# PMATH 321 Non-Euclidean Geometry, Solutions to the Exercises for Chapter 2

**1:** (a) Let $u=(3,4,-1)$ and $v=(4,1,3)$. Find the spherical distance between $u$ and $v$ on the sphere given by $x^2+y^2+z^2=26$.

Solution: The spherical distance is $d=R\,\theta(u,v)=R\cos^{-1}\frac{u\,\bullet\,v}{|u||v|}=\sqrt{26}\,\cos^{-1}\frac{1}{2}=\frac{\sqrt{26}\,\pi}{3}$.

(b) Let $u=\frac{1}{\sqrt{3}}(1,1,-1)$ and $v=\frac{1}{\sqrt{2}}(1,0,1)$. Find a point $w\in\mathbb{S}^2$ such that $L_w$ is the line through $u$ and $v$.

Solution: The line through $u$ and $v$ is the line $L=P\cap\mathbb{S}^2$ where $P$ is the plane through the origin which passes through $u$ and $v$, that is $P=\text{Span}\{u,v\}$. The unit normal vectors of $P$ are $\pm\frac{u\times v}{|u\times v|}$ and so $L=L_w$ where

$$w=\frac{u\times v}{|u\times v|}=\frac{(1,1,-1)\times(1,0,1)}{\big|(1,1,-1)\times(1,0,1)\big|}=\frac{(1,-2,-1)}{\big|(1,-2,-1)\big|}=\tfrac{1}{\sqrt{6}}(1,-2,-1).$$

(c) Let $u=\frac{1}{3}(1,2,2)$ and $v=\frac{1}{\sqrt{5}}(0,1,2)$. Find a point $w\in\mathbb{S}^2$ such that $L_w$ is the line through $u$ which is perpendicular to $L_v$.

Solution: The lines perpendicular to $L_v$ all pass through $v$ and so the line through $u$ which is perpendicular to $L_v$ is the line through $u$ and $v$ which, as above, is the line $L_w$ where

$$w=\frac{u\times v}{|u\times v|}=\frac{(1,2,2)\times(0,1,2)}{\big|(1,2,2)\times(0,1,2)\big|}=\frac{(2,-2,1)}{\big|(2,-2,1)\big|}=\tfrac{1}{3}(2,-2,1).$$

(d) Let $u=\frac{1}{\sqrt{3}}(1,-1,1)$ and $v=\frac{1}{\sqrt{6}}(1,2,1)$. Find the two points of intersection on $\mathbb{S}^2$ of the spherical line $L_u$ and the spherical circle $C\big(v,\frac{\pi}{3}\big)$.

Solution: The line $L_u$ is the set of points $p\in\mathbb{S}^2$ such that $p\bullet u=0$, that is the set of points $(x,y,z)\in\mathbb{S}^2$ with $x-y+z=0$ (1). The circle $C\big(v,\frac{\pi}{3}\big)$ is the set of points $p\in S^2$ such that $p\bullet v=\cos\frac{\pi}{3}=\frac{1}{2}$, that is the set of points $(x,y,z)\in\mathbb{S}^2$ such that $x+2y+z=\frac{\sqrt{6}}{2}$ (2). Solve equations (1) and (2) for $(x,y,z)\in\mathbb{R}^3$ to get $(x,y,z)=\frac{1}{\sqrt{6}}\big((1,1,0)+t(-1,0,1)\big)$ for some $t\in\mathbb{R}$. For these points $(x,y,z)$ we have

$$(x,y,z)\in\mathbb{S}^2\iff x^2+y^2+z^2=1\iff\tfrac{1}{6}\big((1-t)^2+1^1+t^2\big)=1\iff t^2-t-2=0\iff t=-1\text{ or }2.$$

When $t=-1$ we obtain $p=(x,y,z)=\frac{1}{\sqrt{6}}\big((1,1,0)-(-1,0,1)\big)=\frac{1}{\sqrt{6}}(2,1,-1)$ and when $t=2$ we obtain $p=(x,y,z)=\frac{1}{\sqrt{6}}\big((1,1,0)+2(-1,0,1)\big)=\frac{1}{\sqrt{6}}(-1,1,2)$.

(e) Let $u=\frac{1}{\sqrt{2}}(0,1,1)$ and $v=\frac{1}{\sqrt{6}}(1,2,1)$. Find $r>0$ such that $C(u,r)$ is tangent to $L_v$.

Solution: Note that $\cos\theta(u,v)=\frac{u\,\bullet\,v}{|u||v|}=\frac{(0,1,1)\,\bullet\,(1,2,1)}{|(0,1,1)||(1,2,1)|}=\frac{3}{\sqrt{2}\sqrt{6}}=\frac{\sqrt{3}}{2}$ so that $d_S(u,v)=\theta(u,v)=\frac{\pi}{6}$. Look at the sphere with the vector $u\times v$ pointing towards us and the vector $v$ pointing upwards so that the sphere looks like the unit circle, say in the $st$-plane, with $v$ at position $(s,t)=(0,1)$ and $u$ at position $(s,t)=\big(\frac{1}{2},\frac{\sqrt{3}}{2}\big)$. From this viewpoint (you should draw the picture), the line $L_v$ appears as the diameter from $(s,t)=(-1,0)$ to $(s,t)=(1,0)$, and each circle $C(u,r)$ appears as a line segment perpendicular to the vector $(s,t)=\big(\frac{1}{2},\frac{\sqrt{3}}{2}\big)$. In particular, the circle $C\big(u,\frac{\pi}{3}\big)$, which appears as the line segment from $(s,t)=\big(-\frac{1}{2},\frac{\sqrt{3}}{2}\big)$ to $(s,t)=(1,0)$, intersects $L_v$ at the point $(s,t)=(1,0)$, and the circle $C\big(u,\frac{2\pi}{3}\big)$, which appears as the line segment from $(s,t)=(-1,0)$ to $(s,t)=\big(\frac{1}{2},-\frac{\sqrt{3}}{2}\big)$, intersects $L_v$ at the point $(s,t)=(-1,0)$. These are the only two circles $C(u,r)$ which intersect with $L_v$ at exactly one point. Thus the two possible values of $r$ are $r=\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

**2:** (a) Let $u=\frac{1}{\sqrt{5}}(2,0,-1)$, $v=(2,-1,4)$ and $w=(1,3,2)$. Find the oriented angle $\theta_o(v,w)$ from $v$ to $w$ in the tangent space $T_u$.

Solution: We have

$$\cos\theta_o(v,w)=\frac{v\bullet w}{|v||w|}=\frac{(2,-1,4)\bullet(1,3,2)}{\big|(2,-1,4)\big|\big|(1,3,2)\big|}=\frac{7}{\sqrt{21}\sqrt{14}}=\tfrac{1}{\sqrt{6}}$$

so that either $\theta_o(v,w)=\cos^{-1}\frac{1}{\sqrt{6}}$ or $\theta_o(v,w)=2\pi-\cos^{-1}\frac{1}{\sqrt{6}}$, and we have

$$\sqrt{5}\,\det(u,v,w)=\det\begin{pmatrix}2&2&1\\0&-1&3\\-1&4&2\end{pmatrix}=-35<0$$

so that $\sin\theta_o(v,w)<0$, and so $\theta_o(v,w)=2\pi-\cos^{-1}\frac{1}{\sqrt{6}}$.

(b) Let $u=\frac{1}{\sqrt{6}}(1,1,2)$, $v=\frac{1}{\sqrt{14}}(2,1,3)$ and $w=\frac{1}{\sqrt{11}}(1,3,1)$. Find $u_v$ and $u_w$.

Solution: We have

$$\begin{aligned}&\big((1,1,2)\times(2,1,3)\big)\times(1,1,2)=(1,1,-1)\times(1,1,2)=(3,-3,0)=3(1,-1,0)\text{ and}\\&(1,1,2)\times(1,3,1)\big)\times(1,1,2)=(-5,1,2)\times(1,1,2)=(0,12,-6)=6(0,2,-1)\end{aligned}$$

and so $u_v=\frac{1}{\sqrt{2}}(1,-1,0)$ and $u_w=\frac{1}{\sqrt{5}}(0,2,-1)$.

