PMATH 321 Non-Euclidean Geometry, Solutions to the Exercises for Chapter 1

Note: for many of the exercises, in order to better understand the solution, it helps to draw a picture which illustrates the situation. The solutions below do not include pictures, so it is recommended that you draw your own.

1: Let $C = C(0, r)$ be the circle in $\mathbb{R}^2$ of radius r centred at the origin.

(a) Given $a \in \mathbb{R}$ with $0 < a < r$, find the centre $(b, 0)$ and radius s of the circle which passes through the point $(a, 0)$ and intersects the circle C orthogonally.

Solution: Let D be the circle of radius s centred at $(b, 0)$ which passes through $(a, 0)$ and intersects orthogonally with C. Let $u = (0, 0)$, let $v = (b, 0)$ and let w be the point of intersection of C and D which lies above the x-axis. Since $w \in C$ we have $|w - u| = r$, since $w \in D$ we have $|w - v| = s$, and since $u = (0, 0)$ and $v = (b, 0)$ with $b > r > 0$ we have $|v - u| = b$. Since D intersects orthogonally with C we have $\angle vwu = \frac{\pi}{2}$ so, by Pythagoras' Theorem, we have $|v - u|^2 = |w - v|^2 + |u - w|^2$, that is $b^2 = s^2 + r^2$ (1). Let $p = (a, 0)$. Since $p = (a, 0)$ and $v = (b, 0)$ with $b > r > a$ we have $|p - v| = b - a$. Since p lies on the circle D we have $|p - v| = s$, and so $b - a = s$ (2). Solve equations (1) and (2) to get $b = \frac{r^2+a^2}{2a}$ and $s = \frac{r^2-a^2}{2a}$.

(b) Let $u = (0, r)$ and let $v = (x, y)$ be a point on C with $v \neq u$. Find the point of intersection $w = (a, 0)$ of the line through u and v with the x-axis.

Solution: Since w lies on the line through u and v, we have $w = u + t(v - u)$ for some $t \in \mathbb{R}$, For $t \in \mathbb{R}$ we have

$$\begin{aligned} w = u + t(v-u) &\iff (a,0) = (0,r) + t\big((x,y) - (0,r)\big) \\ &\iff (a,0) = \big(tx\,,\, r + t(y-r)\big) \\ &\iff \Big(a = tx \text{ and } 0 = r + t(y-r)\Big). \end{aligned}$$

To get $0 = r + t(y - r)$ we need $t = \frac{r}{r-y}$, and then we have $a = tx = \frac{rx}{r-y}$.

(c) Let a, b, c and d be any 4 distinct points on C such that the line segment $[a, c]$ intersects with the line segment $[b, d]$ at a point p inside C. Show that $|a - p||p - c| = |b - p||p - d|$.

Solution: Since the points b and c both subtend the same chord $[a, d]$ on the circle C (see Corollary 1.52), it follows that $\angle abd = \angle acd$ so we have $\angle abp = \angle dcp$. Since the points a and d both subtend the chord $[b, c]$ on the circle C, it follows that $\angle bac = \angle bdc$ and so we have $\angle pab = \angle pdc$. Since $\angle abp = \angle dcp$ and $\angle pab = \angle pdc$, the triangles $[a, b, p]$ and $[d, c, p]$ are similar, by Corollary 1.50 (Similar Triangles, also called Angle-Angle), meaning that there is a scaling factor $t > 0$ such that $|p - c| = t|p - b|$, $|d - p| = t|a - p|$ and $|c - d| = t|b - a|$. Thus we have $t = \frac{|p-c|}{|p-b|} = \frac{|d-p|}{|a-p|}$ hence $|p - c||a - p| = |p - b||d - p|$, as required.

2: (a) Let $u = (1,0)$, $v = (4,1)$ and $w = (6,5)$. Find the exact values of the interior angles α, β and γ in the triangle $[u,v,w]$.

Solution: Let $p = (3,2)$. Note that p is the point on the line through u and w which is nearest to v. From the right-angled triangle $[u,p,v]$ (which has a right angle at p) we see that

$$\alpha = \angle wuv = \angle puv = \tan^{-1}\tfrac{|p-v|}{|p-u|} = \tan^{-1}\tfrac{\sqrt{2}}{2\sqrt{2}} = \tan^{-1}\tfrac{1}{2}.$$

From the right-angled triangle $[w,p,v]$ (which has a right-angle at p) we see that

$$\gamma = \angle vwu = \angle vwp = \tan^{-1}\tfrac{|v-p|}{|p-w|} = \tan^{-1}\tfrac{\sqrt{2}}{3\sqrt{2}} = \tan^{-1}\tfrac{1}{3}.$$

Let $q = (7,2)$. Note that q is the point on the line through u and v which is nearest to w. From the right-angled triangle $[v,q,w]$ (which has a right-angle at q) we see that

$$\beta = \pi - \angle wvq = \pi - \tan^{-1}\tfrac{|w-q|}{|q-v|} = \pi\tan^{-1}\tfrac{\sqrt{10}}{\sqrt{10}} = \pi\tan^{-1}1 = \pi - \tfrac{\pi}{4} = \tfrac{3\pi}{4}.$$

(b) Let $u = (-1,1)$ and $v = (2,0)$. Find a point $w \in \mathbb{R}^2$ such that the triangle $[u,v,w]$ is positively oriented with $\alpha = \frac{\pi}{4}$ and $\gamma = \tan^{-1}\frac{1}{3}$.

Solution: In order for $[u,v,w]$ to be positively oriented, we need w to lie above the line through u and v. Let $a = (0,4)$ and $b = (3,3)$. Note that the vertices u, v, b and a form a square (with sides of length $\sqrt{10}$) and the line through u and b is a diagonal of the square, and so $\angle buv = \frac{\pi}{4}$. It follows that in order to have $\angle wuv = \alpha = \frac{\pi}{4}$ we must choose w to lie along the line L through u and b.

Let $c = (1,2)$ and note that c is the point on line L which is nearest to v. Note that $[b,c,v]$ is a right-angle with its right angle at c and $|b-c| = |c-v| = \sqrt{5}$. To get $\gamma = \tan^{-1}\frac{1}{3}$ we can choose

$$w = c + 3(b-c) = (7,5)$$

so $[w,c,v]$ is a right-angled triangle with right angle at c and $\gamma = \angle vwu = \angle vwc = \tan^{-1}\frac{|v-c|}{|wic|} = \sqrt{\frac{5}{3\sqrt{5}}} = \frac{1}{3}$.

(c) Let $a = \sqrt{5}$, $b = \sqrt{10}$ and $c = \sqrt{13}$. Find the exact area of the triangle with sides of length a, b and c.

