p,\theta}$ and $R_{q.\phi}$ with $p, q \in \mathbb{S}^2$ and $\theta, \phi \in \mathbb{R}$.

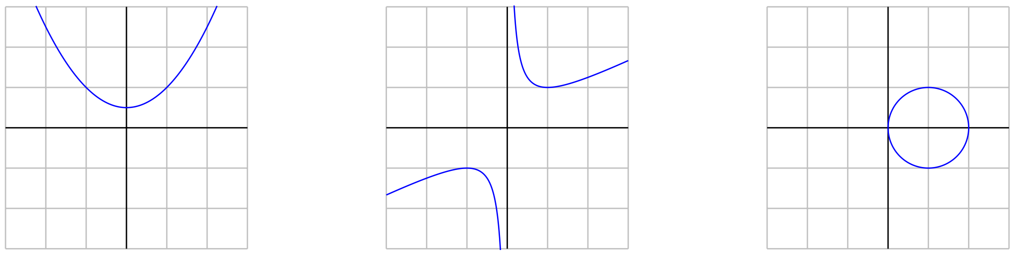
Solution: You should draw a picture to accompany this solution. Let $\hat{x} = \frac{x}{|x|} = \frac{1}{\sqrt{3}}(1,1,1)$ and similarly for $\hat{y}$, $\hat{z}$, $\hat{u}$, $\hat{v}$ and $\hat{w}$. Notice that the triangles $[\hat{x},\hat{y},\hat{z}]$ and $[\hat{u},\hat{v},\hat{w}]$ are congruent, each with 2 right angles. Note that $[\hat{x},\hat{y},\hat{z}]$ is positively oriented while $[\hat{u},\hat{v},\hat{w}]$ is negatively oriented. Note that, because there are right angles at $\hat{u}$ and $\hat{v}$, if we replace $\hat{w}$ by $-\hat{w}$ then we obtain a congruent triangle $[\hat{u},\hat{v},-\hat{w}]$. Of the 8 triangles $[\pm\hat{u},\pm\hat{v},\pm\hat{w}]$, only the 4 triangles $[\hat{u},\hat{v},\pm\hat{w}]$ and $[-\hat{u},-\hat{v},\pm\hat{w}]$ are congruent and, of these 4 triangles, only the 2 triangles $[\hat{u},\hat{v},-\hat{w}]$ and $[-\hat{u},-\hat{v},-\hat{w}]$ are positively oriented. It follows that the two isometries on $\mathbb{P}^2$ which send $[x]$, $[y]$ and $[z]$ to $[u]$, $[v]$ and $[w]$ are induced by the two rotations on $\mathbb{S}^2$ which send the triangle $[\hat{x},\hat{y},\hat{z}]$ to each of the triangles $[\hat{u},\hat{v},-\hat{w}]$ and $[-\hat{u},-\hat{v},-\hat{w}]$.

By inspection (with the help of a picture) we see that the isometry on $\mathbb{S}^2$ which sends $[\hat{x},\hat{y},\hat{z}]$ to $[\hat{u},\hat{v},-\hat{w}]$ is the rotation $R_{p,\frac{2\pi}{3}}$ with $p = \hat{x} = \hat{u} = \frac{1}{\sqrt{3}}(1,1,1)$. By inspection (again using a picture) the isometry on $\mathbb{S}^2$ which sends $[\hat{x},\hat{y},\hat{z}]$ to $[\hat{u},\hat{v},\hat{w}]$ is the reflection in the line $y = z$, that is the reflection F_q with $q = \frac{1}{\sqrt{2}}(0,1,-1)$, and so the isometry on $\mathbb{S}^2$ which sends $[\hat{x},\hat{y},\hat{z}]$ to $[-\hat{u},-\hat{v},-\hat{w}]$ is the rotation $R_{q,\pi} = -F_q$. Thus the two required isometries on $\mathbb{P}^2$ are induced by the rotations $R_{p,\frac{2\pi}{3}}$ with $p = \frac{1}{\sqrt{3}}(1,1,1)$ and $R_{q,\pi}$ with $q = \frac{1}{\sqrt{2}}(0,1,-1)$.

6: For each of the following polynomials $f(x,y)$, find the homogenization $F(x,y,z)$, and make an accurate sketch of the zero sets of the dehomogenizations $f_1(y,z)$, $f_2(x,z)$ and $f_3(x,y)$.

