PMATH 321 Non-Euclidean Geometry, Exercises for Chapter 2: Spherical Geometry

1: (a) Let $u=(3,4,-1)$ and $v=(4,1,3)$. Find the spherical distance between u and v on the sphere given by $x^2+y^2+z^2=26$.
(b) Let $u=\frac{1}{\sqrt{3}}(1,1,-1)$ and $v=\frac{1}{\sqrt{2}}(1,0,1)$. Find a point $w\in\mathbb{S}^2$ such that L_w is the line through u and v.
(c) Let $u=\frac{1}{3}(1,2,2)$ and $v=\frac{1}{\sqrt{5}}(0,1,2)$. Find a point $w\in\mathbb{S}^2$ such that L_w is the line through u which is perpendicular to L_v.
(d) Let $u=\frac{1}{\sqrt{3}}(1,-1,1)$ and $v=\frac{1}{\sqrt{6}}(1,2,1)$. Find the two points of intersection on $\mathbb{S}^2$ of the spherical line L_u and the spherical circle $C\big(v,\frac{\pi}{3}\big)$.
(e) Let $u=\frac{1}{\sqrt{2}}(0,1,1)$ and $v=\frac{1}{\sqrt{6}}(1,2,1)$. Find $r>0$ such that $C(u,r)$ is tangent to L_v.

2: (a) Let $u=\frac{1}{\sqrt{5}}(2,0,-1)$, $v=(2,-1,4)$ and $w=(1,3,2)$. Find the oriented angle $\theta_o(v,w)$ from v to w in the tangent space T_u.
(b) Let $u=\frac{1}{\sqrt{6}}(1,1,2)$, $v=\frac{1}{\sqrt{14}}(2,1,3)$ and $w=\frac{1}{\sqrt{11}}(1,3,1)$. Find u_v and u_w.
(c) Let $u=\frac{1}{3}(-1,2,-2)$, $v=\frac{1}{\sqrt{2}}(1,0,1)$ and $w=\frac{1}{3\sqrt{3}}(5,-1,1)$. Find the interior angles α, β and γ in the ordered triangle $[u,v,w]$.

3: Let $[u,v,w]$ be a triangle with edge lengths a, b and c and interir angles α, β and γ.
(a) Given that $a=\frac{\pi}{3}$, $c=\frac{\pi}{6}$ and $\beta=\frac{2\pi}{3}$, find b.
(b) Given that $a=\frac{\pi}{6}$, $\beta=\frac{5\pi}{6}$ and $\gamma=\frac{\pi}{4}$, find c.
(c) Given that $a=\cos^{-1}\frac{1}{3}$, $b=\frac{\pi}{2}$ and $\alpha=\frac{\pi}{4}$, find all possible values of c.

4: (a) Let $u=\frac{1}{\sqrt{6}}(1,2,1)$, $v=\frac{1}{\sqrt{6}}(1,-1,2)$ and $w=\frac{1}{\sqrt{6}}(2,1,-1)$. Find the centroid of the spherical triangle $[u,v,w]$, that is find the point of intersection of the 3 medians $\big[u,\frac{v+w}{|v+w|}\big]$, $\big[v,\frac{w+u}{|w+u|}\big]$ and $\big[w,\frac{u+v}{|u+v|}\big]$.
(b) Let $u=\frac{1}{3}(2,2,-1)$, $v=\frac{1}{3}(1,2,2)$ and $w=\frac{1}{3}(2,-1,-2)$. Find the centre and radius of the circumscribed circle of the spherical triangle $[u,v,w]$.
(c) Let $u=\frac{1}{\sqrt{3}}(1,1,1)$, $v=\frac{1}{\sqrt{3}}(1,-1,-1)$ and $w=(0,1,0)$. Find the incentre of triangle $[u,v,w]$.

5: (a) Find the perimeter of the square on $\mathbb{S}^2$ with interior angles equal to $\frac{2\pi}{3}$.
(b) Find the area of the regular hexagon on $\mathbb{S}^2$ with sides of length $\ell=\cos^{-1}\frac{2}{3}$.
(c) Find the perimeter and the area of the regular hexagon on $\mathbb{S}^2$ which is inscribed in an equilateral triangle with interior angles $\frac{\pi}{2}$.

6: (a) For a point (x,y,z) on the sphere $x^2+y^2+z^2=R^2$, let $\phi\in[0,\pi]$ measure the angle in $\mathbb{R}^3$ from $(0,0,1)$ to (x,y,z) and let theta $\theta\in\mathbb{R}$ measure the angle in $\mathbb{R}^2$ from $(1,0)$ counterclockwise to (x,y). Given $0\le\phi_1\le\phi_2\le\pi$ and $\theta_1\le\theta_2\le\theta_1+2\pi$, find the area of the portion of the sphere given by $\phi_1\le\phi\le\phi_2$ and $\theta_1\le\theta\le\theta_2$.
(b) A light at position $(0,0,8)$ shines down on a spherical balloon of radius $\sqrt{5}$ centred at $(3,4,3)$. Find the area of the shadow which is cast on the xy-plane (given that the shadow is an ellipse and that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to πab).
(c) A light at position $(0,0,30)$ shines down on a red spherical balloon of radius $\sqrt{10}$ centred at $(0,0,20)$ casting a shadow on a green balloon of radius $6\sqrt{5}$ centred at $(0,0,0)$. Find the area of the illuminated portion of the green balloon.