(c) Let $u=\frac{1}{3}(-1,2,-2)$, $v=\frac{1}{\sqrt{2}}(1,0,1)$ and $w=\frac{1}{3\sqrt{3}}(5,-1,1)$. Find the interior angles $\alpha$, $\beta$ and $\gamma$ in the ordered triangle $[u,v,w]$.

Solution: We have

$$3\cdot3\sqrt{3}\det(u,v,w)=\det\begin{pmatrix}-1&1&5\\2&0&-1\\-2&1&1\end{pmatrix}=9>0$$

so that $[u,v,w]$ is positively oriented. We have $\cos a=v\bullet w=\frac{6}{\sqrt{2}\cdot3\sqrt{3}}=\sqrt{\frac{2}{3}}$, $\cos b=w\bullet u=\frac{-9}{3\sqrt{3}\cdot3}=-\frac{1}{\sqrt{3}}$ and $\cos c=u\bullet v=\frac{-3}{3\cdot\sqrt{2}}=-\frac{1}{\sqrt{2}}$ so that $a=\cos^{-1}\sqrt{\frac{2}{3}}$, $b=\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ and $c=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{3\pi}{4}$. By the First Law of Cosines, we have

$$\begin{aligned}\cos\alpha&=\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{\sqrt{\frac{2}{3}}-\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}}{\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2}}}=\frac{1}{\sqrt{2}}\,,\\\cos\beta&=\frac{\cos b-\cos a\cos c}{\sin a\sin c}=\frac{-\frac{1}{\sqrt{3}}+\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}}=0\,,\\\cos\gamma&=\frac{\cos c-\cos a\cos b}{\sin a\sin c}=\frac{-\frac{1}{\sqrt{2}}+\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{2}{3}}}=-\frac{1}{2}\,,\end{aligned}$$

so that $\alpha=\frac{\pi}{4}$, $\beta=\frac{\pi}{2}$ and $\gamma=\frac{2\pi}{3}$.

**3:** Let $[u,v,w]$ be a triangle with edge lengths $a$, $b$ and $c$ and interior angles $\alpha$, $\beta$ and $\gamma$.

(a) Given that $a=\frac{\pi}{3}$, $c=\frac{\pi}{6}$ and $\beta=\frac{2\pi}{3}$, find $b$.

Solution: By the First Law of Cosines we have $\cos\beta=\dfrac{\cos b-\cos a\cos c}{\sin a\sin c}$ and so

$$\cos b=\cos\beta\sin a\sin c+\cos a\cos c=-\tfrac{1}{2}\cdot\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{\sqrt{3}}{2}=\tfrac{\sqrt{3}}{8}$$

and so $b=\cos^{-1}\frac{\sqrt{3}}{8}$.

(b) Given that $a=\frac{\pi}{6}$, $\beta=\frac{5\pi}{6}$ and $\gamma=\frac{\pi}{4}$, find $c$.

Solution: By the Second Law of Cosines we have $\cos a=\dfrac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}$ and so

$$\cos\alpha=\cos a\sin\beta\sin\gamma-\cos\beta\cos\gamma=\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{2}\cdot\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2}=\tfrac{3\sqrt{6}}{8}.$$

It follows that $\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\frac{54}{64}}=\frac{\sqrt{10}}{8}$. By the Second Cosine Law, we have

$$\cos c=\frac{\cos\gamma+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}=\frac{\frac{\sqrt{2}}{2}-\frac{3\sqrt{6}}{8}\cdot\frac{\sqrt{3}}{2}}{\frac{\sqrt{10}}{8}\cdot\frac{1}{2}}=\tfrac{8\sqrt{2}-9\sqrt{2}}{\sqrt{10}}=-\tfrac{1}{\sqrt{5}}$$

and so $c=\cos^{-1}\frac{-1}{\sqrt{5}}$.

(c) Given that $a=\cos^{-1}\frac{1}{3}$, $b=\frac{\pi}{2}$ and $\alpha=\frac{\pi}{4}$, find all possible values of $c$.

Solution: Since $a=\cos^{-1}\frac{1}{3}$ we have $\cos a=\frac{1}{3}$ and $\sin a=\sqrt{1-\cos^2 a}=\sqrt{1-\frac{1}{9}}=\frac{2\sqrt{2}}{3}$. By the Sine Law, we have

$$\sin\beta=\frac{\sin b\sin\alpha}{\sin a}=\frac{\frac{\sqrt{2}}{2}\cdot 1}{\frac{2\sqrt{2}}{3}}=\tfrac{3}{4}$$

hence

$$\cos\beta=\pm\sqrt{1-\sin^2\beta}=\pm\sqrt{1-\tfrac{9}{16}}=\pm\tfrac{\sqrt{7}}{4}.$$

By the First Law of Cosines, we have

$$\cos\gamma=\frac{\cos c-\cos a\cos b}{\sin a\sin b}=\frac{\cos c-\frac{1}{3}\cdot 0}{\frac{2\sqrt{2}}{3}\cdot 1}=\tfrac{3\sqrt{2}}{4}\cos c$$

and so the Second Law of Cosines gives

$$\cos c=\frac{\cos\gamma+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}=\frac{\frac{3\sqrt{2}}{4}\cos c\pm\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{7}}{4}}{\frac{\sqrt{2}}{2}\cdot\frac{3}{4}}=2\cos c\pm\tfrac{\sqrt{7}}{3}.$$

Thus we have $\cos c=\pm\frac{\sqrt{7}}{3}$. hence $c=\cos^{-1}\left(\pm\frac{\sqrt{7}}{3}\right)$.

**4:** (a) Let $u=\frac{1}{\sqrt6}(1,2,1)$, $v=\frac{1}{\sqrt6}(1,-1,2)$ and $w=\frac{1}{\sqrt6}(2,1,-1)$. Find the centroid of the spherical triangle $[u,v,w]$, that is find the point of intersection of the 3 medians $\left[u,\frac{v+w}{|v+w|}\right]$, $\left[v,\frac{w+u}{|w+u|}\right]$ and $\left[w,\frac{u+v}{|u+v|}\right]$.

Solution: The midpoint of the spherical line segment $[v,w]$ is the point $\frac{v+w}{|v+w|}$. The spherical line $L$ through $u$ and $\frac{v+w}{|v+w|}$ is the intersection of $\mathbb{S}^2$ with the vector space $P=\text{Span}\left\{u,\frac{v+w}{|v+w|}\right\}=\text{Span}\{u,v+w\}$. Note that $u+v+w\in P$ and so the point $\frac{u+v+w}{|u+v+w|}\in P\cap\mathbb{S}^2=L$. Similarly, the point $\frac{u+v+w}{|u+v+w|}$ lies on each of the other 2 medians. Thus the centroid is the point $g=\frac{u+v+w}{|u+v+w|}=\frac{1}{\sqrt6}(2,1,1)$.

(b) Let $u=\frac13(2,2,-1)$, $v=\frac13(1,2,2)$ and $w=\frac13(2,-1,-2)$. Find the centre and radius of the circumscribed circle of the spherical triangle $[u,v,w]$.