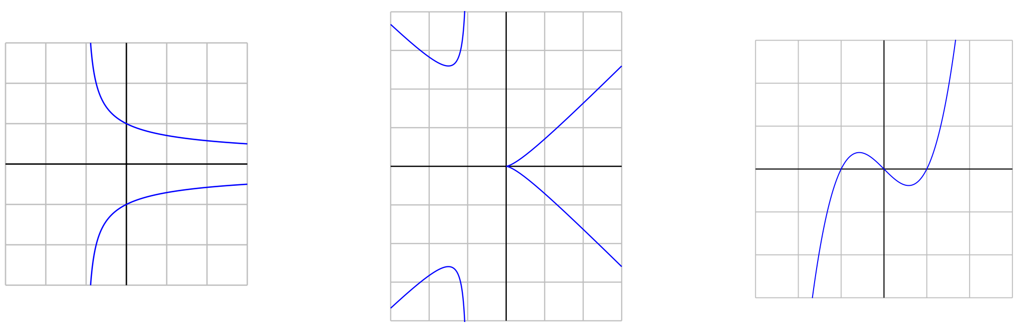
(a) $f(x,y) = x^2 + y^2 - 2x$

Solution: The homogenization of $f(x,y)$ is $F(x,y,z) = x^2 + y^2 - 2xz$ and the dehomogenizations are given by $f_1(y,z) = 1 + y^2 - 2z$, $f_2(x,z) = x^2 + 1 - 2xz$ and $f_3(x,y) = x^2 + y^2 - 2x$. The zero set $Z(f_1)$ is given by $z = \frac{1}{2}(y^2+1)$, the zero set $Z(f_2)$ is given by $z = \frac{x^2+1}{2x}$, and the zero set $Z(f_3)$ is given by $(x-1)^2 + y^2 = 1$. The zero sets are sketched below.



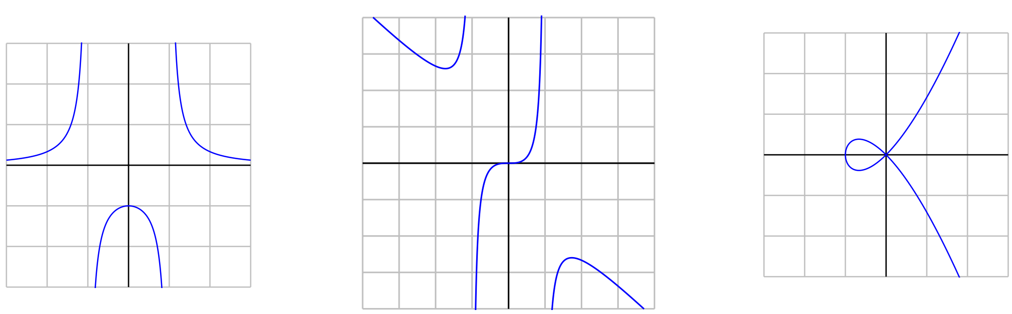
(b) $f(x,y) = y - x^3 + x$

Solution: The homogenization of $f(x,y)$ is $F(x,y,z) = yz^2 - x^3 + xz^2$ and the dehomogenizations are given by $f_1(y,z) = yz^2 - 1 + z^2$, $f_2(x,z) = z^2 - x^3 + xz^2$ and $f_3(x,y) = y - x^3 + x$. $Z(f_1)$ is given by $y = \frac{1}{z^2} - 1$, $Z(f_2)$ is given by $z^2 = \frac{x^3}{1+x}$ and $Z(f_3)$ is given by $y = x(x-1)(x+1)$.



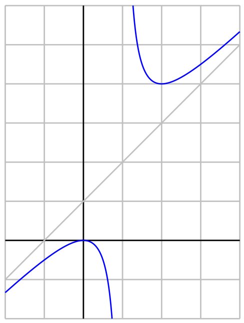
(c) $f(x,y) = x^3 + x^2 - y^2$

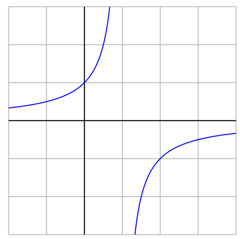
Solution: The homogenization of $f(x,y)$ is $F(x,y,z) = x^3 + x^2z - y^2z$ and the dehomogenizations are given by $f_1(y,z) = 1 + z - y^2z$, $f_2(x,z) = x^3 + x^2z - z$ and $f_3(x,y) = x^3 + x^2 - y^2$. $Z(f_1)$ is given by $z = \frac{1}{y^2-1}$, $Z(f_2)$ is given by $z = \frac{x^3}{1-x^2}$ and $Z(f_3)$ is given by $y^2 = x^2(x+1)$.