Solution: By inspection, we can place vertices u, v and w at positions $u = (0,0)$, $v = (3,2)$ and $w = (1,3)$ to get $a = |w-u| = \sqrt{5}$, $b = |u-v| = \sqrt{10}$ and $c = |v-u| = \sqrt{13}$. Then we can use the formula from Corollary 1.51 to get

$$A = \tfrac{1}{2}\Big|\det(u,v) + \det(v,w) + \det(w,u)\Big| = \tfrac{1}{2}\Big|\det(v,w)\Big| = \tfrac{1}{2}\left|\det\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}\right| = \tfrac{7}{2}.$$

3: (a) Let $u = (6,7)$, $v = (-2,3)$ and $w = (7,0)$. Find the centre o and the radius r of the circumscribed circle of triangle $[u,v,w]$.

Solution: The line segment $[u,v]$ has slope $\frac{v_2-u_2}{v_1-u_1} = \frac{1}{2}$, The perpendicular bisector of $[u,v]$ has slope -2 and passes through the point $\frac{u+v}{2} = (2,5)$ so it has equation $2x + y = 9$ (1). The line segment $[v,w]$ has slope $\frac{w_2-v_2}{w_1-v_1} = -\frac{1}{3}$. The perpendicular bisector of $[v,w]$ has slope 3 and passes through $\frac{v+w}{2} = \left(\frac{5}{2}, \frac{3}{2}\right)$ so it has equation $3x - y = 6$ (2). Solve equations (1) and (2) to get $o = (3,3)$, and the radius is $r = |o - v| = 5$

(b) Let $u = (0,9)$, $v = (12,0)$ and $w = 12,14)$. Find the centre i and the radius r of the inscribed circle of triangle $[u,v,w]$.

Solution: Since $u - v = (-12,9) = 3(-4,3)$, the line from v to u has direction vector $(-4,3)$ and we note that $|(-4,3)| = 5$. The line from v to w has direction vector $(0,5)$ with $|(0,5)| = 5$. The internal angle bisector at v is the line through $v = (12,0)$ with direction vector $(-4,3) + (0,5) = (-4,8) = 4(-1,2)$, so it has equation $2x + y = 24$ (1). The line from w to u has direction vector $u - w = (-12,-5)$ and $|(-12,5)| = 13$. The line from w to v has direction vector $(0,-13)$ with $|(0,-13)| = 13$. The internal angle bisector at w is the line through $w = (12,14)$ with direction vector $(-12,-5) + (0,-13) = (-12,-18) = -6(2,3)$, so it has equation $3x - 2y = 8$ (2). Solve equations (1) and (2) to get $i = (8,8)$, and then the radius r is the distance from i to the line through v and w, that is $r = 4$.

(c) Let $u = (-2,9)$, $v = (2,1)$ and $w = (7,6)$. Find the centres o, g and h of the triangle $[u,v,w]$ and verify that $h = 3g - 2o$.

Solution: We have $g = \frac{1}{3}(u + v + w) = \frac{1}{3}(7,16) = \left(\frac{7}{3}, \frac{16}{3}\right)$. The perpendicular bisector of $[u,v]$ passes through the point $\frac{u+v}{2} = (0,5)$ and has slope $-\frac{v_1-u_1}{v_2-u_2} = \frac{1}{2}$, so it has equation $x - 2y = -10$ (1). The perpendicular bisector of $[v,w]$ passes through $\frac{v+w}{2} = \left(\frac{9}{2}, \frac{7}{2}\right)$ with slope $-\frac{w_1-v_1}{w_2-v_2} = -1$, so it has equation $x + y = 8$ (2). Solve equations (1) and (2) to get $o = (2,6)$. The altitude from w passes through $w = (7,6)$ with slope $-\frac{v_1-u_1}{v_2-u_2} = \frac{1}{2}$, so it has equation $x - 2y = -5$ (3). The altitude from u passes through $u = (-2,9)$ with slope $-\frac{w_1-v_1}{w_2-v_2} = -1$, so it has equation $x + y = 7$ (4). Solve equations (3) and (4) to get $h = (3,4)$. Finally, note that $3g - 2o = (7,16) - (4,12) = (3,4) = h$.

4: (a) Let $u=(3,1)$, $v=(3,6)$ and $i=(2,3)$. Find the point w in $\mathbb{R}^2$ such that i is the incentre of triangle $[u,v,w]$.

Solution: Let $a=(3,3)$ and note that a is the point on $[u,v]$ which is nearest to i. From the right-angled triangle $[u,a,i]$ we see that $\angle vui=\angle aui=\tan^{-1}\frac{|i-a|}{|a-u|}=\tan^{-1}\frac{1}{2}$. Let $b=u+2(i-u)=(1,5)$ and let $c=b+(i-u)^\times=(-1,4)$ so that $[u,b,c]$ is a right angled triangle (with its right angle at b), and $\angle iuc=\angle buc=\tan^{-1}\frac{|c-b|}{|b-u|}=\tan^{-1}\frac{1}{2}$. It follows that w must lie along the ray from u through c. The line through u and c has equation $3x+4y=13$ (1). From the right-angled triangle $[i,v,a]$ we see that $\angle ivu=\angle iva=\tan^{-1}\frac{|i-a|}{|a-v|}=\tan^{-1}\frac{1}{3}$. Let $d=v+3(i-v)=(0,-3)$ and let $e=di(i-v)^\times=(-3,-2)$ so that $[v,d,c]$ is a right-angled triangle (with right angle at d) and $\angle ive=\angle dve=\tan^{-1}\frac{1}{3}$. It follows that w must lie along the ray from v through e. The line through v and e has equation $4x-3y=-6$ (2). Solve equations (1) and (2) to get $w=\left(\frac{3}{5},\frac{14}{5}\right)$.

(b) Let $u=(7,4)$, $o=(1,1)$ and $g=(1,2)$. Find points v and w in $\mathbb{R}^2$ such that o is the circumcentre and g is the centroid of triangle $[u,v,w]$.

Solution: Since g lies $\frac{2}{3}$ of the way along the median from u to the midpoint $m=\frac{v+w}{2}$ it follows that $m=u+\frac{3}{2}(g-u)=(-2,1)$. Since o lies on the perpendicular bisector of $[v,w]$ it follows that the line through v, and w is the line through m which is perpendicular to the vector $o-m=(3,0)$, so the line through v and w has equation $x=-2$ (1). Since o is equidistant from u, v and w the points v and w both lie on the circle centred at o of radius $|o-u|=3\sqrt{5}$, which has equation $(x-1)^2+(y-1)^2=45$ (2). Put $x=-2$ into equation (2) to get $(-3)^2+(y-1)^2=45$ which gives $(y-1)^2=36$ hence $y-1=\pm 6$, that is $y\in\{-5,7\}$. Thus v and w are the two points $(-2,-5)$ and $(-2,7)$.