Solution: The circumcentre is the point $x$ such that $d_S(x,u)=d_S(x,v)=d_S(x,w)$. We have

$$\begin{aligned}
d_S(x,u)=d_S(x,v)=d_S(x,w) &\iff \cos^{-1}x\cdot u=\cos^{-1}x\cdot v=\cos^{-1}x\cdot w \iff x\cdot u=x\cdot v=x\cdot w\\
&\iff x\cdot u=x\cdot v \text{ and } x\cdot u=x\cdot w \iff x\cdot(u-v)=0 \text{ and } x\cdot(u-w)=0\\
&\iff x\cdot(1,-0,-3)=0 \text{ and } x\cdot(0,3,1)=0 \iff x_1-3x_3=0 \text{ and } 3x_2+x_3=0\\
&\iff x=(x_1,x_2,x_3)=t(9,-1,3) \text{ for some } t\in\mathbb{R}\,.
\end{aligned}$$

To get $|x|=1$ we take $t=\frac{1}{\sqrt{91}}$, so the circumcentre is the point $o=\frac{1}{\sqrt{91}}(9,-1,3)$. The radius of the circumcircle is $r=d_S(o,u)=\cos^{-1}(o\cdot u)=\cos^{-1}\frac{13}{3\sqrt{91}}$.

(c) Let $u=\frac{1}{\sqrt3}(1,1,1)$, $v=\frac{1}{\sqrt3}(1,-1,-1)$ and $w=(0,1,0)$. Find the incentre of triangle $[u,v,w]$.

Solution: The vector $u_v$ is in the same direction of the vector

$$u'=\big((1,1,1)\times(1,0,1)\big)\times(1,1,1)=(1,0,-1)\times(1,1,1)=(1,-2,1)$$

and the vector $u_w$ is in the same direction as the vector

$$u''=\big((1,1,1)\times(0,0,-1)\big)\times(1,1,1)=(-1,1,0)\times(1,1,1)=(1,1,-2).$$

Since the vectors $u'$ and $u''$ have the same length, the angle bisector at $u$ is in the direction of the vector

$$u'''=u'+u''=(1,-2,1)+(1,1,-2)=(2,-1,-1).$$

This angle bisector has normal vector

$$m=\sqrt3 u\times u'''=(1,1,1)\times(2,-1,-1)=(0,3,-3).$$

The vector $v_u$ is in the same direction as the vector

$$v'=\big((1,0,1)\times(1,1,1)\big)\times(1,0,1)=(-1,0,1)\times(1,0,1)=(0,2,0)$$

and the vector $v_w$ is in the same direction as the vector

$$v''=\big((1,0,1)\times(0,0,-1)\big)\times(1,0,1)=(0,1,0)\times(1,0,1)=(1,0,-1).$$

Since the vectors $\frac{1}{\sqrt2}v'$ and $v''$ have the same length, the angle bisector at $v$ is in the direction of the vector

$$v'''=\frac{1}{\sqrt2}v'+v''=(0,\sqrt2,0)+(1,0,-1)=(1,\sqrt2,-1).$$

This angle bisector has normal vector

$$n=v\times v'''=\tfrac{1}{\sqrt2}(1,0,1)\times(1,\sqrt2,-1)=(-1,\sqrt2,1).$$

Since the two angle bisectors have normal vectors $n$ and $m$, their point of intersection lies in the direction of the vector

$$x=\tfrac13 m\times n=(0,1,-1)\times(-1,\sqrt2,1)=(1+\sqrt2,1,1).$$

Thus the incentre of the triangle $[u,v,w]$ is the point

$$i=\frac{x}{|x|}=\frac{(1+\sqrt2,1,1)}{\sqrt{(1+\sqrt2)^2+1^2+1^2}}=\tfrac{1}{\sqrt{5+2\sqrt2}}(1+\sqrt2,1,1).$$

**5:** (a) Find the perimeter of the square on $\mathbb{S}^2$ with interior angles equal to $\frac{2\pi}{3}$.

Solution: Let $\ell$ be the length of the sides of the square. Notice that the square can be cut into 4 triangles (with a common vertex at the centre of the square) each of which is congruent to a triangle $[u, v, w]$ with $a = \ell$, $\alpha = \frac{\pi}{2}$ and $\beta = \gamma = \frac{\pi}{3}$. By the Second Law of Cosines, we have

$$\cos \ell = \cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma} = \frac{0 + \frac{1}{2} \cdot \frac{1}{2}}{\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}} = \frac{1}{3}.$$

Thus the perimeter of the square is $L = 4\ell = 4 \cos^{-1} \frac{1}{3}$.

(b) Find the area of the regular hexagon on $\mathbb{S}^2$ with sides of length $\ell = \cos^{-1} \frac{2}{3}$.

Solution: Note that the hexagon can be cut into 6 triangles (with a common vertex at the centre of the hexagon) each of which is congruent to a triangle $[u, v, w]$ with $a = \ell = \cos^{-1} \frac{2}{3}$, $\alpha = \frac{\pi}{3}$ and $\beta = \gamma$. The Second Law of Cosines gives

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \Longrightarrow \frac{2}{3} = \frac{\frac{1}{2} + \cos^2 \beta}{\sin^2 \beta} \Longrightarrow \frac{2}{3}(1 - \cos^2 \beta) = \frac{1}{2} + \cos^2 \beta$$
$$\Longrightarrow \frac{1}{6} = \frac{5}{3} \cos^2 \beta \Longrightarrow \cos^2 \beta = \frac{1}{10} \Longrightarrow \beta = \cos^{-1} \frac{1}{\sqrt{10}}.$$

Thus the area of the hexagon is

$$A = 6\big(\alpha + \beta + \gamma - \pi\big) = 6\big(\frac{\pi}{3} + 2\cos^{-1} \frac{1}{\sqrt{10}} - \pi\big) = 12 \cos^{-1} \frac{1}{\sqrt{10}} - 4\pi.$$

(c) Find the perimeter and the area of the regular hexagon on $\mathbb{S}^2$ which is inscribed in an equilateral triangle with interior angles $\frac{\pi}{2}$.

Solution: Let $\ell$ be the length of the sides of the equilateral triangle. By the Second Law of Cosines, we have

$$\cos \ell = \frac{\cos \frac{\pi}{2} + \cos^2 \frac{\pi}{2}}{\sin^2 \frac{\pi}{2}} = 0$$

so that $\ell = \frac{\pi}{2}$. Let $y$ be the length of the sides of the hexagon and let $y + 2x = \ell = \frac{\pi}{2}$ so that the original triangle is the union of the hexagon together with three triangles each of which is congruent to a triangle with $\alpha = \frac{\pi}{2}$, $a = y$ and $b = c = x$. The First Law of Cosines gives

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \Longrightarrow 0 = \frac{\cos y - \cos^2 x}{\sin^2 x} \Longrightarrow \cos y = \cos^2 x.$$

Since $y + 2x = \frac{\pi}{2}$ this gives

$$\cos^2 x = \cos y = \cos\left(\frac{\pi}{2} - 2x\right) = \sin 2x = 2 \sin x \cos x$$

so we have $\cos x = 0$ or $\cos x = 2 \sin x$. Since $x < \frac{\pi}{2}$ so that $\cos x \neq 0$, we must have $\cos x = 2 \sin x$, so that $\tan x = \frac{1}{2}$ hence $x = \tan^{-1} \frac{1}{2}$. Thus

$$y = \frac{\pi}{2} - 2x = \frac{\pi}{2} - 2\tan^{-1} \frac{1}{2} = \frac{\pi}{2} - \tan^{-1} \frac{4}{3} = \tan^{-1} \frac{3}{4}.$$

Note that the hexagon can be cut into 6 triangles each of which is congruent to a triangle with $a = y = \tan^{-1} \frac{3}{4}$, $\alpha = \frac{\pi}{3}$ and $\beta = \gamma$. The Second Law of Cosines gives

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \Longrightarrow \frac{4}{5} = \frac{\frac{1}{2} + \cos^2 \beta}{\sin^2 \beta} \Longrightarrow \frac{4}{5}\big(1 - \cos^2 \beta\big) = \frac{1}{2} + \cos^2 \beta$$
$$\Longrightarrow \frac{3}{10} = \frac{9}{5} \cos^2 \beta \Longrightarrow \cos^2 \beta = \frac{1}{6} \Longrightarrow \cos \beta = \frac{1}{\sqrt{6}} \Longrightarrow \beta = \cos^{-1} \frac{1}{\sqrt{6}}.$$