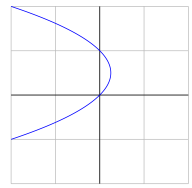


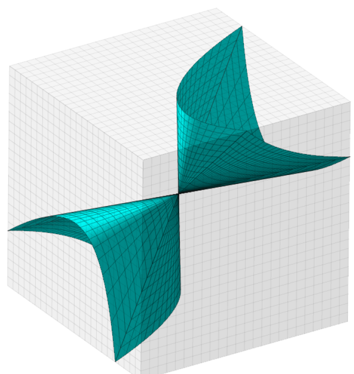
7: (a) Let $f(x,y)=x+y^2-y$. Find the homogenization $F(x,y,z)$, sketch the zero sets of the dehomogenizations $f_1(y,z)$, $f_2(x,z)$ and $f_3(x,y)$, then use your artistic talent to sketch the zero set $Z(F)$.

Solution: The homegenization is $F(x,y,z)=xz+y^2-yz$. The 3 dehomogenizations are $f_1(y,z)=z+y^2-yz$, $f_2(x,z)=xz+1-z$ and $f_3(x,z)=f(x,z)=x+y^2-y$. The zero set $Z(f_1)$ is given by $z+y^2=yz$, that is $z=\frac{y^2}{y-1}$, the zero set $Z(f_2)$ is given by $xz+1=z$ that is $z=\frac{1}{1-x}$, and the zero set $Z(f_3)=Z(f)$ is given by $x+y^2=y$ that is $x=y-y^2=y(1-y)$. The zero sets are shown below. The zero set $Z(F)$ can be drawn by drawing the zero sets $Z(f_1)$, $Z(f_2)$ and $Z(f_3)$ on the faces $x=1$, $y=1$ and $z=1$ of the cube, giving a curve on the front, right, and top faces of the cube, and then $Z(f)$ consists of the lines through 0 which pass through this curve.









(b) Let $f(x,y)=y^2-x+1$, let $g(x,y)=y-x+7$ and let R be the region in the xy-plane which lies between the zero sets $Z(f)$ and $Z(g)$. By finding the homogenizations $F(x,y,z)$ and $G(x,y,z)$ and the dehomogenizations $f_1(y,z)$ and $g_1(y,z)$, find the area of the image of R under the composite $\phi\circ\psi$ where $\psi(x,y)=[x,y,1]\in\mathbb{P}^2$ and $\phi([x,y,z])=\left(\frac{y}{x},\frac{z}{x}\right)$.

Solution: Note that ϕ is the gnomic projection ϕ_1 and ψ is the inverse of the gnomic projection ϕ_3, so the composite $\phi\circ\psi$ first projects from the xy-plane to projective space $\mathbb{P}^2$ and then from $\mathbb{P}^2$ to the yz-plane. The homogenizations are $F(x,y,z)=y^2-xz+z^2$ and $G(x,y,z)=y-x+7z$ and the dehomogenizations are $f_1(y,z)=y^2-z+z^2$ and $g_1(y,z)=y-1+7z$. The zero set $Z(f_1)$ is given by $y^2+z-z^2=0$ or equivalently by $y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$, so it is the circle C in the yz-plane centred at $u=\left(0,\frac{1}{2}\right)$ of radius $r=\frac{1}{2}$ (with the point $(0,0)$ removed because it corresponds to the point at infinity on the projective completion of the parabola). The zero set $Z(g_1)$ is given by $y-1+7z$, so it is the line L in the yz-plane with equation $y=1-7z$. The image of the given region in the xy-plane is the region R in the yz-plane which lies inside the circle C and above the line L (not below L because the point $(0,0)$ corresponds to the point at infinity). To find the points of intersection of C and L we solve the two equations $y^2+z^2-z=0$ (1) and $y=1-7z$ (2) to get the intersection points $v=\left(-\frac{2}{5},\frac{1}{5}\right)$ and $w=\left(\frac{3}{10},\frac{1}{10}\right)$. Notice that the line from u to v has slope $\frac{3}{4}$ and the line from u to w has slope $-\frac{4}{3}$ so the angle at u in the triangle $[u,v,w]$ is $\alpha=\frac{\pi}{2}$. Thus the area A of the region R is equal to $\frac{3}{4}$ of the area of the circle C plus the area of the triangle $[u,v,w]$, that is

$$A=\frac{3}{4}\cdot\frac{\pi}{4}+\frac{1}{8}=\frac{3\pi}{16}+\frac{1}{8}.$$

8: (a) Find $p \in \mathbb{R}^3$, $u \in \mathbb{S}^2$ and $\phi \in \left(0, \frac{\pi}{2}\right)$ such that $V(p,u,\phi)$ intersects the xy-plane in the parabola $y = 3x^2$.