5: (a) Let L be the line $x+y=1$, let M be the line $3y=2x+6$, and let N be the line $2x+y=6$. Find points $u,v,w\in\mathbb{R}^2$ such that, in the triangle $[u,v,w]$, L is the perpendicular bisector of $[u,v]$, M is the median from u, and N is the altitude from v.

Solution: Let $[u,v,w]$ be a triangle which satisfies the required conditions. Let $p=(5,-4)$ and note that $L\cap N=\{p\}$. Since N is the altitude from v we have $v\in N$ so N is the line through p and v. Let K be the line through p and u. Let m be the midpoint $m=\frac{u+v}{2}$. Since L is the perpendicular bisector of $[u,v]$ the triangles $[p,v,m]$ and $[p,u,m]$ are congruent right-angled triangles (with the right angles at m) and so we have $\angle vpm=\angle upm$. It follows that K is the line through p with $K\neq L$ for which $\theta(N,L)=\theta(L,K)$ (so L is the angle bisector of N and K at p). Let $a=(3,0)$ and $b=(2,-1)$ so that $a\in N$ and b is the point on L nearest to a. Then we have $\theta(N,L)=\angle apb=\tan^{-1}\frac{|a-b|}{|b-p|}=\tan^{-1}\frac{1}{3}$. Let $c=b+(b-a)=(1,-2)$ so that $[p,b,a]$ is a right-angled triangle (with right angle at b) and $\angle cpb=\tan^{-1}\frac{|c-b|}{|b-p|}=\tan^{-1}\frac{1}{3}$. It follows that the K is the line through p and c, and so K has equation $x+2y=-3$. Since M is the median from u we must have $u\in M$. Since $u\in M$ and $u\in K$ we can solve the equations for M and K to get $u=(-3,0)$. The midpoint m is the point on L nearest to u, which is the point $m=(-1,2)$, and then the point v is given by $v=u+2(m-u)=(1,4)$. Since N is the altitude from v, the line through u and w is the line through u perpendicular to N, which has equation $x-2y=-3$ (1). Since M is the median from u we must have $\frac{v+w}{2}\in M$. For $w=(x,y)$ we have $\frac{v+w}{2}=\frac{(1,4)+(x,y)}{2}=\left(\frac{x+1}{2},\frac{y+4}{2}\right)$ and by putting this in the equation of M we have

$$\tfrac{v+w}{2}\in M\iff 3\left(\tfrac{y+4}{2}\right)=2\left(\tfrac{x+1}{2}\right)+6\iff 3(y+4)=2(x+1)+12\iff 2x-3y=-2\ (2)$$

Solve equations (1) and (2) to get $w=(2,2)$.

(b) Let L be the line $2y=x+4$, let M be the line $x+y=4$, and let N be the line $y+8=3x$. Find points $u,v,w\in\mathbb{R}^2$ such that, in the triangle $[u,v,w]$, L is the angle bisector at u, M is the altitude from v, and N is the median from w.

Solution: Let $[u,v,w]$ be a triangle with the required properties. Let K be the line through u and w. Since M is the altitude from v, the line K is perpendicular to M. Since M has normal vector $(1,1)$, K has direction vector $(1,1)$. Let J be the line through u and v. Then L is an angle bisector of the lines J and K at the point u. Let us determine the direction of the line J. Since K has direction vector $(1,1)$ and L has direction vector $(2,1)$ we have $\theta(L,K)=\theta\big((2,1),(1,1)\big)=\cos^{-1}\frac{3}{\sqrt{10}}=\tan^{-1}\frac{1}{3}$. Since L has direction vector $(2,1)$, to get $\theta(J,L)=\theta(L,K)=\tan^{-1}\frac{1}{3}$, the line J must be in the direction of the vector $3(2,1)+(1,-2)=(7,1)$. Since the line J through u and v has direction vector $(7,1)$, we can write $v=u+t(7,1)$ for some $t\in\mathbb{R}$, and then the midpoint m of u and v is $m=\frac{u+v}{2}=u+\frac{t}{2}(7,1)$. Let us now calculate the coordinates of the point u. Let $u=(x,y)$. Then $m=u+\frac{t}{2}(7,1)=(x+\frac{7t}{2},y+\frac{t}{2})$ and $v=u+t(7,1)=(x+7t\,,\,y+t)$. Put v into the equation for M to get $(x+7t)+(y+t)=4$ that is $x+y+8t=4$ (1). Put m into the equation for N to get $(y+\frac{t}{2})+8=3(x+\frac{7t}{2})$, that is $3x-y+10t=8$ (2). Multiply equation (2) times 4 and subtract 5 times equation (1) to get $7x-9y=12$ (3). Since u also lies on L, which has equation $2y=x+4$ (4), we can solve equations (3) and (4) to get $u=(12,8)$. Put $(x,y)=(12,8)$ into equation (1) to get $t=-2$, and so $v=u+t(7,1)=(12,8)-2(7,1)=(-2,6)$. The line K through u and w is the line through $u=(12,8)$ with direction vector $(1,1)$, so K has equation $x-y=4$ (5). The point w also lies on the line N, which has equation $3x-y=8$ (6). Solve equations (5) and (6) to get $w=(2,-2)$.

6: (a) Let $u=(1,1,2)$, $v=(2,1,3)$ and $x=(4,1,-1)$. Find $\text{Proj}_U(x)$ where $U=\text{Span}\{u,v\}$.

Solution: Let $w=u\times v=(1,1,2)\times(2,1,3)=(1,1,-1)$. Then

$$\begin{aligned}\text{Proj}_U(x)&=x-\text{Proj}_w(x)=x-\frac{x\cdot w}{|w|^2}w=(4,1,-1)-\frac{(4,1,-1)\times(1,1,-1)}{\big|(1,1,-1)\big|^2}(1,1,-1)\\&=(4,1,-1)-\tfrac{6}{3}(1,1,-1)=(4,1,-1)-(2,2,-2)=(2,-1,1).\end{aligned}$$

(b) Let $a=(2,1,3)$, $b=(1,2,1)$, $u=(1,3,2)$ and $v=(2,0,1)$. Find the distance between the line $x=a+tu$ and the line $x=b+tv$.