Thus the perimeter and the area of the hexagon are equal to

$$L = 6a = 6y = 6 \tan^{-1} \frac{3}{4} \text{ , and}$$
$$A = 6\big(\alpha + \beta + \gamma - \pi\big) = 6\big(\frac{\pi}{3} + 2\cos^{-1} \frac{1}{\sqrt{6}} - \pi\big) = 12 \cos^{-1} \frac{1}{\sqrt{6}} - 4\pi.$$

**6:** (a) For a point $(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = R^2$, let $\phi \in [0, \pi]$ measure the angle in $\mathbb{R}^3$ from $(0, 0, 1)$ to $(x, y, z)$ and let theta $\theta \in \mathbb{R}$ measure the angle in $\mathbb{R}^2$ from $(1, 0)$ counterclockwise to $(x, y)$. Given $0 \le \phi_1 \le \phi_2 \le \pi$ and $\theta_1 \le \theta_2 \le \theta_1 + 2\pi$, find the area of the portion of the sphere given by $\phi_1 \le \phi \le \phi_2$ and $\theta_1 \le \theta \le \theta_2$.

Solution: Since $\phi$ measures the angle between $(0, 0, 1)$ and $(x, y, z)$, we have $z = R\cos\phi$ and so the portion of the sphere given by $\phi_1 \le \phi \le \phi_2$ is the portion of the sphere which lies between the planes $z = R\cos\phi_1$ and $z = R\cos\phi_2$, and its area is

$$S = 2\pi R\,\Delta = 2\pi R\big(R\cos\phi_1 - R\cos\phi_2\big) = 2\pi R^2(\cos\phi_1 - \cos\phi_2).$$

By looking at the sphere with the $z$-axis pointing towards us and the $x$-axis pointing to the right, we see that area $A$ of the portion of the sphere given by $\phi_1 \le \phi \le \phi_2$ and $\theta_1 \le \theta \le \theta_2$ is equal to $\frac{\theta_2 - \theta_1}{2\pi}$ times the area of the above area, that is

$$A = \frac{\theta_2 - \theta_1}{2\pi}\,S = \frac{\theta_2 - \theta_1}{2\pi} \cdot 2\pi R^2(\cos\phi_1 - \cos\phi_2) = R^2(\theta_2 - \theta_1)(\cos\phi_1 - \cos\phi_2).$$

(b) A light at position $(0, 0, 8)$ shines down on a spherical balloon of radius $\sqrt{5}$ centred at $(3, 4, 3)$. Find the area of the shadow which is cast on the $xy$-plane (given that the shadow is an ellipse and that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to $\pi ab$).

Solution: Draw a picture to represent the situation with the vector $(3, 4, 0)$ pointing to the right and the vector $(0, 0, 1)$ pointing upwards. From this viewpoint the ballon looks like the circle, say in the $sz$-plane, of radius $\sqrt{5}$ centred at $(s, z) = (5, 3)$, and the light is at position $(s, z) = (0, 8)$. Rays of light shine down following the lines $z = 8 - 2s$ and $z = 8 - \frac{1}{2}s$, which are tangent to the circle at $(s, z) = (3, 2)$ and $(s, t) = (6, 4)$, and meet the $s$-axis at $s = 4$ and $s = 16$. This shows that elliptical shadow has major axis equal to 12 units.

Draw another picture with the vector $(3, 4, 0)$ pointing towards us and the vector $(0, 0, 1)$ pointing upwards. From this viewpoint, the balloon looks like the circle, say in the $tz$-plane, of radius $\sqrt{5}$ centred at $(t, z) = (0, 3)$, and the light is positioned at $(t, z) = (0, 8)$. Rays shine down following the lines $z = 8 \pm 2t$ which are tangent to the circle at $(t, z) = (\pm 2, 4)$ and meet the $t$-axis at $t = \pm 4$. This shows that the elliptical shadow has minor axis equal to 8 units. Thus the elliptical shadow has the same shape as the ellipse $\frac{x^2}{6^2} + \frac{y^2}{4^2} = 1$ and its area is $A = \pi \cdot 6 \cdot 4 = 24\pi$.

(c) A light at position $(0, 0, 30)$ shines down on a red spherical balloon of radius $\sqrt{10}$ centred at $(0, 0, 20)$ casting a shadow on a green balloon of radius $6\sqrt{5}$ centred at $(0, 0, 0)$. Find the area of the illuminated portion of the green balloon.

Solution: Draw a picture to represent the situation with the $x$-axis pointing towards us and the $z$-axis pointing upwards. From this viewpoint, the red ballon looks like the circle, in the $yz$-plane, of radius $\sqrt{10}$ centred at $(y, z) = (0, 30)$, the green balloon looks like the circle of radius $6\sqrt{5}$ centred at $(y, z) = (0, 0)$, and the light is at position $(y, z) = (0, 30)$. Two rays shine down following the lines $z = 30 \pm 2y$ which are tangent to the green balloon at $(y, z) = (\pm 12, 6)$, and two rays shine down following the lines $z = 30 - \pm 3y$ which are tangent to the red balloon at $(y, z) = (\pm 3, 21)$ and continue down to intersect the green ballon at the points $(y, z) = (\pm 6, 12)$. This shows that the illuminated portion of the green balloon is the portion which lies between $z = 6$ and $z = 12$. The area of this portion is

$$A = 2\pi R\,\Delta = 2\pi \cdot 6\sqrt{5} \cdot (12 - 6) = 72\sqrt{5}\,\pi.$$

**7:** (a) Find the radius $R$ of a sphere on which there is an equilateral triangle with sides of length $\pi$ and angles equal to $\frac{5\pi}{6}$.

Solution: The Second Cosine Law, modified for a sphere of radius $R$, gives

$$\cos\frac{\pi}{R} = \frac{\cos\frac{5\pi}{6} + \cos^2\frac{5\pi}{6}}{\sin^2\frac{5\pi}{6}} = \frac{-\frac{\sqrt{3}}{2} + \frac{3}{4}}{\frac{1}{4}} = 3 - 2\sqrt{3}$$

so that

$$R = \frac{\pi}{\cos^{-1}(3 - 2\sqrt{3})}.$$

(b) Find an approximate value for the radius $R$ of a sphere on which there is a circle of radius 2 and circumference $\frac{215\pi}{54}$.

Solution: The circumference $L$ of a circle of radius $r$ on a sphere of radius $R$ is given by

$$L = 2\pi R \sin\tfrac{r}{R} \cong 2\pi R\Big(\tfrac{r}{R} - \tfrac{1}{6}\tfrac{r^3}{R^3}\Big) = 2\pi r - \tfrac{\pi}{3}\tfrac{r^3}{R^2}.$$

Putting in $L = \frac{215\pi}{54}$ and $r = 2$ gives

$$\tfrac{215\pi}{54} \cong 4\pi - \tfrac{8\pi}{3R^2} \Longrightarrow \tfrac{8}{3R^2} \cong 4 - \tfrac{215}{54} = \tfrac{1}{54} \Longrightarrow R^2 \cong \tfrac{8\cdot 54}{3} = 144 \Longrightarrow R \cong 12.$$

(c) Let $R$ be the radius of the Earth, in meters ($R \cong 6,370,000$). We describe the position of a point on the Earth in terms of its longitude $\theta$ (with $\theta = 0$ at Greenwitch, England and $\theta = \frac{\pi}{2}$ somewhere in Bangladesh) and its latitude $\phi$ (with $\phi = 0$ at the equator and $\phi = \frac{\pi}{2}$ at the north pole). Find the distance (expressed as a multiple of $R$) and the bearing (expressed as an angle north of east) from the point at $(\theta, \phi) = (\frac{\pi}{3}, \frac{\pi}{6})$ to the point at $(\theta, \phi) = (\frac{\pi}{2}, \frac{\pi}{4})$.

