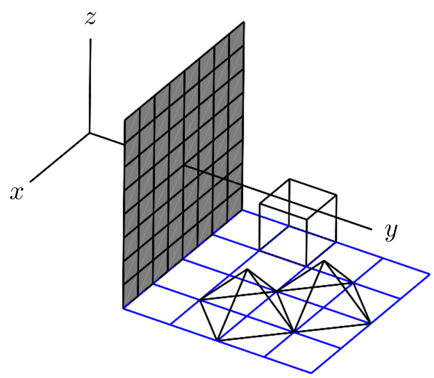
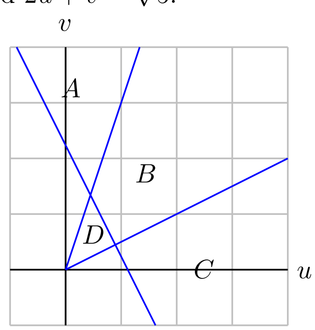
PMATH 321 Non-Euclidean Geometry, Exercises for Chapter 3: Projective Geometry

1: (a) Let $x = (-2, 1, 3)$ and $y = (5, 1, -4)$. Find $d_P\big([x], [y]\big)$.
(b) What fraction of the area of $\mathbb{P}^2$ is covered by a triangle with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$?
(c) Find the area of the circle on $\mathbb{P}^2$ which is circumscribed around a square on $\mathbb{P}^2$ with sides of length $\frac{\pi}{3}$.

2: Make an accurate sketch of the artist's perspective drawing (the image under the gnomic projection) of the scene shown below. The artist's eye is at the origin, the (transparent) canvas is at the plane $y = 1$, the floor lies along the plane $z = -1$, the blue grid lines on the floor are spaced $\frac{1}{2}$ units apart, the cube has sides of length $\frac{1}{2}$ and the two pyramids have height $\frac{1}{2}$.



3: Find the areas of the inverse images in $U_3 \subseteq \mathbb{P}^2$ under the gnomic projection ϕ_3, which is given by $\phi_3\big([x, y, z]\big) = \big(\frac{x}{z}, \frac{y}{z}\big)$, of the regions A, B, C and D in the uv-plane as shown below. The regions are bounded by the lines $v = 3u$, $u = 2v$ and $2u + v = \sqrt{5}$.



4: (a) Let $u = \frac{1}{\sqrt{2}}(0, 1, 1)$, $v = \frac{1}{\sqrt{3}}(1, 1, 1)$ and $\beta = \frac{2\pi}{3}$. Find $p \in \mathbb{S}^2$ and $\theta \in [0, \pi]$ so that $F_u R_{v,\beta} = R_{p,\theta}$ as an isometry on $\mathbb{P}^2$.
(b) Let $u = (0, 0, 1)$, $v = \frac{1}{\sqrt{2}}(1, 1, 0)$ and $w = \frac{1}{\sqrt{2}}(0, 1, 1)$. Find $p \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ such that $F_u F_v F_w = R_{p,\theta}$, as an isometry on $\mathbb{P}^2$.

5: (a) Let $u = (1, 1, -1)$, $v = (0, 1, 1)$, $w = (1, -1, 0)$, $x = (1, 1, 1)$, $y = (0, 1, -1)$ and $z = w$. There are two isometries F on $\mathbb{P}^2$ such that $F\big([u]\big) = [x]$, $F\big([v]\big) = [y]$ and $F\big([w]\big) = [z]$. Express these two isometries in the form $R_{p,\theta}$ and $R_{q,\phi}$ with $p, q \in \mathbb{S}^2$ and $\theta, \phi \in [0, \pi]$.
(b) Let $x = (1, 1, 1)$, $y = (1, -1, 1)$, $z = (1, 0, -1)$, $u = x$, $v = (1, 1, -1)$ and $w = (1, -1, 0)$. There are two isometries F on $\mathbb{P}^2$ such that $F\big([x]\big) = [u]$, $F\big([y]\big) = [v]$ and $F\big([z]\big) = [w]$. Express these two isometries in the form $R_{p,\theta}$ and $R_{q.\phi}$ with $p, q \in \mathbb{S}^2$ and $\theta, \phi \in \mathbb{R}$.

6: For each of the following polynomials $f(x, y)$, find the homogenization $F(x, y, z)$, and make an accurate sketch of the zero sets of the dehomogenizations $f_1(y, z)$, $f_2(x, z)$ and $f_3(x, y)$.
(a) $f(x, y) = x^2 + y^2 - 2x$
(b) $f(x, y) = y - x^3 + x$
(c) $f(x, y) = x^3 + x^2 - y^2$

7: (a) Let $f(x,y) = x+y^2-y$. Find the homogenization $F(x,y,z)$, sketch the zero sets of the dehomogenizations $f_1(y,z)$, $f_2(x,z)$ and $f_3(x,y)$, then use your artistic talent to sketch the zero set $Z(F)$.

(b) Let $f(x,y) = y^2 - x + 1$, let $g(x,y) = y - x + 7$ and let R be the region in the xy-plane which lies between the zero sets $Z(f)$ and $Z(g)$. By finding the homogenizations $F(x,y,z)$ and $G(x,y,z)$ and the dehomogenizations $f_1(y,z)$ and $g_1(y,z)$, find the area of the image of R under the composite $\phi \circ \psi$ where $\psi(x,y) = [x,y,1] \in \mathbb{P}^2$ and $\phi([x,y,z]) = \left(\frac{y}{x}, \frac{z}{x}\right)$.

8: (a) Find $p \in \mathbb{R}^3$, $u \in \mathbb{S}^2$ and $\phi \in \left(0, \frac{\pi}{2}\right)$ such that $V(p,u,\phi)$ intersects the xy-plane in the parabola $y = 3x^2$.

(b) Find $p \in \mathbb{R}^3$, $u \in \mathbb{S}^2$ and $\phi \in \left(0, \frac{\pi}{2}\right)$ such that the intersection of $V(p,u,\phi)$ with the xy-plane is the hyperbola given by $\frac{x^2}{12} - \frac{y^2}{4} = 1$.

(c) Find $p \in \mathbb{R}^3$ and $u \in \mathbb{S}^2$ such that the intersection of $V\left(p,u,\frac{\pi}{4}\right)$ with the xy-plane is the ellipse $\frac{x^2}{12} + \frac{y^2}{6} = 1$.

(d) Verify that the conclusion of Pascal's Theorem holds for the points $u_1 = (-2,3)$, $u_2 = (-1,1)$, $u_3 = (0,0)$, $u_4 = (1,0)$, $u_5 = (4,6)$ and $u_6 = (5,10)$ which all lie on the parabola $x + 2y = x^2$.