Solution: In general, when L is the line $x=a+su$, $s\in\mathbb{R}$, and M is the line $x=b+tv$, $t\in\mathbb{R}$, the Euclidean distance between L and M is given by

$$\begin{aligned}d_E(L,M)&=\min\big\{d(a+su,b+tv)\,\big|\,s,t\in\mathbb{R}\big\}\\&=\min\big\{|(a+su)-(b+tv)|\,\big|\,s,t\in\mathbb{R}\big\}\\&=\min\big\{|(a-b)-(-su+tv)|\,\big|\,s,t\in\mathbb{R}\big\}\\&=\min\big\{d(a-b,w)\,\big|\,w\in\text{Span}\{u,v\}\big\}\\&=d_E(a-b,U)\ \text{ where } U=\text{Span}\{u,v\}\\&=\Big|\text{Proj}_{u\times v}(a-b)\Big|\end{aligned}$$

For this particular problem, note that $u\times v=(1,3,2)\times(2,0,1)=(3,3,-6)$ and let $w=\frac{1}{3}(u\times v)=(1,1,-2)$. The distance d between the two lines is

$$d=\big|\text{Proj}_{u\times v}(a-b)\big|=\big|\text{Proj}_w(a-b)\big|=\frac{|(a-b)\cdot w|}{|w|}=\frac{\big|(1,-1,2)\cdot(1,1,-2)\big|}{\big|(1,1,-2)\big|}=\frac{4}{\sqrt6}=\frac{2\sqrt6}{3}.$$

(c) Referring to Theorem 1.22 in the Lecture Notes (Properties of the Cross Product), use Properties 1, 2, 3 and 5 to prove Property 4.

Solution: Let $u,v,w,x\in\mathbb{R}^3$. Then

$$\begin{aligned}(u\times v)\cdot(w\times x)&=\big((u\times v)\times w\big)\cdot x&&\text{by Theorem 1.22 Part 5}\\&=\big((u\cdot w)v-(v\cdot w)u\big)\cdot x&&\text{by Theorem 1.22 Part 3}\\&=(u\cdot w)(v\cdot x)-(v\cdot w)(u\cdot x)&&\text{by Theorem 1.22 Part 1}\end{aligned}$$

7: Let $u_1=(1,0,1,-1)$, $u_2=(2,1,1,0)$, $u_3=(1,-3,2,1)$, $\mathcal{B}=\{u_1,u_2,u_3\}$, $U=\text{Span}(\mathcal{B})$, and $x=(1,1,7,3)$. Find $\text{Proj}_U(x)$ in the following three ways.

(a) Let $A=\big(u_1,u_2,u_3\big)\in M_{4\times 3}$ so $\text{Proj}_U(x)=At$ where t is the solution to $A^TA\,t=A^Tx$.

Solution: We have

$$A^TA=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\begin{pmatrix}1&2&1\\0&1&-3\\1&1&2\\-1&0&1\end{pmatrix}=\begin{pmatrix}3&3&2\\3&6&1\\2&1&15\end{pmatrix}$$

$$A^Tx=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\begin{pmatrix}1\\1\\7\\3\end{pmatrix}=\begin{pmatrix}5\\10\\15\end{pmatrix}$$

$$\big(A^TA\big|A^Tx\big)=\left(\begin{array}{ccc|c}3&3&2&5\\3&6&1&10\\2&1&15&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\3&6&1&10\\2&1&15&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\0&0&40&40\\0&3&-41&-35\end{array}\right)$$
$$\sim\left(\begin{array}{ccc|c}1&2&-13&-10\\0&3&-41&-35\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&0&3\\0&3&0&6\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&0&3\\0&1&0&2\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&-1\\0&1&0&2\\0&0&1&1\end{array}\right)$$

and so

$$t=\begin{pmatrix}-1\\2\\1\end{pmatrix}\ \text{ and } \text{Proj}_U(x)=At=\begin{pmatrix}1&2&1\\0&1&-3\\1&1&2\\-1&0&1\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

(b) Apply the Gram-Schmidt Procedure to the basis $\mathcal{B}$ to obtain an orthogonal basis $\mathcal{C}=\{v_1,v_2,v_3\}$ for U, so that $\text{Proj}_U(x)=\sum\limits_{i=1}^{3}\frac{x\,\bullet\,v_i}{|v_i|^2}\,v_i$.

Solution: We let

$$v_1=u_1=\begin{pmatrix}1\\0\\1\\-1\end{pmatrix},\qquad v_2=u_2-\frac{u_2\bullet v_1}{|v_1|^2}\,v_1=\begin{pmatrix}2\\1\\1\\0\end{pmatrix}-\frac{3}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix},$$

$$v_3=u_3-\frac{u_3\bullet v_1}{|v_1|^2}\,v_1-\frac{u_3\bullet v_2}{|v_2|^2}\,v_2=\begin{pmatrix}1\\-3\\2\\1\end{pmatrix}-\frac{2}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2\\-8\\4\\6\end{pmatrix}=\frac{2}{3}\begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Then

$$\text{Proj}_U(x)=\frac{x\bullet v_1}{|v_1|^2}\,v_1-\frac{x\bullet v_2}{|v_2|^2}\,v_2-\frac{x\bullet v_3}{|v_3|^2}\,v_3=\frac{5}{3}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}+\frac{5}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}+\frac{20}{30}\begin{pmatrix}1\\-4\\2\\3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}12\\-3\\9\\6\end{pmatrix}=\begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

(c) Find $w\in\mathbb{R}^4$ so $\{w\}$ is a basis for $U^\perp$, so $\text{Proj}_U(x)=x-\text{Proj}_w(x)=x-\frac{x\,\bullet\,w}{|w|^2}\,w$.

Solution: Let $A=(u_1,u_2,u_3)\in M_{4\times 3}$. We wish to find a basis for $U^\perp=(\text{ColA})^\perp=(\text{RowA}^\text{T})^\perp=\text{NullA}^\text{T}$. We have

$$A^T=\begin{pmatrix}1&0&1&-1\\2&1&1&0\\1&-3&2&1\end{pmatrix}\sim\begin{pmatrix}1&0&1&-1\\0&1&-1&2\\0&3&-1&-2\end{pmatrix}\sim\begin{pmatrix}1&0&1&-1\\0&1&-1&2\\0&0&2&-8\end{pmatrix}\sim\begin{pmatrix}1&0&0&3\\0&1&0&-2\\0&0&1&-4\end{pmatrix}$$

so we can take $w=(-3,2,4,1)^T$. Thus

$$\text{Proj}_U(x)=x-\frac{x\bullet w}{|w|^2}\,w=\begin{pmatrix}1\\1\\7\\3\end{pmatrix}-\frac{30}{30}\begin{pmatrix}-3\\2\\4\\1\end{pmatrix}=\begin{pmatrix}4\\-1\\3\\2\end{pmatrix}.$$

8: (a) Let $a = (2,1,3)$, $b = (4,1,2)$ and $c = (1,-1,5)$. Find the area of the triangle in $\mathbb{R}^3$ with vertices at a, b and c.