Solution: Consider the spherical triangle with vertices at $u$, $v$ and $w$ where $u$ is given by $(\theta, \phi) = \big(\frac{\pi}{3}, \frac{\pi}{6}\big)$, $v$ is given by $(\theta, \phi) = \big(\frac{\pi}{2}, \frac{\pi}{4}\big)$, and $w$ is the north pole, which is given by $\phi = \frac{\pi}{2}$. For this triangle we have $a = R \cdot \frac{\pi}{4}$, $b = R \cdot \frac{\pi}{3}$, and $\gamma = \frac{\pi}{6}$. The First Law of Cosines, modified for a sphere of radius $R$, gives

$$\cos\gamma = \frac{\cos(c/R) - \cos(b/R)\cos(a/R)}{\sin(b/R)\sin(a/R)}$$

so

$$\begin{aligned}\cos(c/R) &= \cos\gamma\sin(b/R)\sin(a/R) + \cos(b/R)\cos(a/R)\\ &= \cos\tfrac{\pi}{6}\sin\tfrac{\pi}{4}\sin\tfrac{\pi}{3} + \cos\tfrac{\pi}{4}\cos\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2}\cdot\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\cdot\tfrac{1}{2} = \tfrac{5\sqrt{2}}{8}\,.\end{aligned}$$

Thus the required distance is $c = R\cos^{-1}\big(\frac{5\sqrt{2}}{8}\big)$. The Law of Sines, modified for a sphere of radius $R$, gives

$$\frac{\sin\alpha}{\sin(a/R)} = \frac{\sin\gamma}{\sin(c/R)}$$

so we have

$$\sin\alpha = \frac{\sin(a/R)\sin\gamma}{\sin(c/R)} \cong \frac{\sin\frac{\pi}{6}\sin\frac{\pi}{4}}{\sqrt{1 - \big(\frac{5\sqrt{2}}{8}\big)^2}} = \frac{\frac{1}{2}\cdot\frac{\sqrt{2}}{2}}{\frac{\sqrt{14}}{8}} = \frac{2}{\sqrt{7}}\,.$$

Thus the bearing is $\theta$ east of north, where

$$\theta = \tfrac{\pi}{2} - \alpha = \tfrac{\pi}{2} - \sin^{-1}\big(\tfrac{2}{\sqrt{7}}\big) = \cos^{-1}\big(\tfrac{2}{\sqrt{7}}\big)\,.$$

**8:** (a) Let $u = \frac{1}{\sqrt{6}}(1,-1,2)$. Express the isometry $F_u$ in matrix form.

Solution: We have

$$F_u = I - 2uu^T = I - \tfrac{1}{3}\begin{pmatrix}1\\-1\\2\end{pmatrix}\begin{pmatrix}1 & -1 & 2\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \tfrac{1}{3}\begin{pmatrix}1&-1&2\\-1&1&-2\\2&-2&4\end{pmatrix} = \tfrac{1}{3}\begin{pmatrix}2&1&-2\\1&2&2\\-2&2&-1\end{pmatrix}.$$

(b) Let $u = \frac{1}{3}(1,2,-2)$ and let $\theta = \frac{\pi}{2}$. Express the isometry $R_{u,\theta}$ in matrix form.

Solution: Let $v = \frac{1}{3}(2,1,2)$ and $w = u \times v = \frac{1}{3}(2,-2,-1)$ so that $\{u,v,w\}$ is an orthonormal basis for $\mathbb{R}^3$. Then

$$R_{u,\theta} = \big(u,v,w\big)\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}\big(u,v,w\big)^T = \tfrac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\-2&2&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\cdot\tfrac{1}{3}\begin{pmatrix}1&2&-2\\2&1&2\\2&-2&-1\end{pmatrix}$$

$$= \tfrac{1}{9}\begin{pmatrix}1&2&2\\2&1&-2\\-2&2&-1\end{pmatrix}\begin{pmatrix}1&2&-2\\-2&2&1\\2&1&2\end{pmatrix} = \tfrac{1}{9}\begin{pmatrix}1&8&4\\-4&4&-7\\-8&-1&4\end{pmatrix}.$$

(c) Let $u = \frac{1}{\sqrt{3}}(1,1,1)$, $\theta = \frac{\pi}{3}$, $v = \frac{1}{\sqrt{6}}(2,-1,1)$ and $x = \frac{1}{\sqrt{2}}(1,0,1)$. Find $R_{u,\theta}F_v(x)$.

Solution: Let

$$w = F_v(x) = x - 2x \bullet v\, v = \tfrac{1}{\sqrt{2}}\Big((1,0,1) - 2\cdot\tfrac{3}{6}(2,-1,1)\Big) = \tfrac{1}{\sqrt{2}}(-1,1,0).$$

Note that $u \bullet w = 0$ and let $y = u \times w = \frac{1}{\sqrt{6}}(-1,-1,2)$ so that $\{u,w,y\}$ is an orthonormal basis for $\mathbb{R}^3$. Then

$$R_{u,\theta}F_v(x) = R_{u,\theta}(w) = \cos\theta\cdot w + \sin\theta\cdot y = \tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}(-1,1,0) + \tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{\sqrt{6}}(-1,-1,2) = \tfrac{1}{\sqrt{2}}(-1,0,1).$$

(d) Find $u \in \mathbb{S}^2$ and $\theta \in [0,\pi]$ such that, in matrix form, we have $-R_{u,\theta} = \frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}$.

Solution: Let $A = -R_{u,\theta}$. Since $A$ is symmetric, it is orthogonally diagonalizable with real eigenvalues, since $A$ is orthogonal, its eigenvalues have norm 1 so they are equal to $\pm 1$, since $\det A = -1$, the eigenvalue $-1$ occurs with multiplicity 1 or 3, and since $A \neq -I$, the eigenvalue $-1$ occurs with multiplicity 1. Thus $A$ is orthogonally similar to the diagonal matrix with diagonal entries $1,-1,-1$ (this can also be verified by calculating the characteristic polynomial of $A$). It follows that $A$ is equal to a reflection $A = F_u$ and hence that $R_{u,\theta} = -F_u = R_{u,\pi}$. Thus the rotation angle is $\theta = \pi$. The axis vector $u$ is a unit eigenvector for the eigenvalue 1. We have

$$I - R_{u,\pi} = \tfrac{1}{3}\begin{pmatrix}4&2&2\\2&4&-2\\2&-2&4\end{pmatrix} \sim \begin{pmatrix}2&1&1\\1&2&-1\\1&-1&2\end{pmatrix} \sim \begin{pmatrix}1&-1&2\\1&2&-1\\1&-1&2\end{pmatrix} \sim \begin{pmatrix}1&-1&2\\0&3&-3\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&-1\\0&0&0\end{pmatrix}$$

Thus one eigenvector is $(-1,1,1)$ and so we can take $u = \pm\frac{1}{\sqrt{3}}(-1,1,1)$.

**9:** (a) Let $u = \frac{1}{\sqrt{2}}(1,1,0)$, $w = \frac{1}{\sqrt{3}}(1,-1,1)$ and $\theta = \frac{\pi}{3}$. Find $v \in \mathbb{S}^2$ such that $F_u F_v = R_{w,\theta}$.

Solution: Note that $u \cdot w = 0$ so that $u \in T_w$. By the Product of Two Reflections Theorem, to get $F_u F_v = R_{w,\theta}$ we can choose $v \in T_w$ so that $\theta_o(v,u) = \frac{\theta}{2} = \frac{\pi}{6}$. To do this, let $x = w \times u = \frac{1}{\sqrt{6}}(-1,1,2)$ so that $\{w,u,x\}$ is an orthonormal basis for $\mathbb{R}^3$ and then let

$$v = R_{w,-\frac{\pi}{6}}(u) = \frac{\sqrt{3}}{2}\,u - \frac{1}{2}\,x = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}(1,1,0) - \frac{1}{2} \cdot \frac{1}{\sqrt{6}}(-1,1,2) = \frac{1}{\sqrt{6}}(2,1,-1).$$

(b) Let $u = \frac{1}{\sqrt{2}}(1,0,-1)$, $v = \frac{1}{\sqrt{6}}(-1,2,1)$, $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$. Find $w$ and $\gamma$ such that $R_{u,2\alpha} R_{v,2\beta} R_{w,2\gamma} = I$.

