PMATH 321 Non-Euclidean Geometry, Exercises for Chapter 1: Euclidean Geometry

1: Let $C = C(0, r)$ be the circle in $\mathbb{R}^2$ of radius r centred at the origin.

(a) Given $a \in \mathbb{R}$ with $0 < a < r$, find the centre $(b, 0)$ and radius s of the circle which passes through the point $(a, 0)$ and intersects the circle C orthogonally.

(b) Let $u = (0, r)$ and let $v = (x, y)$ be a point on C with $v \neq u$. Find the point of intersection $w = (a, 0)$ of the line through u and v with the x-axis.

(c) Let a, b, c and d be any 4 distinct points on C such that the line segment $[a, c]$ intersects with the line segment $[b, d]$ at a point p inside C. Show that $|a - p||p - c| = |b - p||p - d|$.

2: (a) Let $u = (1, 0)$, $v = (4, 1)$ and $w = (6, 5)$. Find the exact values of the interior angles α, β and γ in the triangle $[u, v, w]$.

(b) Let $u = (-1, 1)$ and $v = (2, 0)$. Find a point $w \in \mathbb{R}^2$ such that the triangle $[u, v, w]$ is positively oriented with $\alpha = \frac{\pi}{4}$ and $\gamma = \tan^{-1} \frac{1}{3}$.

(c) Let $a = \sqrt{5}$, $b = \sqrt{10}$ and $c = \sqrt{13}$. Find the exact area of the triangle with sides of length a, b and c.

3: (a) Let $u = (6, 7)$, $v = (-2, 3)$ and $w = (7, 0)$. Find the centre o and the radius r of the circumscribed circle of triangle $[u, v, w]$.

(b) Let $u = (0, 9)$, $v = (12, 0)$ and $w = 12, 14)$. Find the centre i and the radius r of the inscribed circle of triangle $[u, v, w]$.

(c) Let $u = (-2, 9)$, $v = (2, 1)$ and $w = (7, 6)$. Find the centres o, g and h of the triangle $[u, v, w]$ and verify that $h = 3g - 2o$.

4: (a) Let $u = (3, 1)$, $v = (3, 6)$ and $i = (2, 3)$. Find the point w in $\mathbb{R}^2$ such that i is the incentre of triangle $[u, v, w]$.

(b) Let $u = (7, 4)$, $o = (1, 1)$ and $g = (1, 2)$. Find points v and w in $\mathbb{R}^2$ such that o is the circumcentre and g is the centroid of triangle $[u, v, w]$.

5: (a) Let L be the line $x + y = 1$, let M be the line $3y = 2x + 6$, and let N be the line $2x + y = 6$. Find points $u, v, w \in \mathbb{R}^2$ such that, in the triangle $[u, v, w]$, L is the perpendicular bisector of $[u, v]$, M is the median from u, and N is the altitude from v.

(b) Let L be the line $2y = x + 4$, let M be the line $x + y = 4$, and let N be the line $y + 8 = 3x$. Find points $u, v, w \in \mathbb{R}^2$ such that, in the triangle $[u, v, w]$, L is the angle bisector at u, M is the altitude from v, and N is the median from w.

6: (a) Let $u = (1, 1, 2)$, $v = (2, 1, 3)$ and $x = (4, 1, -1)$. Find $\text{Proj}_U(x)$ where $U = \text{Span}\{u, v\}$.

(b) Let $a = (2, 1, 3)$, $b = (1, 2, 1)$, $u = (1, 3, 2)$ and $v = (2, 0, 1)$. Find the distance between the line $x = a + tu$ and the line $x = b + tv$.

(c) Referring to Theorem 1.22 in the Lecture Notes (Properties of the Cross Product), use Properties 1, 2, 3 and 5 to prove Property 4.

7: Let $u_1 = (1, 0, 1, -1)$, $u_2 = (2, 1, 1, 0)$, $u_3 = (1, -3, 2, 1)$, $\mathcal{B} = \{u_1, u_2, u_3\}$, $U = \text{Span}(\mathcal{B})$, and $x = (1, 1, 7, 3)$. Find $\text{Proj}_U(x)$ in the following three ways.

(a) Let $A = \big(u_1, u_2, u_3\big) \in M_{4\times 3}$ so $\text{Proj}_U(x) = At$ where t is the solution to $A^T A\, t = A^T x$.

(b) Apply the Gram-Schmidt Procedure to the basis $\mathcal{B}$ to obtain an orthogonal basis $\mathcal{C} = \{v_1, v_2, v_3\}$ for U, so that $\text{Proj}_U(x) = \sum\limits_{i=1}^{3} \frac{x \bullet v_i}{|v_i|^2}\, v_i$.

(c) Find $w \in \mathbb{R}^4$ so $\{w\}$ is a basis for $U^\perp$, so $\text{Proj}_U(x) = x - \text{Proj}_w(x) = x - \frac{x \bullet w}{|w|^2}\, w$.

8: (a) Let $a = (2, 1, 3)$, $b = (4, 1, 2)$ and $c = (1, -1, 5)$. Find the area of the triangle in $\mathbb{R}^3$ with vertices at a, b and c.