Solution: Let $u = b-a = (2,0,-1)$ and $v = c-a = (-1,-2,2)$. The area A of the triangle with vertices a,b,c is equal to the area of the triangle with vertices $0,u,v$ which is equal to half of the area of the parallelogram with vertices at $0,u,v,u+v$, that is

$$A = \tfrac{1}{2}|u\times v| = \tfrac{1}{2}\big|(2,0,-1)\times(-1,-2,2)\big| = \tfrac{1}{2}\big|(-2,-3,-4)\big| = \tfrac{\sqrt{29}}{2}.$$

(b) (a) Let $a = (1,0,1)$, $b = (2,2,1)$, $c = (0,1,4)$ and $d = (-1,1,2)$. Find the volume of the tetrahedron with vertices at a, b, c and d (given that the volume of a tetrahedron is equal to $\frac{1}{3}Ah$ where A is the base area and h is the altitude).

Solution: Let $u = b-a = (1,2,0)$, $v = c-a = (-1,1,3)$ and $w = d-a = (-2,1,1)$. Then the volume V of the tetrahedron with vertices a,b,c,d is equal to the volume of the tetrahedron with vertices $0,u,v,w$, which is equal to one sixth of the volume of the parallelotope with vertices at $0,u,v,w,u+v,u+w,v+w,u+v+w$ (because the volume of the tetrahedron is $\frac{1}{3}Ah$, and the volume of the paralleloptope is Bh, where both have the same height h, and A is the area of the base triangle with vertices at $0,u,v$ and B is the areaa of the base parallelogram with vertices at $0,u,v,u+v$, and we have $B = 2A$). By Part 3 of Theorem 1,25, the volume of the tetrahedron is

$$V = \tfrac{1}{6}\big|\det(u,v,w)\big| = \tfrac{1}{6}\left|\det\begin{pmatrix}1 & -1 & -2\\ 2 & 1 & 1\\ 0 & 3 & 1\end{pmatrix}\right| = \tfrac{1}{6}\big|1-12-3+2\big| = 2.$$

(c) Find the volume of the regular icosahedron whose 12 vertices are at $\pm(1,\pm a,0)$, $\pm(0,1,\pm a)$ and $\pm(\pm a,0,1)$, where $a>0$ is chosen so that the edges all have equal length.

Solution: First let us find $a>0$ so the side lengths are equal. The distance from $(1,a,0)$ to $(1,-a,0)$ is equal to 2a and the distance from $(1,a,0)$ to $(0,1,a)$ is $\sqrt{1+(a-1)^2+a^2} = \sqrt{2a^2-2a+2}$, and so, for the edges to be equal, we need

$$\sqrt{2a^2-2a+2} = 2a \Longrightarrow 2a^2-2a+2 = 4a^2 \Longrightarrow 2a^2+2a-2 = 0 \Longrightarrow a^2+a-1 = 0 \Longrightarrow a = \tfrac{-1+\sqrt{5}}{2}.$$

The icosahedron can be cut into 20 congruent tetrahedra each of which is congruent to the tetrahedron with vertices at $0,u,v,w$ with $u = (1,a,0)$, $v = (0,1,a)$ and $w = (a,0,1)$, and so the volume is

$$V = 20\cdot\tfrac{1}{6}\big|\det(u,v,w)\big| = \tfrac{10}{3}\left|\det\begin{pmatrix}1 & 0 & a\\ a & 1 & 0\\ 0 & a & 1\end{pmatrix}\right| = \tfrac{10}{3}\big|1+a^3\big|.$$

Since $a^2+a-1=0$ we have $a^2 = 1-a$ hence $a^3 = a-a^2 = a-(1-a) = 2a-1 = (-1+\sqrt{5})-1 = \sqrt{5}-2$ and so

$$V = \tfrac{10}{3}(1+a^3) = \tfrac{10}{3}(\sqrt{5}-1).$$

9: Let x and p be points in $\mathbb{R}^3$, let u and v be unit vectors in $\mathbb{R}^3$ with $v \neq \pm u$, and let P and Q be the planes in $\mathbb{R}^3$ through p with unit normal vectors u and v,. The (Euclidean) **distance** between x and P is $d(x,P) = \big|\text{Proj}_u(x-p)\big| = \big|(x-p)\cdot u\big|$, the **angle** between P and Q is $\theta(P,Q) = \cos^{-1}|u \cdot v| \in \big[0, \frac{\pi}{2}\big]$, and the **angle bisectors** of P and Q are the two planes in $\mathbb{R}^3$ through p with normal vectors $u+v$ and $u-v$.

(a) Show that x lies on one of the two angle bisectors of P and Q if and only if we have $d(x,P) = d(x,Q)$.

Solution: Let B and C be the two angle bisectors of P and Q, in other words let B and C be the planes in $\mathbb{R}^3$ through p with normal vectors $u+v$ and $u-v$. Then

$$\begin{aligned} d(x,P) = d(x,Q) &\iff \big|(x-p)\cdot u\big| = \big|(x-p)\cdot v\big| \iff (x-p)\cdot u = \pm(x-p)\cdot v \\ &\iff (x-p)\cdot(u \pm v) = 0 \iff x \in B \text{ or } x \in C. \end{aligned}$$

(b) Let B be a plane in $\mathbb{R}^3$. Show that B is equal to one of the two angle bisectors of P and Q if and only if we have $P \cap Q \subseteq B$ and $\theta(B,P) = \theta(B,Q)$.

Solution: Let B be one of the two angle bisectors of P and Q, say B is the plane through p with normal vector $u+v$ (the case that B has normal vector $u-v$ is similar). Then $P \cap Q \subseteq B$ because

$$\begin{aligned} x \in P \cap Q &\implies \big(x \in P \text{ and } x \in Q\big) \implies \Big((x-p)\cdot u = 0 \text{ and } (x-p)\cdot v = 0\Big) \\ &\implies (x-p)\cdot(u+v) = 0 \implies x \in B, \end{aligned}$$

and we have

$$\theta(B,P) = \cos^{-1}\big|(u+v)\cdot u\big| = \cos^{-1}\big|u\cdot u + u\cdot v\big| = \cos^{-1}\big|v\cdot v + u\cdot v\big| = \cos^{-1}\big|(u+v)\cdot v\big| = \theta(B,Q).$$

Conversely, let B be any plane in $\mathbb{R}^3$ with $P \cap Q \subseteq B$ such that $\theta(B,P) = \theta(B,Q)$. Let w be a unit normal vector for B. Note that $P \cap Q$ is equal to the line through p in the direction of the vector $u \times v$. Since $P \cap Q \subseteq B$, it follows that B is parallel to $u \times v$, and so the normal vector w must be perpendicular to $u \times v$, and hence $w \in \text{Span}\{u,v\}$. Also note that in $\text{Span}\{u,v\}$ we have $(u+v)\cdot(u-v) = u\cdot u - v\cdot v = 1 - 1 = 0$ and so $\big\{u+v, u-v\big\}$ is an orthogonal basis for $\text{Span}\{u,v\}$. We have

$$\begin{aligned} \theta(B,P) = \theta(B,Q) &\implies \cos\theta(B,P) = \cos\theta(B,Q) \implies |w\cdot u| = |w \cdot v| \\ &\implies w\cdot u = \pm w\cdot v \implies \Big(w\cdot(u+v) = 0 \text{ or } w\cdot(u-v) = 0\Big). \end{aligned}$$

In the case that $w\cdot(u+v) = 0$, since $\big\{u+v, u-v\big\}$ is an orthogonal basis for $\text{Span}\{u,v\}$ it follows that $w = \pm\frac{u-v}{|u-v|}$, and so B is the plane through p with normal vector $u-v$. Similarly, in the case that $w\cdot(u-v) = 0$, B is the plane through p with normal vector $u+v$.