Solution: Note that when $[u,v,w]$ is a positively oriented triangle in $\mathbb{S}^2$ with angles $\alpha$, $\beta$ and $\gamma$, if we let $L$, $M$ and $N$ be the lines in $\mathbb{S}^2$ which contain the edges $[v,w]$, $[w,u]$ and $[u,v]$ respectively, then we have

$$R_{u,2\alpha} R_{v,2\beta} R_{w,2\gamma} = F_M F_N F_N F_L F_L F_M = I.$$

Thus it suffices to find $w$ and $\gamma$ so that $[u,v,w]$ is a triangle with angles $\alpha$, $\beta$ and $\gamma$. We can find $w$ and $\gamma$ using spherical trigonometry. In the triangle $[u,v,w]$ we have $\cos c = u \cdot v = -\frac{1}{\sqrt{3}}$. By the Second Law of cosines we then have

$$\cos\gamma = \cos c \sin\alpha \sin\beta - \cos\alpha\cos\beta = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = -\frac{\sqrt{2}}{2}$$

so that $\gamma = \frac{3\pi}{4}$. Using the Second Law of Cosines twice more gives

$$\cos a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma} = \frac{\frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{3}}\,, \text{ and}$$

$$\cos b = \frac{\cos\beta + \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} = \frac{\frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}} = 0.$$

Let $w = (x,y,z)$. To get $\cos b = 0$ we need $u \cdot w = 0$, that is $x - z = 0$ (1). To get $\cos a = \frac{1}{\sqrt{3}}$ we need $v \cdot w = \frac{1}{\sqrt{3}}$, that is $-x + 2y + z = \sqrt{2}$ (2). Solve (1) and (2) to get $(x,y,z) = \left(t, \frac{1}{\sqrt{2}}, t\right)$. To get $|w| = 1$ we need $t^2 + \frac{1}{2} + t^2 = 1$, that is $2t^2 = \frac{1}{2}$ so we need $t = \pm\frac{1}{2}$. Take $t = \frac{1}{2}$ to get $w = \frac{1}{2}(1, \sqrt{2}, 1)$.

(c) Find $p \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ so that $F = -R_{p,\theta}$ where $F$ is the isometry such that $F(u) = u'$, $F(v) = v'$ and $F(w) = w'$ where $u = \frac{1}{\sqrt{2}}(1,1,0)$, $v = (0,1,0)$, $w = \frac{1}{\sqrt{3}}(1,1,1)$, $u' = \frac{1}{\sqrt{2}}(0,-1,1)$, $v' = (0,0,1)$ and $w' = \frac{1}{\sqrt{3}}(1,-1,1)$.

Solution: We provide a solution which follows the proof of the Congruent Triangles and Isometries Theorem (Theorem 2.36) and the proof of the Geometric Classification of Isometries Theorem (Theorem 2.39).

Let $L$ be the line $y = 0$, which is the perpendicular bisector of $w$ and $w'$. Let $u_1 = F_L(u) = \frac{1}{\sqrt{2}}(1,-1,0)$, $v_1 = F_L(v) = (0,-1,0)$ and $w_1 = F_L(w) = w' = \frac{1}{\sqrt{3}}(1,-1,1)$. Let $M$ be the line $y + z = 0$ , which is the perpendicular bisector of $v_1$ and $v'$. Let $u_2 = F_M(u_1) = \frac{1}{\sqrt{2}}(1,0,1)$, $v_2 = F_M(v_1) = v'$, and $w_2 = F_M(w_1) = w_2 = w'$. Let $N$ be the line $x + y = 0$, which is the perpendicular bisector of $u_2$ and $u'$. Then $F_N(u_2) = u'$, $F_N(v_2) = v_2 = v'$ and $F_N(w_2) = w_2 = w'$. It follows that $F = F_N F_M F_L$.

By the Product of Two Reflections Theorem (Theorem 2.33), we see that $F_M F_L = R_{e_1,\frac{\pi}{2}}$ where $e_1 = (1,0,0)$ $\big($because $L = L_a$ with $a = (0,1,0)$ and $M = L_b$ with $b = \frac{1}{\sqrt{2}}(0,1,1)$ and in the tangent space $T_{e_1}$ we have $\varphi(a,b) = \frac{\pi}{4}\big)$. Let $N' = N$, let $M'$ be the line $z = 0$ which is the line through $e_1$ which is orthogonal to $N'$, and let $L' = M$ so that, by Theorem 2.33, we have $F_{M'} F_{L'} = R_{e_1,\frac{\pi}{2}} = F_M F_L$ $\big($because $M' = L_{e_3}$ with $e_3 = (0,0,1)$ and $L' = L_c$ with $c = \frac{1}{\sqrt{2}}(0,1,1)$ and in the tangent space $T_{e_1}$ we have $\theta_o(c,e_3) = \frac{\pi}{4}\big)$. It follows that $F = F_{N'} F_{M'} F_{L'}$. Since $M' \cap N' = \{u_1\}$ where $u_1 = \frac{1}{\sqrt{2}}(1,-1,0)$ and $M'$ is perpendicular to $N'$, by Theorem 2.33 we have $F_{N'} F_{M'} = R_{u_1,\pi}$. Let $L'' = L'$, let $N''$ be the line through $u_1$ orthogonal to $L''$, and let $M''$ be the line through $u_1$ which is orthogonal to $N''$ so that $F_{N''} F_{M''} = R_{u_1,\pi} = F_{N'} F_{M'}$. It follows that $F = F_{N''} F_{M''} F_{L''}$.

Note that $L'' = L_r$ where $r = \frac{1}{\sqrt{2}}(0,1,1)$, and $N'' = L_p$ where $p = \frac{1}{\sqrt{3}}(1,1,-1)$ and $M'' = L_s$ where $s = \frac{1}{\sqrt{6}}(1,1,2)$. In the tangent space $T_p$ we have $\theta_o(r,s) = \frac{\pi}{6}$ so, by Theorem 2.33, $F_{M''} F_{L''} = R_{p,\frac{\pi}{3}}$. Thus

$$F = F_{N''} F_{M''} F_{L''} = F_p R_{p,\frac{\pi}{3}} = -R_{p,\frac{4\pi}{3}}\,.$$

**10:** (a) Show that for every $w \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ there exist $u, v \in \mathbb{S}^2$ such that $R_{u,\pi}R_{v,\pi} = R_{w,\theta}$.

Solution: Let $w \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$. Choose lines $L$ and $M$ through $w$ so that $R_{w,\theta} = F_L F_M$. Let $N$ be a line which is perpendicular to both $L$ and $M$. Choose $u, v \in \mathbb{S}^2$ so that $L \cap N = \{\pm u\}$ and $M \cap N = \{\pm v\}$. Then

$$R_{w,\theta} = F_L F_M = F_L F_N F_N F_M = R_{u,\pi} R_{v,\pi}.$$

(b) Let $u \in \mathbb{S}^2$ and let $L$ be a line in $\mathbb{S}^2$. Show that $\big(F_L R_{u,\pi}\big)^2 = I$ if and only if either $u \in L$ or $L = L_u$.