Solution: From Part 1 of Lemma 3.32, when $p = (0,0,-h)$ and $u = \frac{1}{\sqrt{2}}(0,1,1)$ and $\phi = \frac{\pi}{4}$, the intersection of $V(p,u,\phi)$ with the xy-plane is the parabola $y = \frac{1}{2h}x^2$, so we take $p = \left(0,0,-\frac{1}{6}\right)$, $u = \frac{1}{\sqrt{2}}(0,1,1)$ and $\phi = \frac{\pi}{4}$.

(b) Find $p \in \mathbb{R}^3$, $u \in \mathbb{S}^2$ and $\phi \in \left(0, \frac{\pi}{2}\right)$ such that the intersection of $V(p,u,\phi)$ with the xy-plane is the hyperbola given by $\frac{x^2}{12} - \frac{y^2}{4} = 1$.

Solution: From Part 2 of Lemma 3.32, when $p = (0,0,h)$, $u = (1,0,0)$ and $\phi \in \left(0, \frac{\pi}{2}\right)$, the intersection of $V(p,u,\phi)$ with the xy-plave is the hyperbola $\frac{x^2}{h^2 \cot^2 \phi} - \frac{y^2}{h^2} = 1$. If we take $h = 2$ and $\varphi = \frac{\pi}{6}$ so that $\cot \varphi = \sqrt{3}$ then we obtain the desired hyperbola $\frac{x^2}{12} - \frac{y^2}{4} = 1$, so take $p = (0,0,2)$, $u = (1,0,0)$ and $\varphi = \frac{\pi}{6}$.

(c) Find $p \in \mathbb{R}^3$ and $u \in \mathbb{S}^2$ such that the intersection of $V\left(p,u,\frac{\pi}{4}\right)$ with the xy-plane is the ellipse $\frac{x^2}{12} + \frac{y^2}{6} = 1$.

Solution: From Part 3 of Lemma 3.32, when $q = (0,0,h)$, $u = (\sin\theta, 0, \cos\theta)$ and $\phi = \frac{\pi}{4}$, the intersection of $V(q,u,\phi)$ with the xy-plane is the ellipse $\frac{(x+h\tan 2\theta)^2}{h^2 \sec^2 2\theta} + \frac{y^2}{h^2 \sec 2\theta} = 1$. If we take $h = \sqrt{3}$ and $\theta = \frac{\pi}{6}$ so that $\tan 2\theta = \sqrt{3}$ and $\sec 2\theta = 2$, then for $q = (0,0,h) = (0,0,\sqrt{3})$ and $u = (\sin\theta, 0, \cos\theta) = \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$, the intersection of the cone $V\left(q,u,\frac{\pi}{4}\right)$ is the ellipse $\frac{(x+3)^2}{12} + \frac{y^2}{6} = 1$. To obtain the desired ellipse we need to translate 3 units to the right, so we use $p = (3,0,\sqrt{3})$ and $u = \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$.

(d) Verify that the conclusion of Pascal's Theorem holds for the points $u_1 = (-2,3)$, $u_2 = (-1,1)$, $u_3 = (0,0)$, $u_4 = (1,0)$, $u_5 = (4,6)$ and $u_6 = (5,10)$ which all lie on the parabola $x + 2y = x^2$.

Solution: The line u_1, u_2 has equation $2x + y = -1$ and the line u_4, u_5 has equation $2x + y = 2$ and these two lines intersect at $a = \left(\frac{1}{4}, -\frac{3}{2}\right)$. The line u_2, u_3 has equation $x + y = 0$ and the line u_5, u_6 has equation $4x - y = 10$ and these two lines intersect at $b = (2,-2)$. The line u_3, u_4 has equation $y = 0$ and the equation u_6, u_1 has equation $x - y = -5$ and these two lines intersect at $c = (-5,0)$. Note that a, b and c all lie on the line with equation $2x + 7y = -10$.