(c) Let $a = (-4,-3,1)$, $b = (8,3,1)$, $c = (2,6,1)$ and $d = (4,3,3)$. Find the centre of the inscribed sphere of the tetrahedron with vertices at a, b, c and d.

Solution: Let F_a, F_b, F_c and F_c be the faces opposite the vertices a, b, c and d so for example F_a is the plane through b, c and d). Let B_{pq} denote the internal angle bisector of the planes F_p and F_q, so for example B_{ab} is the angle bisector of F_a and F_b (which intersect along the edge through c and d). Outwards pointing normal vectors to these faces are given by

$$\begin{aligned} (c-b)\times(d-b) &= (-6,3,0)\times(-4,0,2) = (6,12,12) = 6(1,2,2) \\ (d-a)\times(c-a) &= (8,6,2)\times(6,9,0) = (-18,12,36) = 6(-3,2,6) \\ (b-a)\times(d-a) &= (12,6,0)\times(8,6,2) = (12,-24,24) = 12(1,-2,2) \\ (c-a)\times(b-a) &= (6,9,0)\times(12,6,0) = (0,0,-72) = 72(0,0,-1) \end{aligned}$$

and so the outwards pointing unit normal vectors are $u_a = \frac{1}{3}(1,2,2)$, $u_b = \frac{1}{7}(-3,2,6)$, $u_c = \frac{1}{3}(1,-2,2)$ and $u_d = (0,0,-1)$. Plane B_{ad} is the plane through $c = (2,6,1)$ with normal vector $u_a - u_d = \frac{1}{3}(1,2,5)$ so it has equation $x + 2y + 5z = 19$ (1). Plane B_{bd} is the plane through $a = (-4,-3,1)$ with normal vector $u_b - u_d = \frac{1}{7}(-3,2,13)$ so it has equation $-3x+2y+13z = 19$ (2). Plane B_{cd} is the plane through $a = (-4,-3,1)$ with normal vector $u_c - u_d = \frac{1}{3}(1,-2,5)$ so it has equation $x - 2y + 5z = 7$ (3). We solve these three equations.

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 19 \\ -3 & 2 & 13 & 19 \\ 1 & -2 & 5 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 19 \\ 0 & 8 & 28 & 76 \\ 0 & 4 & 0 & 12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 19 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 7 & 19 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 5 & 13 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 7 & 13 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{26}{7} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{13}{7} \end{array}\right)$$

Thus the centre of the inscribed sphere is the point $\big(\frac{26}{7}, 3, \frac{13}{7}\big)$.

10: (a) Let $u=(1,8)$, $v=(2,1)$ and $w=(3,4)$. Find the image of the triangle $[u,v,w]$ under the isometry $G_{(-2,-6),y=3x-8}R_{(3,8),\frac{\pi}{2}}$.

Solution: Let $R=R_{(3,8),90^\circ}$ and $G=G_{(-2,-6),y=3x-8}$. Then the given isometry is the composite $S=GR$. The image of the triangle $[u,v,w]$ is the triangle $[u',v',w']$ where $u'=S(u)=G(R(1,8))=G(3,6)=(4,-1)$, $v'=S(v)=G(R(2,1))=G(10,7)=(-1,4)$ and $w'=S(w)=G(R(3,4)=G(7,8)=(2,3)$.

(b) Let $a=(2,-1)$ and $b=(3,2)$. Draw a picture, or give an accurate description of the set

$$\left\{p\in\mathbb{R}^2\middle|R_{p,\theta}(a)=b \text{ for some } \theta\in\mathbb{R}\right\}.$$

Solution: Since $R_{p,\theta}$ is an isometry and $R_{p,\theta}(p)=p$, it follows that in order to have $R_{p,\theta}(a)=b$ we must have $|p-a|=\left|R_{p,\theta}(p)-R_{p,\theta}(a)\right|=|p-b|$ and so the point p must lie on the perpendicular bisector of $[a,b]$. Conversely, if p lies on the perpendicular of $[a,b]$ and $\theta=\theta_0(a-p,b-p)$ then we have $R_\theta(a-p)=b-p$ and so $R_{p,\theta}(a)=b$. Thus the set of all points p for which $R_{p,\theta}(a)=R_{p,\theta}(b)$ for some angle θ is equal to the perpendicular bisector of $[a,b]$. In the case that $a=(2,-1)$ and $b=(3,2)$, the given set is the line $x+3y=4$.

(c) Let $a=(-1,2)$ and $b=(3,4)$. Draw a picture, or give an accurate description, of the set

$$\left\{0\neq u\in\mathbb{R}^2\middle|G_{u,L}(a)=b \text{ for some line } L \text{ with direction vector } u\right\}.$$

Solution: When $0\neq u\in\mathbb{R}^2$ and L is a line in $\mathbb{R}^2$ with direction vector u and $a\in\mathbb{R}^2$ and $b=G_{u,L}(a)$, the points a, $T_u(a)=a+u$, $F_L(a)$ and $G_{u,L}(a)=b$ are the vertices of a rectangle (unless $a\in L$ in which cas $F_L(a)=a$ and $T_u(a)=a+u=b$). It follows that the circle with diameter a,b passes through the point $T_u(a)=a+u$ (and also the point $F_L(a)$). By translating by $-a$ we see that u lies on the circle with diameter $0,b-a$. Conversely, suppose that u lies on the circle with diameter $0,,b-a$. Then the point $T_u(a)=a+u$ lies on the circle with diameter a,b. If $u=b-a$ so that $T_u(a)=b$ then we can take L to be the line through a and b to get $G_{u,L}(a)=F_L(a+u)=F_L(b)=b$. If $u\neq b-a$ then we can take L to be the perpendicular bisector of $[a+u,b]$ to get $G_{u,L}(a)=F_L(a+u)=b$. Thus the set of all points $0\neq u\in\mathbb{R}^2$ such that $G_{u,L}(a)=b$ for some line L with direction vector u is equal to the circle in $\mathbb{R}^2$ with diameter $0,b-a$ (with the point 0 removed). In the case that $a=(-1,2)$ and $b=(3,4)$, the given set is the circle with diameter from $(0,0)$ to $(4,2)$, that is the circle $(x-2)^2+(y-1)^2=5$, with the point $(0,0)$ removed.