Solution: Note first that

$$(F_L R_{u,\pi})^2 = I \iff F_L R_{u,\pi} = (F_L R_{u,\pi})^{-1} \iff F_L R_{u,\pi} = {R_{u,\pi}}^{-1} {F_L}^{-1} \iff F_L R_{u,\pi} = R_{u,\pi} F_L.$$

Suppose that $u \in L$. Let $M$ be the line through $u$ which is perpendicular to $L$ so that, by Theorem 2.33, we have $F_L F_M = R_{u,\pi} = F_M F_L$. Then

$$F_L R_{u,\theta} = F_L F_L F_M = F_M = F_M F_L F_L = R_{u,\pi} F_L.$$

Now suppose that $L = L_u$ so that $F_L = F_u$ then, by Part (2) of Theorem 5.38, we have

$$F_L R_{u,\pi} = F_u R_{u,\pi} = R_{u,\pi} F_u = R_{u,\pi} F_L.$$

Finally suppose that $u \notin L$ and $L \neq L_u$. Choose $v \in \mathbb{S}^2$ so that $L = L_v$ and note that $v \neq \pm u$ since $L \neq L_u$. Let $M$ be the line through $u$ which is perpendicular to $L$ and choose $p \in \mathbb{S}^2$ so that $M = L_p$. Choose $w \in \mathbb{S}^2$ so that $L \cap M = \{\pm w\}$ and note that $w \neq \pm u$ since $u \notin L$. Let $N$ be the line through $u$ perpendicular to $M$ so that $R_{u,\pi} = F_N F_M = F_M F_N$, and note that $N \neq L$ since $u \in M$ but $u \notin L$. Note that since $u, w \in M = L_p$ we have $u, w \in T_p$, and so $F_N F_L = R_{p,\theta}$ where $\theta = 2\varphi = \varphi(u, w)$. Note that since $u \neq \pm w$ we have $\theta \neq 2\pi k$ for $k \in \mathbb{Z}$, and since $u \neq \pm v$ we have $\theta \neq \pi + 2\pi k$ for $k \in \mathbb{Z}$. We have

$$F_L R_{u,\pi} = F_L F_N F_M = R_{p,\theta} F_p \quad \text{and} \quad R_{u,\pi} F_L = F_M F_N F_L = F_p R_{p,-\theta} = R_{p,-\theta} F_p.$$

Since $\theta \neq \pi k$ for $k \in \mathbb{Z}$ we have $\theta \neq -\theta + 2\pi k$ for $k \in \mathbb{Z}$, so $R_{p,\theta} \neq R_{p,-\theta}$ hence $R_{u,\pi} F_L \neq F_L R_{u,\pi}$.

(c) Let $[u, v, w]$ be a positively oriented triangle with circumcentre $p$ and let $L$, $M$ and $N$ are the perpendicular bisectors of edges $[v, w]$, $[w, u]$ and $[u, v]$ respectively. Show that $F_L F_M F_N = F_N F_M F_L = F_K$ where $K$ is the line through $v$ and $p$.

Solution: For a reflection $F$ we have $\det(F) = -1$ and it follows that $\det(F_L F_M F_N) = -1$. From the Geometric Classification of Isometries, it follows that $F_L F_M F_N$ is either of the form $F_J$ for some line $J$ or of the form $-R_{p,\theta}$ for some $p \in \mathbb{S}^2$ and some $\theta \in \mathbb{R}$. Since $L$, $M$ and $N$ are the perpendicular bisectors of the edges of $[u, v, w]$, we have $F_L F_M F_N(v) = F_L F_M(w) = F_L(u) = u$. Since $p \in L$, $p \in M$ and $p \in N$ we have $F_L F_M F_N(p) = F_L F_M(p) = F_L(p) = p$. Since the isometry $F_L F_M F_N$ fixes $v$ and $p$ it cannot be of the form $-R_{p,\theta}$ (since $R_{p,\theta}$ has no fixed points) so it must be equal $F_J$ for some line $J$. Since $F_J(v) = v$ and $F_J(p) = p$ we have $v \in J$ and $p \in J$, so $J = K$. A similar argument shows that $F_N F_M F_L = F_K$.

**11:** (a) Let $L$ be the line segment in $\mathbb{R}^2$ from $\left(\frac{1}{2},0\right)$ to $\left(\frac{1}{2},\frac{1}{2}\right)$. Find the arclength of the inverse image of $L$ under the orthogonal projection $\phi(x,y,z)=(x,y)$.

Solution: The inverse image of the given line segment is the arc $A$ from $a=\phi^{-1}\left(\frac{1}{2},0\right)=\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)$ to $b=\phi^{-1}\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ along the circle $C=\mathbb{S}^2\cap P$ where $P$ is the plane $x=\frac{1}{2}$. As a Euclidean circle in the plane $P$, the centre of $C$ is at the point $c=\left(\frac{1}{2},0,0\right)$ and the radius is $r=|a-c|=\left|\left(0,0,\frac{\sqrt{3}}{2}\right)\right|=\frac{\sqrt{3}}{2}$. Let $u=a-c=\left(0,0,\frac{\sqrt{3}}{2}\right)$ and $v=b-c=\left(0,\frac{1}{2},\frac{1}{\sqrt{2}}\right)$. Then the arc $A$ along $C$ from $a$ to $b$ subtends the angle $\theta=\theta(u,v)$, and we have

$$\cos\theta=\frac{u\cdot v}{|u||v|}=\frac{\frac{\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}=\frac{\sqrt{2}}{\sqrt{3}}$$

so the length of the arc $A$ is equal to

$$L=r\theta=\tfrac{\sqrt{3}}{2}\cos^{-1}\tfrac{\sqrt{2}}{\sqrt{3}}.$$

(b) Let $C$ be the circular disc in $\mathbb{R}^2$ centred at $\left(\frac{1}{2},0\right)$ of radius $\frac{1}{2}$. Find the area of the inverse image of $C$ under the orthogonal projection $\phi(x,y,z)=(x,y)$.

Solution: The circle of radius $\frac{1}{2}$ centred at $\left(\frac{1}{2},0\right)$ has equation $x^2+y^2=x$ which can be written in polar coordinates as $r^2=r\cos\theta$, or by $r=\cos\theta$. It follows that the circular disc $C$ corresponds to the polar coordinates region $R=\left\{(r,\theta)\middle|-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\,,\,0\le r\le\cos\theta\right\}$ and so, by the formula in Note 2.42, the area of the inverse image of $C$ under $\phi$ is

$$\begin{aligned}A&=\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{\cos\theta}\frac{r}{\sqrt{1-r^2}}\,dr\,d\theta=\int_{\theta=-\pi/2}^{\pi/2}\Big[-\sqrt{1-r^2}\Big]_{r=0}^{\cos\theta}d\theta=\int_{\theta=-\pi/2}^{\pi/2}1-\sqrt{1-\cos^2\theta}\;d\theta\\&=\int_{\theta=-\pi/2}^{\pi/2}1-\big|\sin\theta\big|\;d\theta=2\int_{\theta=0}^{\pi/2}1-\sin\theta\;d\theta=2\Big[\theta+\cos\theta\Big]_{\theta=0}^{\pi/2}=2\left(\tfrac{\pi}{2}-1\right)=\pi-2.\end{aligned}$$

(c) Let $R=\left\{(u,v)\in\mathbb{R}^2\middle|0\le u\le 1,1\le v\right\}$. Find the perimeter and the area of the inverse image of the set $R$ under the gnomic projection $\phi(x,y,z)=\left(\frac{x}{z},\frac{y}{z}\right)$.