(b) (a) Let $a = (1, 0, 1)$, $b = (2, 2, 1)$, $c = (0, 1, 4)$ and $d = (-1, 1, 2)$. Find the volume of the tetrahedron with vertices at a, b, c and d (given that the volume of a tetrahedron is equal to $\frac{1}{3}Ah$ where A is the base area and h is the altitude).

(c) Find the volume of the regular icosahedron whose 12 vertices are at $\pm(1, \pm a, 0)$, $\pm(0, 1, \pm a)$ and $\pm(\pm a, 0, 1)$, where $a > 0$ is chosen so that the edges all have equal length.

9: Let x and p be points in $\mathbb{R}^3$, let u and v be unit vectors in $\mathbb{R}^3$ with $v \neq \pm u$, and let P and Q be the planes in $\mathbb{R}^3$ through p with unit normal vectors u and v,. The (Euclidean) **distance** between x and P is $d(x,P) = \left|\text{Proj}_u(x-p)\right| = \left|(x-p)\cdot u\right|$, the **angle** between P and Q is $\theta(P,Q) = \cos^{-1}|u\cdot v| \in \left[0,\frac{\pi}{2}\right]$, and the **angle bisectors** of P and Q are the two planes in $\mathbb{R}^3$ through p with normal vectors $u+v$ and $u-v$.

(a) Show that x lies on one of the two angle bisectors of P and Q if and only if we have $d(x,P) = d(x,Q)$.

(b) Let B be a plane in $\mathbb{R}^3$. Show that B is equal to one of the two angle bisectors of P and Q if and only if we have $P\cap Q \subseteq B$ and $\theta(B,P) = \theta(B,Q)$.

(c) Let $a = (-4,-3,1)$, $b = (8,3,1)$, $c = (2,6,1)$ and $d = (4,3,3)$. Find the centre of the inscribed sphere of the tetrahedron with vertices at a, b, c and d.

10: (a) Let $u = (1,8)$, $v = (2,1)$ and $w = (3,4)$. Find the image of the triangle $[u,v,w]$ under the isometry $G_{(-2,-6),y=3x-8}R_{(3,8),\frac{\pi}{2}}$.

(b) Let $a = (2,-1)$ and $b = (3,2)$. Draw a picture, or give an accurate description of the set

$$\left\{p \in \mathbb{R}^2 \middle| R_{p,\theta}(a) = b \text{ for some } \theta \in \mathbb{R}\right\}.$$

(c) Let $a = (-1,2)$ and $b = (3,4)$. Draw a picture, or give an accurate description, of the set

$$\left\{0 \neq u \in \mathbb{R}^2 \middle| G_{u,L}(a) = b \text{ for some line } L \text{ with direction vector } u\right\}.$$

11: (a) Let L be the line $x+3y=2$. Find the equation of the line M such that $F_M F_L = T_{(1,3)}$.

(b) Let L be the line $2x-3y=1$. Find the equation of the line M with $F_M F_L = R_{(2,1),\frac{\pi}{2}}$.

(c) Let $p\in\mathbb{R}^2$ and let L, M and N be any lines in $\mathbb{R}^2$ with $p \in L\cap M\cap N$. Show that $F_N F_M F_L = F_L F_M F_N$.

12: (a) Express the isometry $R_{(4,4),\frac{\pi}{2}}F_{x+3y=6}$ as a glide reflection.

(b) Express the isometry $F_{y=3x}T_{(-2,3)}G_{(2,1),x+2=2y}$ as a rotation.

(c) Express the isometry $G_{(2,0),y=\sqrt{3}}F_{\sqrt{3}x+y=2\sqrt{3}}$ as a rotation.

13: Given a point $p\in\mathbb{R}^2$ and a real number $k\neq 0,1$, we define the **dilation** about p with scaling factor k to be the map $D_{p,k}:\mathbb{R}^2\to\mathbb{R}^2$ given by $D_{p,k}(x) = p + k(x-p)$ for $x\in\mathbb{R}^2$.

(a) Show that if $q = p$ and $k\ell = 1$ then $D_{q,\ell}D_{p.k}$ is the identity.

(b) Show that if $q\neq p$ and $k\ell = 1$ then $D_{q,\ell}D_{p,k}$ is a translation.

(c) Show that if $q\neq p$ and $k\ell\neq 1$ then $D_{q,\ell}D_{p,k}$ is a dilation.

14: (a) Show that the composite of a translation and a dilation is a dilation.

(b) Show that given $k\in\mathbb{R}$ with $k\neq 0,1$ and given $a,b,c,d\in\mathbb{R}^2$ with $(d-c) = k(b-a)$ there exists a unique point $p\in\mathbb{R}^2$ such that $D_{p,k}(a) = c$ and $D_{p,k}(b) = d$.

(c) Show that a dilation maps a circle to a circle, and find two dilations which send the circle $(x-2)^2+y^2=5$ to the circle $(x-6)^2+(y-2)^2=20$.