11: (a) Let L be the line $x+3y=2$. Find the equation of the line M such that $F_M F_L = T_{(1,3)}$.

Solution: By Theorem 1.83, the line M is the line obtained by translating the line L by $\frac{1}{2}(1,3)=\left(\frac{1}{2},\frac{3}{2}\right)$. The line L passes through the point $(2,0)$ so the line M passes through the point $(2,0)+\left(\frac{1}{2},\frac{3}{2}\right)=\left(\frac{5}{2},\frac{3}{2}\right)$. Thus M is the line through $\left(\frac{5}{2},\frac{3}{2}\right)$ parallel to L, so M has equation $x+3y=7$.

(b) Let L be the line $2x-3y=1$. Find the equation of the line M with $F_M F_L = R_{(2,1),\frac{\pi}{2}}$.

Solution: By Theorem 1.83, the line N is the line obtain by rotating the line L by $\frac{\pi}{4}$ about the point $(2,1)$. Notice that the points $(2,1)$, $(5,3)$, $(3,6)$ and $(0,4)$ form a square, so the line N is the diagonal which passes through $(2,1)$ and $(3,6)$. Thus N is the line $y=5x-9$.

(c) Let $p\in\mathbb{R}^2$ and let L, M and N be any lines in $\mathbb{R}^2$ with $p\in L\cap M\cap N$. Show that $F_N F_M F_L = F_L F_M F_N$.

Solution: Let $\theta=\theta_0(L,M)$ so that $F_M F_L = R_{p,2\theta}$ and $F_L F_M = R_{p,-2\theta}$. Let K be the line obtained by rotating the line N about p by $-\theta$ so that $F_N F_K = R_{p,2\theta}$ and $F_K F_N = R_{p,-2\theta}$. Then we have

$$F_N F_M F_L = F_N R_{p,2\theta} = F_N F_N F_K = F_K,$$
$$F_L F_M F_N = R_{p,-2\theta} F_N = F_K F_N F_N = F_K.$$

12: (a) Express the isometry $R_{(4,4),\frac{\pi}{2}} F_{x+3y=6}$ as a glide reflection.

Solution: Note that for a glide-reflection $S = G_{u,L}$ and a point $a \in \mathbb{R}^2$, the points a, $F_u(a)$, $S(a)$ and $F_L(a)$ form a rectangle, and the reflection line L passes through the centre of this rectangle, which is the midpoint of $[a, S(a)]$. Let $S = R_{(4,4),90^\circ} F_{x+3y=6}$. Choose $a = (0,2)$ and $b = (3,1)$ (we could have chosen any two points a and b). We have

$$S(a) = R_{(4,4,),90^\circ} F_{x+3y=6}(0,2) = R_{(4,4),90^\circ}(0,2) = (6,0),$$
$$S(b) = R_{(4,4),90^\circ} F_{x+3y=6}(3,1) = R_{(4,4),90^\circ}(3,1) = (7,3).$$

The midpoint of a and $S(a)$ is $(3,1)$ and the midpoint of b and $S(b)$ is $(5,2)$. To have $S = G_{u,L}$, the reflection line L must pass through these two midpoints, so L is the line $x - 2y = 1$. The translation vector is the vector $u = S(a) - F_L(a) = (6,0) - (2,-2) = (4,2)$. Thus $S = G_{(4,2),x-2y=1}$.

(b) Express the isometry $F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}$ as a rotation.

Solution: Note that for a rotation $S = R_{p,\theta}$ and a point $a \in \mathbb{R}^2$, since a and $S(a)$ are equidistant form p, the point rotation p must lie on the perpendicular bisector of $[a, S(a)]$. Let $S = F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}$. Choose $a = (0,1)$ and $b = (2,2)$. We have

$$S(a) = F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}(0,1) = F_{y=3x} T_{(-2,3)}(2,2) = F_{y=3x}(0,5) = (3,4),$$
$$S(b) = F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}(2,2) = F_{y=3x} T_{(-2,3)}(4,3) = F_{y=3x}(2,6) = (2,6).$$

The perpendicular bisector of a and $S(a)$ is the line $x + y = 4$, and the perpendicular bisector of b and $S(b)$ is the line $y = 4$. As explained above, the rotation point is the point of intersection of these two perpendicular bisectors, which is the point $p = (0,4)$. The rotation angle θ is the angle from the vector $b - a = (2,1)$ to the vector $S(b) - S(a) = (-1,2)$, that is $\theta = \frac{\pi}{2}$. Thus $S = R_{(0,4),\frac{\pi}{2}}$.

(c) Express the isometry $G_{(2,0),y=\sqrt{3}} F_{\sqrt{3}x+y=2\sqrt{3}}$ as a rotation.

Solution: To help to visualize the relative points and lines in this problem, it helps to draw a grid of equilateral triangles with vertices at points $s(2,0) + t(1,\sqrt{3})$ with $s,t \in \mathbb{Z}$. Let $S = G_{(2,0),y=\sqrt{3}} F_{\sqrt{3}x+y=2\sqrt{3}}$. Choose $a = (2,0)$ and $b = (0,2\sqrt{3})$ (these two points lie on the line $\sqrt{3}x + y = 2\sqrt{3}$). Then

$$S(a) = G_{(2,0),y=\sqrt{3}} F_{\sqrt{3}x+y=2\sqrt{3}}(2,0) = G_{(2,0),y=\sqrt{3}}(2,0) = F_{y=\sqrt{3}}(4,0) = (4,2\sqrt{3}),$$
$$S(b) = G_{(2,0),y=\sqrt{3}} F_{\sqrt{3}x+y=2\sqrt{3}}(0,2\sqrt{3}) = G_{(2,0),y=\sqrt{3}}(0,2\sqrt{3}) = F_{y=\sqrt{3}}(2,2\sqrt{3}) = (2,0).$$

The perpendicular bisector of a and $S(a)$ is the line $x + \sqrt{3}y = 6$ (1) and the perpendicular bisector of b and $S(b)$ is the line $x - \sqrt{3}y = -2$ (2). As explained in Part (b), the rotation point p is the point of intersection of these two perpendicular bisectors, so we solve (1) and (2) to get $p = \left(2, \frac{4}{3}\sqrt{3}\right)$. The rotation angle θ is the oriented angle from the vector $b - a = 2(-1,\sqrt{3})$ to the vector $S(b) - S(a) = 2(-1,-\sqrt{3})$, that is $\theta = \frac{2\pi}{3}$.

13: Given a point $p \in \mathbb{R}^2$ and a real number $k \neq 0, 1$, we define the **dilation** about p with scaling factor k to be the map $D_{p,k} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $D_{p,k}(x) = p + k(x - p)$ for $x \in \mathbb{R}^2$.