Solution: The region $R$ is bounded by the lines $u=0$, $u=1$ and $v=1$. The inverse image of the line $v=1$ in $\mathbb{R}^2$ is the line $\frac{y}{z}=1$, that is the line $y=z$ in $\mathbb{S}^2$ intersected with the upper hemisphere $H$, the inverse image of the line $u=1$ is the line $\frac{x}{z}=1$, that is the line $x=z$ in $\mathbb{S}^2$ intersected with $H$, and the inverse image of the line $u=0$ is the line $x=0$ in $\mathbb{S}^2$ intersected with $H$. Let $k=\phi^{-1}(0,1)=\frac{1}{\sqrt{2}}(0,1,1)$, $l=\phi^{-1}(1,1)=\frac{1}{\sqrt{3}}(1,1,1)$ and let $m=\lim\limits_{t\to\infty}\phi^{-1}(0,t)=\lim\limits_{t\to\infty}\frac{1}{1+y^2}(0,t,1)=(0,1,0)$. Then the inverse image of the region $R$ is the triangle $T=[k,l,m]$. The side lengths $a$, $b$ and $c$ are given by $\cos a=l\cdot m=\frac{1}{\sqrt{3}}$, $\cos b=m\cdot k=\frac{1}{\sqrt{2}}$ and $\cos c=k\cdot l=\frac{\sqrt{2}}{\sqrt{3}}$, and so the perimeter of $T$ is

$$L=(a+c)+b=\left(\cos^{-1}\tfrac{1}{\sqrt{3}}+\cos^{-1}\tfrac{\sqrt{2}}{\sqrt{3}}\right)+\cos^{-1}\tfrac{1}{\sqrt{2}}=\tfrac{\pi}{2}+\tfrac{\pi}{4}=\tfrac{3\pi}{4}.$$

By the First Law of Cosines, the angles $\alpha$, $\beta$ and $\gamma$ are given by

$$\begin{aligned}\cos\alpha&=\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{3}}}{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}}=0,\\ \cos\beta&=\frac{\cos b-\cos c\cos a}{\sin c\sin a}=\frac{\frac{1}{\sqrt{2}}-\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}}=\tfrac{1}{2}\text{ and}\\ \cos\gamma&=\frac{\cos c-\cos a\cos b}{\sin a\sin b}=\frac{\frac{\sqrt{2}}{\sqrt{3}}-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{3}}}=\tfrac{1}{\sqrt{2}},\end{aligned}$$

and so the area of $T$ is

$$A=(\alpha+\beta+\gamma)-\pi=\left(\tfrac{\pi}{2}+\tfrac{\pi}{3}+\tfrac{\pi}{4}\right)-\pi=\tfrac{\pi}{12}.$$

**12:** (a) Let $w = \frac{1}{\sqrt{3}}(1,-1,1)$. Find the area of the image of the circle $C\left(w, \frac{\pi}{6}\right)$ under the stereographic projection $\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

Solution: Note that $C\left(w, \frac{\pi}{6}\right) = \mathbb{S}^2 \cap P$ where $P$ is the plane in $\mathbb{R}^3$ with equation $p \bullet w = \cos\frac{\pi}{6}$ where $p \in \mathbb{R}^3$, which we can also write as $\frac{1}{\sqrt{3}}\,(x,y,z) \bullet (1,-1,1) = \frac{\sqrt{3}}{2}$ where $(x,y,z) \in \mathbb{R}^3$, that is $x - y + z = \frac{3}{2}$. Letting $(a,b,c,d) = \left(1,-1,1,-\frac{3}{2}\right)$ and applying the formula which occurs at the end of the proof of Theorem 2.55, we see that the image of the circle $C\left(w, \frac{\pi}{6}\right)$ under the stereographic projection $\phi$ is the circle $D$ in the $uv$-plane centred at $(u,v) = \left(\frac{-a}{c+d}\,,\,\frac{-b}{c+d}\right) = (2,-2)$ of radius $r = \frac{\sqrt{a^2+b^2+c^2-d^2}}{|c+d|} = \sqrt{3}$. The area of the image circle $D$ is $A = \pi r^2 = 3\pi$.

(b) Let $T$ be the triangle on $\mathbb{S}^2$ with vertices at $\frac{1}{\sqrt{2}}(1,-1,0)$, $\frac{1}{\sqrt{3}}(1,1,1)$ and $\frac{1}{\sqrt{3}}(1,-1,1)$. Find the area of the image of $T$ under the stereographic projection $\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

Solution: Let $a = \phi(u) = \left(\frac{1/\sqrt{2}}{1-0}, \frac{-1/\sqrt{2}}{1-0}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $b = \phi(v) = \left(\frac{1/\sqrt{3}}{1-1/\sqrt{3}}, \frac{1/\sqrt{3}}{1-1/\sqrt{3}}\right) = \left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}+1}{2}\right)$ and $c = \phi(w) = \left(\frac{1/\sqrt{3}}{1-1/\sqrt{3}}, \frac{-1/\sqrt{3}}{1-1/\sqrt{3}}\right) = \left(\frac{\sqrt{3}+1}{2}, -\frac{\sqrt{3}+1}{2}\right)$. Let $L$, $M$ and $N$ be the lines in $\mathbb{S}^2$ containing the edges $[v,w]$, $[w,u]$ and $[u,v]$ respectively. Since $v \times w$ is in the direction of $(1,0,-1)$, we have $L = \mathbb{S}^2 \cap P$ where $P$ is the plane $x - z = 0$. Since $w \times u$ is in the direction of $(1,1,0)$, we have $M = \mathbb{S}^2 \cap Q$ where $Q$ is the plane $x + y = 0$. Since $u \times v$ is in the direction of $(-1,-1,2)$ we have $N = \mathbb{S}^2 \cap R$ where $R$ is the plane $x + y - 2z = 0$. Since $Q$ passes through $(0,0,1)$, the map $\phi$ maps points in $Q$ to points in $Q$, so the image of $M$ is equal to $Q \cap \mathbb{R}^2$ which is the line $x + y = 0$ in the $xy$-plane, that is the line $u + v = 0$ in the $uv$-plane. Recall that the proof of Theorem 2.55 shows that the intersection of $\mathbb{S}^2$ with the plane $ax + by + cz = d$ with $c \neq d$ is mapped by $\phi$ to the circle centred at $\left(\frac{a}{d-c}, \frac{b}{d-c}\right)$ of radius $\frac{\sqrt{a^2+b^2+c^2-d^2}}{|d-c|}$. It follows that the image $\phi(L)$ is the circle centred at $p = (1,0)$ of radius $r = \sqrt{2}$ and that the image $\phi(N)$ is the circle centred at $q = \left(\frac{1}{2}, \frac{1}{2}\right)$ of radius $s = \frac{\sqrt{6}}{2}$. The figure below shows the images $\phi(L)$, $\phi(M)$ and $\phi(N)$ in grey and outlines the image of $[u,v,w]$ in blue.

The area of the image of $[u,v,w]$ is

$$A = B + C - D - E$$

where $B$ is the area of the sector of the circle centred at $p$ from $c$ to $b$, and $C$ is the area of the quadrilateral $0,c,p,b$ which is equal to twice the area of triangle $0,c,p$, and $D$ is the area of the sector of the circle centred at $q$ from $a$ to $b$, and $E$ is the area of triangle $0,a,q$. It is easy to check that $C = \frac{\sqrt{3}+1}{2}$ and $E = \frac{1}{2\sqrt{2}}$. Since $b - p = \left(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}+1}{2}\right)$, by symmetry we have $\theta = \angle cpb = 2\tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 \cdot \frac{5\pi}{12} = \frac{5\pi}{6}$ and so $B = \frac{1}{2}\,r^2\theta = \frac{5\pi}{6}$. Since $\angle 0qa = \tan^{-1}\sqrt{2}$ we have $\phi = \angle aqb = \pi - \tan^{-1}\sqrt{2}$ and so $D = \frac{1}{2}\,s^2\phi = \frac{3}{4}\phi = \frac{3\pi}{4} - \frac{3}{4}\tan^{-1}\sqrt{2}$. Thus

$$A = B + C - D - E = \frac{5\pi}{6} + \frac{\sqrt{3}+1}{2} - \frac{3\pi}{4} + \frac{3}{4}\tan^{-1}\sqrt{2} - \frac{1}{2\sqrt{2}} = \frac{\pi}{12} + \frac{3}{4}\tan^{-1}\sqrt{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} - \frac{1}{2\sqrt{2}}.$$