7: (a) Find the radius R of a sphere on which there is an equilateral triangle with sides of length π and angles equal to $\frac{5\pi}{6}$.
(b) Find an approximate value for the radius R of a sphere on which there is a circle of radius 2 and circumference $\frac{215\pi}{54}$.
(c) Let R be the radius of the Earth, in meters ($R\cong 6,370,000$). We describe the position of a point on the Earth in terms of its longitude θ (with $\theta=0$ at Greenwitch, England and $\theta=\frac{\pi}{2}$ somewhere in Bangladesh) and its latitude ϕ (with $\phi=0$ at the equator and $\phi=\frac{\pi}{2}$ at the north pole). Find the distance (expressed as a multiple of R) and the bearing (expressed as an angle north of east) from the point at $(\theta,\phi)=(\frac{\pi}{3},\frac{\pi}{6})$ to the point at $(\theta,\phi)=(\frac{\pi}{2},\frac{\pi}{4})$.

8: (a) Let $u = \frac{1}{\sqrt{6}}(1,-1,2)$. Express the isometry F_u in matrix form.

(b) Let $u = \frac{1}{3}(1,2,-2)$ and let $\theta = \frac{\pi}{2}$. Express the isometry $R_{u,\theta}$ in matrix form.

(c) Let $u = \frac{1}{\sqrt{3}}(1,1,1)$, $\theta = \frac{\pi}{3}$, $v = \frac{1}{\sqrt{6}}(2,-1,1)$ and $x = \frac{1}{\sqrt{2}}(1,0,1)$. Find $R_{u,\theta}F_v(x)$.

(d) Find $u \in \mathbb{S}^2$ and $\theta \in [0,\pi]$ such that, in matrix form, we have $-R_{u,\theta} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

9: (a) Let $u = \frac{1}{\sqrt{2}}(1,1,0)$, $w = \frac{1}{\sqrt{3}}(1,-1,1)$ and $\theta = \frac{\pi}{3}$. Find $v \in \mathbb{S}^2$ such that $F_u F_v = R_{w,\theta}$.

(b) Let $u = \frac{1}{\sqrt{2}}(1,0,-1)$, $v = \frac{1}{\sqrt{6}}(-1,2,1)$, $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$. Find w and γ such that $R_{u,2\alpha}R_{v,2\beta}R_{w,2\gamma} = I$.

(c) Find $p \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ so that $F = -R_{p,\theta}$ where F is the isometry such that $F(u) = u'$, $F(v) = v'$ and $F(w) = w'$ where $u = \frac{1}{\sqrt{2}}(1,1,0)$, $v = (0,1,0)$, $w = \frac{1}{\sqrt{3}}(1,1,1)$, $u' = \frac{1}{\sqrt{2}}(0,-1,1)$, $v' = (0,0,1)$ and $w' = \frac{1}{\sqrt{3}}(1,-1,1)$.

10: (a) Show that for every $w \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ there exist $u, v \in \mathbb{S}^2$ such that $R_{u,\pi}R_{v,\pi} = R_{w,\theta}$.

(b) Let $u \in \mathbb{S}^2$ and let L be a line in $\mathbb{S}^2$. Show that $\big(F_L R_{u,\pi}\big)^2 = I$ if and only if either $u \in L$ or $L = L_u$.

(c) Let $[u,v,w]$ be a positively oriented triangle with circumcentre p and let L, M and N are the perpendicular bisectors of edges $[v,w]$, $[w,u]$ and $[u,v]$ respectively. Show that $F_L F_M F_N = F_N F_M F_L = F_K$ where K is the line through v and p.

11: (a) Let L be the line segment in $\mathbb{R}^2$ from $\left(\frac{1}{2},0\right)$ to $\left(\frac{1}{2},\frac{1}{2}\right)$. Find the arclength of the inverse image of L under the orthogonal projection $\phi(x,y,z) = (x,y)$.

(b) Let C be the circular disc in $\mathbb{R}^2$ centred at $\left(\frac{1}{2},0\right)$ of radius $\frac{1}{2}$. Find the area of the inverse image of C under the orthogonal projection $\phi(x,y,z) = (x,y)$.

(c) Let $R = \left\{(u,v) \in \mathbb{R}^2 \middle| 0 \le u \le 1, 1 \le v\right\}$. Find the perimeter and the area of the inverse image of the set R under the gnomic projection $\phi(x,y,z) = \left(\frac{x}{z},\frac{y}{z}\right)$.

12: (a) Let $w = \frac{1}{\sqrt{3}}(1,-1,1)$. Find the area of the image of the circle $C\left(w,\frac{\pi}{6}\right)$ under the stereographic projection $\phi(x,y,z) = \left(\frac{x}{1-z},\frac{y}{1-z}\right)$.

(b) Let T be the triangle on $\mathbb{S}^2$ with vertices at $\frac{1}{\sqrt{2}}(1,-1,0)$, $\frac{1}{\sqrt{3}}(1,1,1)$ and $\frac{1}{\sqrt{3}}(1,-1,1)$. Find the area of the image of T under the stereographic projection $\phi(x,y,z) = \left(\frac{x}{1-z},\frac{y}{1-z}\right)$.