(a) Show that if $q = p$ and $k\ell = 1$ then $D_{q,\ell} D_{p.k}$ is the identity.

Solution: Suppose that $q = p$ and that $k\ell = 1$. Then for $x \in \mathbb{R}^2$ we have

$$D_{q,\ell} D_{p,k}(x) = D_{p,\ell}\big(p + k(x-p)\big) = p + \ell\big(p + k(x-p) - p\big) = p + \ell k(x-p) = p + x - p = x$$

(b) Show that if $q \neq p$ and $k\ell = 1$ then $D_{q,\ell} D_{p,k}$ is a translation.

Solution: Suppose that $q \neq p$ and that $k\ell = 1$. Then for $x \in \mathbb{R}^2$ we have

$$\begin{aligned} D_{q,\ell} D_{p,k}(x) &= D_{q,\ell}\big(p + k(x-p)\big) = q + \ell\big(p + k(x-p) - q\big) \\ &= q + \ell(p - q) + (x - p) = x + (1 - \ell)(q - p) = T_u(x) \end{aligned}$$

where $u = (1 - \ell)(q - p)$.

(c) Show that if $q \neq p$ and $k\ell \neq 1$ then $D_{q,\ell} D_{p,k}$ is a dilation.

Solution: Suppose that $k\ell \neq 1$. Then for $x \in \mathbb{R}^2$ we have

$$\begin{aligned} D_{q,\ell} D_{p,k}(x) &= D_{q,\ell}\big(p + k(x-p)\big) = q + \ell\big(p + k(x-p) - q\big) \\ &= q + \ell(p - q) - k\ell p + k\ell x \end{aligned}$$

and for $r \in \mathbb{R}^2$ we have

$$D_{r,k\ell}(x) = r + k\ell(x - r) = r(1 - k\ell) + k\ell x.$$

and so we obtain $D_{q,\ell} D_{p,k}(x) = D_{r,k\ell}(x)$ by choosing

$$r = \frac{q + \ell(p - q) - k\ell p}{1 - k\ell}.$$

14: (a) Show that the composite of a translation and a dilation is a dilation.

Solution: Given $u \in \mathbb{R}^2$, $p \in \mathbb{R}^2$ and $k \in \mathbb{R}$ with $k \neq 0, 1$, we have

$$T_u D_{p,k}(x) = T_u\big(p + k(x-p)\big) = p + k(x-p) + u = u + p - kp + kx$$

and given $r \in \mathbb{R}^2$ we have

$$D_{r,k}(x) = r + k(x-r) = r(1-k) + kx$$

and so we obtain $T_u D_{p,k}(x) = D_{r,k}(x)$ by choosing

$$r = \frac{u + p - kp}{1-k} = p + \tfrac{1}{1-k}\,u.$$

Similarly, one can obtain $D_{p,k} T_u(x) = D_{r,k}(x)$ when $r = \frac{p+ku-kp}{1-k} = p + \frac{k}{1-k}\,u$.

(b) Show that given $k \in \mathbb{R}$ with $k \neq 0, 1$ and given $a, b, c, d \in \mathbb{R}^2$ with $(d-c) = k(b-a)$ there exists a unique point $p \in \mathbb{R}^2$ such that $D_{p,k}(a) = c$ and $D_{p,k}(b) = d$.

Solution: Let $k \in \mathbb{R}$ with $k \neq 0, 1$, let $a, b, c, d \in \mathbb{R}^2$ with $0 \neq (d-c) = k(b-a)$. We have $D_{p,k}(a) = c$ when $c = p + k(a-p) = p(1-k) + ka$ and this occurs when $p = \frac{c-ka}{1-k}$. We must check that with this choice of p we also have $D_{p,k}(b) = d$. Indeed, when $p = \frac{c-ka}{1-k}$ we have

$$\begin{aligned} D_{p,k}(b) = p + k(b-p) &= \tfrac{c-ka}{1-k} + k\Big(b - \tfrac{c-ka}{1-k}\Big) = \tfrac{c-ka}{1-k} + k\Big(\tfrac{b-kb-c+ka}{1-k}\Big) \\ &= \frac{c - ka + kb - k^2 b - kc + k^2 a}{1-k} = \frac{(c-ka+kb)(1-k)}{1-k} \\ &= c - ka + kb = c + k(b-a) = c + (d-c) = c. \end{aligned}$$

(c) Show that a dilation maps a circle to a circle, and find two dilations which send the circle $(x-2)^2 + y^2 = 5$ to the circle $(x-6)^2 + (y-2)^2 = 20$.

Solution: We claim that the dilation $D_{p,k}$ maps the circle C centred at a of radius r to the circle C' centred at $D_{p,k}(a)$ of radius $|k|r$. For $x \in \mathbb{R}^2$ we have

$$\Big|D_{p,k}(x) - D_{p,k}(a)\Big| = \Big|\big(p + k(x-p)\big) - \big(p + k(a-p)\big)\Big| = \big|k(x-a)\big| = |k||x-a|$$

and so

$$x \in C \iff |x-a| = r \iff \Big|D_{p,k}(x) - D_{p,k}(a)\Big| = |k|r \iff D_{p,k}(x) \in C'.$$

Thus the dilation $D_{p,k}$ maps the circle C to the circle C', as claimed.

Now let C be the circle $(x-2)^2 + y^2 = 5$ and let C' be the circle $(x-6)^2 + (y-2)^2 = 20$. The circle C has centre $a = (2,0)$ and radius $r = \sqrt{5}$ and the circle C' has centre $b = (6,2)$ and radius $s = 2\sqrt{5}$. By our above calculation, in order for $D_{p,k}$ to send C to C' we must have $s = |k|r$, that is $2\sqrt{5} = |k|\sqrt{5}$, that is $k = \pm 2$, and in this case the dilation $D_{p,k}$ sends C to C' provided that $D_{p,k}(a) = b$. As we just showed in Part (b), we have $D_{p,k}(a) = b$ when $b = p + k(a-p) = (1-k)p + ka$, that is when $p = \frac{b-ka}{1-k}$. In the case that $k = 2$ we obtain $p = \frac{b-ka}{1-k} = \frac{b-2a}{-1} = 2a - b = 2(2,0) - (6,2) = (-2,-2)$. In the case that $k = -2$ we obtain $p = \frac{b-ka}{1-k} = \frac{b+2a}{3} = \frac{1}{3}\big((6,2) + 2(2,0)\big) = \big(\frac{10}{3}, \frac{2}{3}\big)$. Thus the two dilations which send the circle C to the circle C' are the dilations $D_{(-2,-2),2}$ and $D_{(\frac{10}{3},\frac{2}{3}),-2}$.